UNDERSTANDING RESURGENCE

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This is a summary of Écalle's theory of resurgence. This is about regularization of divergent series.

1. Borel resummation

You want to make sense of the sum of a divergent series expansion. Like $f(x) = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1}$. This has radius of convergence 0. An idea is that you can

(1) formally transform the divergent series into another series,
(2) hope that it's not divergent everywhere there,
(3) and transform back.

It's called **Borel resummable** if this thing works for **Laplace transform** (or its inverse actually). Recall that the Laplace transform $\mathcal{L}F$ of F is

$$\mathcal{L}F(x) = \int_0^\infty e^{-xt} F(t)dt.$$

This really sends $x^n \mapsto n!/t^{n+1}$.

Example 1.1. Well, what is g such that formally $\mathcal{L}g(x) = f(1/x)$? It should be $g(t) = \sum_{n=0}^{\infty} (-1)^n t^n$. So $g(t)$ is summable, radius of convergence 1, to $\frac{1}{1+t}$. Now

$$\mathcal{L}g(x) = \int_0^\infty \frac{e^{-xt}}{1+t} dt.$$

This is a convergent integral if $\mathrm{Re}(x) > 0$.

The process works for a divergent series where $|a_n|$ grows, at worst, as fast as $n!$ times a geometric series. Such formal power series is called of **Gevrey order** ≤ 1. Similarly **Gevrey order** α means $|a_n|$ grows as fast as $(n!)^\alpha$ times a geometric series.

Definition 1.1. *(1) The* ***minor*** *of $\varphi \in \mathbb{C}[[x^{-1}]]$ is the Borel transform of $\varphi - a_0$, where a_0 is the constant term of φ. This is because 1 cannot really be Borel-transformed. A series with $a_0 = 0$ is called* ***small****.*

(2) A germ of analytic functions at the origin is ***endlessly continuable*** *if for every $L > 0$, there is a finite set $\Omega_L(\varphi) \subset \mathbb{C}$,* L-accessible singularities*, such that $\underline{\varphi}$ has an analytic continuation along any path missing $\Omega_L(\varphi)$.*

(3) A germ of analytic functions at the origin that is endlessly continuable has ***simple singularities*** *if, along any path approaching to a singularity, the singulaity is of form (simple pole)$+$(log)$+$* (analytic)*.*

(4) A ***simple resurgent function*** *is a formal power series of Gevrey order ≤ 1 whose minor is endlessly continuable function with simple singularities. The set of simple resurgent functions is a subalgebra ${}^{+}\mathcal{R}(1) \subset \mathbb{C}[[x^{-1}]]$.*

Here the multiplication is just a usual multiplication of functions. Thus the last statement needs a verification. This holds because

- multiplcation is sent to convolution,
- convolution of two endlessly continuable function is endlessly continuable (with even a description of possible positions of singulairities; namely

$$\Omega(\underline{\varphi} * \underline{\psi})_L \subset \bigcup_{L_1+L_2\leq L} \left(\Omega(\underline{\varphi})_{L_1} + \Omega(\underline{\psi})_{L_2}\right),$$

- and a convolution of two functions with simple singularities has simple singularities.

Furthermore, ${}^{+}\mathcal{R}(1)$ is stable under natural differentiation $\frac{d}{dx}$, because it is sent via Borel transform to the operator ∂, where $\partial\underline{\varphi}(\xi) = -\xi\underline{\varphi}(\xi)$.

There is more complication in trying to substitute the variable into a function, but then some cases are okay, like $\exp\varphi$, $\log$, or $(1-(-))^{-1}$.

Considering expanding at a different point other than 0, one is led to consider the definition of an **elementary resurgent symbol** which is just of form φe^{-xw} for $w \in \mathbb{C}$, which is called the **support** of the symbol. The Borel transform sends φe^{-xw} to $\underline{\varphi}(x-w)$.

Remark 1.1. Although this might seem meaningless at the first sight, remember that you work with expansion at $x = \infty$, so e^{-xw} has essential singularity at the point of our expansion. In other words, in a usual language where we work with $\mathbb{C}[[x]]$, we put $e^{-w/x}$ as a symbol.

So far the discussion has been purely formal. To really try to do resummation, one has to consider the integrals of the form

$$s_\alpha g(x) = \int_0^{+\alpha\infty} e^{-xt} g(t)dt,$$

where α is a direction (e.g. $\alpha \in S^1 \subset \mathbb{C}$) and $+\alpha\infty$ is the ray from 0 to the direction of α. This might not be well-defined precisely because g might have a singularity on the ray. If there is a singularity, then one can consider $s_{\alpha+}g$ and $s_{\alpha-}g$, where these two integrals have contour along the same ray but missing the singularities slightly by having a detour. $+$ means the detour is always clockwise (choosing detour making argument larger than $\arg\alpha$) and $-$ means the detour is always counterclockwise.

Each $s_\alpha g(x)$ is absolutely convergent as an integral for $\mathrm{Re}(\alpha x) > 0$, and if there is no singularity between the cone between the ray of α and that of α', then $s_\alpha g(x)$ has $s_{\alpha'}g(x)$ coincide on the overlap of region of absolute convergence. So, if you imagine α going through the circle once, the functions $s_\alpha g(x)$ can possibly change after passing a singularity. This is called the **Stokes phenomenon**. Because the contour for $s_{\alpha+}g(x) - s_{\alpha-}g(x)$ is just the same as small (positive) simple loops around the singularities, it is really easy to calculate what the effect of passing a singular direction is. One calls s_α ($s_{\alpha+}$, $s_{\alpha-}$, respectively) the (right lateral, left lateral, respectively) **Borel sum** of g in the direction α. In this regard, if you instead start with a Gevrey order r formal power series φ with $r > 1$, then its minor is an order $r-1$ function (in the sense of complex analysis), and the Borel transform $s_\alpha\underline{\varphi}(x)$ would absolutely converge for $\mathrm{Re}(x\alpha) > r-1$ (here $|\alpha| = 1$ is normalized). Also one can think of resurgent symbol for these. From now on s_α (and its variants) will both take functions in ξ (namely, actual analytic function germs at 0) and functions in x (namely, formal power series of finite Gevrey order).

Remark 1.2. Here something needs to be clarified. When you are calculating the integral for $s_{\alpha_+} g$ and $s_{\alpha_-} g$, you are going through different paths avoiding the singularities, so the integrands are different! They are different analytic continuations, even though they are evaluated at the "same point". They are either considered as multivalued functions on $\mathbb{C}$ minus singularities, or just a function on the universal cover of that domain. For example, when you are passing a log singularity, then depending on whether you choose $+$ direction or $-$ direction, the actual log function changes.

Suppose φ is a formal power series such that its minor $\underline{\varphi}$ is endlessly continuable with simple singularities (abbreviated to **ECSS** from now on) has only one singularity w along the direction α. Then, around w, $\underline{\varphi}$ has of form

$$\underline{\varphi}(\xi) = \frac{a_w}{2\pi i(\xi - w)} + \underline{\varphi}_w(\xi - w)\frac{\log(\xi - w)}{2\pi i} + h_w(\xi - w),$$

where $\underline{\varphi}_w$ and h_w are holomorphic around w. Then,

$$(s_{\alpha_+} - s_{\alpha_-})\varphi(x) = \int_{C_{w_\alpha}} e^{-x\xi}\underline{\varphi}(\xi)d\xi,$$

where C_{w_α} is the contour coming from $+\alpha\infty$, go around w once clockwise, and going to $+\alpha\infty$ from there. So the difference is precisely coming from the difference of integrand we talked above (namely, coming from different branches of log). So

$$\int_{C_w} e^{-x\xi}\underline{\varphi}(\xi)d\xi = -e^{-xw}(a_w + s_\alpha \underline{\varphi}_w) = s_\alpha((a_w + \varphi_w)e^{-xw}),$$

if you accept the definition that φ_w is the formal power series that will give rise to $\underline{\varphi}_w$ as its minor, and s_α is trivially extended to apply to resurgent symbols as above. So one can think of the operator of simple resurgent functions

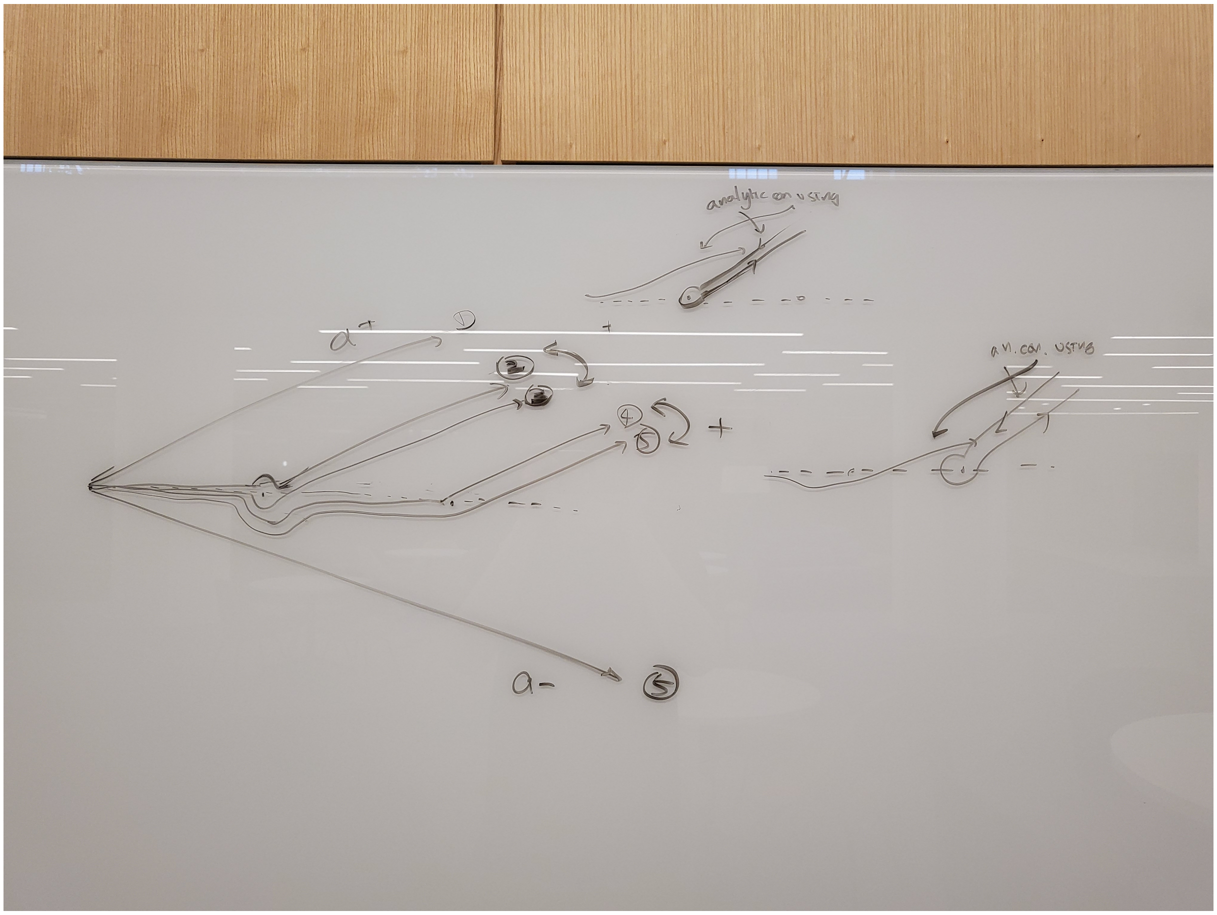
$$\mathcal{S}_w : {}^+\mathcal{R}(1) \to {}^+\mathcal{R}(1), \varphi \mapsto a_w + \varphi_w,$$

called **hold of singularity at** w, and let $\dot{\mathcal{S}}_w = e^{-xw}\mathcal{S}_w$, then we have

$$s_{\alpha_+}(1 + \dot{\mathcal{S}}_w) = s_{\alpha_-}.$$

This analysis extends to the case where there are many (yet discrete) singularities on the ray of direciton α. Basically one defines $\mathcal{S}_{\alpha_+}\varphi = \sum_w \dot{\mathcal{S}}_{w+}\varphi$, where the functions $a_w + \varphi_w$ are taken through the analytic continuation on the right, then $s_{\alpha_+} = (1 + \mathcal{S}_{\alpha_+}) = s_{\alpha_-}$ and so on.

Example 1.2. In the case of Euler series, its minor $g(\xi) = \frac{1}{1+\xi}$ has singularity at $\xi = -1$, with $\mathcal{S}_{-1}(f) = 2\pi i$.



Now one can work with resurgent symbols with many supports, $\dot{\varphi} = \sum_{\omega\in\Omega(\dot{\varphi})} \dot{\varphi}^{\omega}$. To have meaningful Stokes phenomenon, we want the supports $\Omega(\dot{\varphi})$ to be contained in a half-plane; $\dot{\varphi}$ is said to be a **resurgent symbol in the codirection** A if A is an arc of S^1 and $\Omega(\dot{\varphi})$ is contained in the intersection of half-planes which cover A. The space of resurgent symbolds in the codirection A is denoted ${}^{+}\dot{\mathcal{R}}(1)(A)$, which is an algebra because $\Omega(fg) \subset \Omega(f) + \Omega(g)$. These define a sheaf ${}^{+}\dot{\mathcal{R}}(1)$ over S^1, and its stalk at $\alpha \in S^1$ is the set $\dot{\mathcal{R}}_{\alpha}$ of resurgent symbols in the direction α. One has obvious extension of $\mathcal{S}_{\alpha}$ and $s_{\alpha_{\pm}}$ to ${}^{+}\dot{\mathcal{R}}(1)(A)$.

Of course, there is a subtlety, because now the resummation process involves extra infinite sum, namely the sum over the suppport (discrete yet may be countably infinite).

Theorem 1.1. *The homomorphism* $\underline{S}_{\alpha} = 1 + \mathcal{S}_{\alpha_+} : {}^{+}\dot{\mathcal{R}}(1)(A) \to {}^{+}\dot{\mathcal{R}}(1)(A)$ *is an automorphism, and* $\dot{s}_{\alpha_+}\underline{S}_{\alpha} = \dot{s}_{\alpha_-}$.

This is called the connection automorphism because it connects wall-crossing phenomena in Stokes phenomenon.

2. Alien derivatives

Formally $f(x+1) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} = \exp(\frac{d}{dx})f(x)$, which gives formally $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} A^n f(x)$, where $Af(x) = f(x+1) - f(x)$. Similarly one can try to define a differentiation Δ_{α} of ${}^{+}\dot{\mathcal{R}}(1)(A)$ by $\underline{S}_{\alpha} = \exp\Delta_{\alpha}$, or $\Delta_{\alpha} = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \mathcal{S}_{\alpha_+}^{n}$. This is called the **directional differentiation**. This can be decomposed into the sum of **alien derivatives** (différentiel étranger)

$$\Delta_\alpha = \sum_{w\in(0,+\alpha\infty)} \dot{\Delta}_w, \quad \dot{\Delta}_w = e^{-xw}\Delta_w.$$

Actual definition would be

$$\dot{\Delta}_w = \sum_{w_1+\cdots+w_n=w, w_i\in(0,+\alpha\infty)} \frac{(-1)^{n-1}}{n}\dot{\mathcal{S}}_{w_1+}\cdots\dot{\mathcal{S}}_{w_n+}$$

Interestingly, this alien derivative is not very much related to the usual derivative. One has a relation as follows: $\frac{d}{dx}$ extends to ${}^+\dot{\mathcal{R}}(1)(A)$ via

$$\frac{d}{dx}\left(\sum_{w\in\Omega(\dot{\varphi})}\varphi^w e^{-xw}\right) := \sum_{x\in\Omega(\dot{\varphi})}(-w\varphi^w + \frac{d}{dx}\varphi^w)e^{-xw},$$

and this commutes with $s_{\alpha_\pm}$. Thus, this commutes with $\dot{\mathcal{S}}_{\alpha_\pm}$'s, and so $\dot{\Delta}_w$ commutes with $\frac{d}{dx}$. So for Δ_w, $[\Delta_w, \frac{d}{dx}] = -w\Delta_w$.

Example 2.1. Using the alien derivative formula, we have $\dot{\Delta}_w f = 2\pi i$, $\Delta_w f = e^x 2\pi i$. Remember that we are working at $x = \infty$ though.

One can conversely define a power of $\underline{S}_\alpha$, by

$$\underline{S}_\alpha^\nu = \exp\left(\nu\Delta_\alpha\right),$$

for any $\nu \in \mathbb{C}$.

If you try to do resummation of a divergent series, even through this process, there are multitude of answers because of Stokes phenomenon. This may be fine if you're trying to find solutions to a differential equation. But if there is a **reality condition**, namely if every coefficient and the point you're evaluating at is real, then you would want the result to be real; and there you have some sort of "canonical answer," called **median resummation**,

$$\dot{s}_{med} = \dot{s}_{0_+} \circ \underline{S}_0^{1/2}.$$

This definition may be justified by the fact that complex conjugation intertwines $\dot{s}_{0_+}$ and $\dot{s}_{0_-}$, and $\underline{S}_0^\nu$ with $\underline{S}_0^{-\nu}$.

3. Acceleration and multisummability

There is a connection with the asymptotic expansion theory. Recall that there is a similar story for a differential equation with irregular singularities that, around such singularity, you have a formal solution, a divergent series, and one can divide the neighborhood of the singularity into sectorial neighborhoods, divided by Stokes lines, such that on each sectorial neighborhood there is an acutal holomorphic solution whose asymptotic expansion agrees with the given divergent series. And that the solutions change after crossing Stokes lines is also called the Stokes phenomenon.

Now here comes the connection: the whole Borel process gives a way to obtain the solutions from the asymptotic expansion. On the other hand, the Borel/Laplace transform only works for Gevrey order 1 series. One can amp up the process with any finite Gevrey order series by just using substitution $x \mapsto x^{1/n}$, and this is called the **acceleration operator** of Écalle (note that this conjugated by Laplace transform rotates the direction α slightly, so the actual acceleration operator is a slightly adjusted version of this). One has fractional/real powers for the Borel transform but this is not a problem, one can use Gamma function instead of factorial. The upshots are

- a formal solution to a differential equation, with coefficients being convergent power series, is of finite Gevrey order,
- and any finite Gevrey order series is "summable" after applying finitely many procedures as above (namely, repeated application of acceleration). This is needed because you might get many different $e^{f(x^{-1})}$ in the formal solution as in the Levelt–Hukuhara–Turritin; recall that one acceleration process (say k-acceleration) can only give terms of form e^{a/x^k}. This is called **multisummable**.
- By this connection, one can see Stokes matrices as being elements of differential Galois group.