IRREGULAR SINGULARITY AND STOKES PHENOMENON

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1. Formal reduction theory

Reference: Turrittin, *Convergent solutions of ordinary linear homogeneous differential equations in the neighborhood of an irregular singular point*

Think about formally solving a linear ODE in one variable,

$$\frac{du}{dz} = A(z)u$$

for $u = (u_i)_{1\le i\le n}$, given a matrix $A = (a_{ij})_{1\le i,j\le n}$ of meromorphic functions. Fuchs' theorem says this is a regular singularity if A has at worst simple pole, so suppose it has a pole of order ≥ 2. Write it as

$$A = A_0 z^{-r} + A_1 z^{-r+1} + \cdots$$

Theorem 1.1 (**Spoiler alert**)**.** *It has a formal fundamental solution of the form*

$$H(z)z^J \exp(Q(z))$$

where for some positive integer p,

- *$Q(z)$ is a diagonal matrix with diagonals polynomials of degree at most $p(r-1)$ with variable $z^{-1/p}$*
- *J is a constant matrix commuting with every $Q(z)$,*
- *$H(z)$ is a formal series in the variable $z^{1/p}$, where $H(z)^{-1}$ can also be written as a formal Laurent series in the variable $z^{1/p}$.*

I will copy the way it's written in Turrittin. If you transform $u' = Au$ to $v = gu$ (gauge transformation) then the new $v' = Bv$ is with $B = gAg^{-1} + g'g^{-1}$.

(1) $r = 0$. There is no worry because just plugging $u = \sum_{k=0}^{\infty} H_k z^k$ will give a recursive relation so that H_0 will determine $H_1, H_2, \cdots$ in order.

(2) $r > 0, n = 1$. This is also not super mysterious because one can separate variables. Let's do this in a little more suggestive way:

- Substitute

$$u = \exp\left(A_{r-1}\log z - \frac{A_{r-2}}{z} - \frac{A_{r-3}}{2z^2} - \cdots \frac{A_0}{(r-1)z^{r-1}}\right) v$$

  Then if you expand, basically everything bad goes away:

$$\frac{dv}{dz} = (A_r + A_{r+1}z + A_{r+2}z^2 + \cdots)v$$

- Then it becomes Case 1.

(3) For the rest of the cases, if we make any transformation of the sort

$$u = Pv$$

where P is a constant nonsingular matrix, then the original equation will be written as $\frac{dv}{dz} = P^{-1}APv$. Thus one can WLOG assume that A_0 is in the Jordan normal form. We assume

$$A_0 = \begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & M_m \end{pmatrix}$$

where each M_i is of form

$$M_i = \begin{pmatrix} \rho_i & 0 & 0 & \cdots & 0 \\ \beta_i & \rho_i & 0 & \cdots & 0 \\ 0 & \beta_i & \rho_i & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & \beta_i & \rho_i \end{pmatrix}$$

where β_i is either 0 or 1.

Now consider the case $m = 1$ and $\beta_1 = 0$. Namely suppose $A_0 = \rho_1 I$ is a scalar matrix. Then we make the following transformation:

$$u = v \exp\left(-\frac{\rho_1}{(r-1)z^{r-1}}\right)$$

if $r > 1$, and

$$u = v \exp(\rho_1 \log z)$$

if $r = 1$. Basically this removes the lowest order term and transforms into

$$\frac{dv}{dz} = (A_1 z^{-r+1} + A_2 z^{-r+2} + \cdots)v.$$

This is then subject to induction.

(4) Now the next natural thing is to subdivide everything into Jordan blocks. This is possible because if you use the transformation

$$u = (I + z^k Q_k)v,$$

then the ODE transforms into

$$\frac{dv}{dz} = z^{-r}(A_0 + \cdots + A_{k-1}z^{k-1} + C_k z^k + C_{k+1}z^{k+1} + \cdots)v,$$

where $C_k = A_k + A_0 Q_k - Q_k A_0$ if $r > 1$ and $C_k = A_k + A_0 Q_k - Q_k A_0 - kQ_k$ if $r = 1$. The point is that it does not change $A_0, \cdots, A_{k-1}$. If you do a small calculation then you can discover that, upon a good choice of Q_k, you can make C_k into a block diagonal matrix where each block corresponds to an eigenvalue of A_0. If $g = 1$, then one can't eliminate the (r, s)-block where $k = \rho_r - \rho_s$ but other off-diagonal blocks are eliminated. So using a transformation

$$u = ((I + zQ_1)(I + z^2 Q_2)(I + z^3 Q_3) \cdots)v$$

one can subdivide the ODE into Jordan blocks of A_0.

(5) If $g = 1$ and ρ_i's are the same to ρ, then we can assume $\beta_i = 1$ for some i. Let $A_0 = \rho I + E$. Then one just dictates that there should be a solution of form

$$u(z) = (H_0 + H_1 z + \cdots) \exp((\rho I + E) \log z)$$

If you put this, you get

$$\frac{du}{dz} = \sum_{k=0}^{\infty} ((k+1) H_{k+1} z^k + H_k z^{k+1} (\rho I + E)) \exp((\rho I + E) \log z)$$

If you equate both sides, then you get $H_0 = I$, $H_1 + H_1 E = E H_1 + A_1, \cdots, k H_k + H_k E = E H_k + A_1 H_{k-1} + A_2 H_{k-2} + \cdots + A_{k-1} H_1 + A_k$. In general you can uniquely solve H for

$$kH + HE = EH + A$$

given A. So you can solve $H_1, H_2, \cdots$. This if you write out means u_i is

$$z^\rho (\text{formal power series involving } \log(z) \text{ to the power at most } n)$$

which is what I know from regular singularities knowledge.

(6) If $g = 1$ and eigenvalues differ by integers, then you can still apply the same simplification process

$$u = ((I + zQ_1)(I + z^2 Q_2)(I + z^3 Q_3) \cdots) v$$

but some of the off-diagonal matrices survive. WLOG we order ρ_i's in an increasing order. Then the simplified ODE becomes

$$\frac{dv}{dz} = \frac{A_0 + zD + K}{z} v$$

where D is the diagonal term and K is the off-diagonal term. Because we ordered eigenvalues, K is upper triangular, and the (r, s)-block is a constant matrix K_{rs} times $z^{-\rho_s + \rho_r}$.

Now we use the transform

$$v = \mathrm{diag}(z^{\rho_1} I_1, \cdots, z^{\rho_m} I_m) w$$

Then every off-diagonal term becomes a constant matrix! More precisely the ODE becomes

$$\frac{dw}{dz} = \frac{D_3 + zD}{z} w$$

where D_3 is a constant block-upper triangular matrix where the diagonal terms are subdiagonal part of Jordan blocks and upper triangular part is precisely K_{rs}'s. In particular all generalized eigenvalues of D_3 are 0! Now it is reduced to the Case 5.

(7) Now what remains is the $g > 1$ case where all eigenvalues are the same. A similar substitution can enable us to assume that $\rho_i = 0$. WLOG we order so that nontrivial Jordan blocks are ordered in the bottom side and larger such block is located in the latter part of

the order. For example an example of a right ordering is

$$\left(\begin{array}{cc|cc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

Then again

$$u = ((I + zQ_1)(I + z^2Q_2)(I + z^3Q_3)\cdots)v$$

can do a certain job. In the above example each A_k, for $k \geq 1$, will be reduced into a form

$$\left(\begin{array}{cc|cc|cccc} * & * & 0 & * & 0 & 0 & 0 & * \\ * & * & 0 & * & 0 & 0 & 0 & * \\ \hline * & * & 0 & * & 0 & 0 & 0 & * \\ * & * & 0 & * & 0 & 0 & 0 & * \\ \hline * & * & * & * & 0 & 0 & 0 & * \\ * & * & * & * & 0 & 0 & 0 & * \\ * & * & * & * & 0 & 0 & 0 & * \\ * & * & * & * & 0 & 0 & 0 & * \end{array}\right)$$

Now for $\mu > 0$ consider making a sheraing transform $v = \mathrm{diag}(z^{\mu(n-1)}, \cdots, z^{\mu}, 1)w$. Then it has an effect (more or less) that (i,j)-part is multiplied by $z^{\mu(i-j)}$. In particular the subdiagonal parts appearing in A_0 will now have z^{μ}. If the above-diagonal parts have power series starting with large enough power, then we can take $\mu = 1$ so that the resulting matrix is actually a multiple of z! One can talk about the **crticial value** μ_0 of μ where z^{μ} gets equal to the decreased leading power in the upper triangle. So if $\mu_0 \geq 1$, then we can take $\mu = 0$ to reduce r to $r-1$.

(8) Now if $\mu_0 < 1$, then $\mu_0 = \frac{q}{p}$ for some positive integers $q < p$, $(q,p) = 1$. Then now we go to a cover $t = z^{1/p}$. The order gets larger but weirdly one can prove that one can repeat the above process again and again and basically you terminate after a finite number of steps..! It is hard to imagine that it leads anywhere, but here is an example. Suppose you have, for $a \neq 0$,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} z^{-2} + \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} z^{-1}.$$

Then the gauge transformation for $g = \begin{pmatrix} z^{-2/3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z^{2/3} \end{pmatrix}$ yields a new B with

$$B = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} z^{-5/3} + \begin{pmatrix} -1/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} z^{-1}$$

so the leading term actually is semisimple..!

The idea behind this termination is that the **dimension of the nilpotent orbit increases** after each process.

There are two fancy recast of the theory:

- If you see $u' = Au$ as the solution of $\nabla_{d/dz} u = 0$ where $\nabla_{d/dz} = \frac{d}{dz} - A$, then this becomes a flat meromorphic connection over some base.
- If one is interested in local behavior, then one gets a differential module over $\mathbb{C}_{z,\mathrm{cgt}}$ where $\mathbb{C}_{z,\mathrm{cgt}}$ is the field of convergent Laurent series. It has not only an interpretation as the germs of holomorphic vector bundle with meromorphic connection but also equipped with a gauge adjoint action of $\mathrm{GL}_n(\mathbb{C}_{z,\mathrm{cgt}})$ and also one could just make it algebraic and just study differential modules.

In terms of gauge equivalence class, one have the following classical theorem.

Definition 1.1. *Let K be an algebraically closed field of characteristic zero, and let $K_{z,\infty} = \cup_{n\geq 1} K[[z^{1/n}]][z^{-1/n}]$. A **canonical form** is an element of $\mathfrak{gl}_n(K_{z,\infty})$ of the form*

$$B = D_1 z^{r_1} + \cdots + D_m z^{r_m} + z^{-1} C$$

*where $r_1 < \cdots < r_m < -1$ are rational numbers, $C, D_1, \cdots, D_m \in \mathfrak{g}_n(K)$ commute with each other, and D_j's are nonzero and diagonal if $m \neq 0$. The r_j's are called **canonical levels**, r_1 is the **principal level=Katz invariant**. $D_1 z^{r_1} + \cdots + D_m z^{r_m}$ is the **irregular part** of B. We say B is unramified if all r_j's are integers. It is **defined over** $K_{z,b}$ if r_j's are in $\frac{1}{b}\mathbb{Z}$. The smallest such b is called the **ramification index**. For such b, it is b-**reduced** all eigenvalues of C has real parts $0 \leq \operatorname{Re} \lambda < \frac{1}{b}$.*

Theorem 1.2 (Hukuhara, Levelt, Turrittin)**.** *Any $A \in \mathfrak{gl}_n(K_{z,\infty})$ is gauge equivalent to a canonical form whose canonical levels depend only on the gauge equivalence class of A (namely, any two canonical forms that are gauge equivalent have the same canonical levels). If $A \in \mathfrak{gl}_n(K_z)$ and b is the ramification index of the canonical form to which A is equivalent, then one can find a gauge equivalent b-reduced canonical form. Two gauge-equivalent b-reduced canonical forms over $K_{z,b}$ are conjugate by ana element in $\mathrm{GL}_n(K)$. Finally, if $A \in \mathfrak{gl}_n(K_z)$, then every canonical level has denominator $\leq n$, so that its canonical forms are defined over $K_{z,n!}$ and are $n!$-reduced, and the gauge transformation traking A to its canonical form may be chosen to be in $\mathrm{GL}_n(K_{z,n!})$.*

In this context the shearing transformation of $A \in \mathfrak{gl}_n(K_z)$ with nilpotent leading term is basically by z^{qH} for some $q \in \mathbb{Q}$ where H is the H coming from applying Jacobson-Morozov to the leading term. This gives that one can gauge-transform so that either the leading term has two distinct eigenvalues or it is still nilpotent but lies in an affine subspace of $\mathfrak{gl}_n(K)$ that contains the leading term but the space is **transversal to the nilpotent orbit** and the new nilpotent leading term is obtained by deforming along the affine subspace while staying nilpotent throughout the deformation. This implies that if it stays nilpotent the dimension of the nilpotent orbit has to increase.

Theorem 1.3 (Babbitt-Varadarajan)**.** *If $A \in \mathfrak{gl}_n(\mathbb{C}_{z,\infty})$ is a connection of order r for $r < -1$ (i.e. the smallest power of z), and let $M = n(|r| - 1)$, then if $A \equiv B (\operatorname{mod} z^M)$, then the irregular parts of the canonical forms of A and B are the same. Moreover, we can find a rational number $k \geq 0$ depending only on the A_{r+s}, $0 \leq s \leq M$, such that if $A \equiv B (\operatorname{mod} z^{M+k})$, then A and B are gauge equivalent.*

2. Stokes phenomenon

Now we see that for irregular singularities one needs things like $z^{1/p}$ or worse $e^{z^{1/p}}$. This is why formal structure around irregular singularity does not determine the analytic structure. However if we restrict to a small sector one can talk about **asymptotic solution**. Here a sector means a fan of form $r > 0, a < \theta < b$. Given a sector Γ, let Γ_δ be $\Gamma \cap \{r < \delta\}$. An open set $\Omega \subset \Gamma$ is **asymptotic** to the sector Γ if for each sector $\Gamma' \Subset \Gamma$ there is $\delta > 0$ such that $\Gamma'_\delta \subset \Omega$. Let $\mathcal{A}(\Gamma)$ be the $\mathbb{C}$-algebra of germs of analytic functions f defined on open sets asymptotic to Γ, two such functions defining the same germ if thy coincide on an open set asymptotic to Γ, such that there is an element $\widetilde{f} \in \mathbb{C}_z$ which is asymptotic, which means that for any integer $N \geq 1$ and sector $\Gamma' \Subset \Gamma$, we have $f(z) = \sum_{r \leq N} f_r z^r + O(|z|^{N+1})$ uniformly in Γ' as $z \to 0$. It is a differential $\mathbb{C}$-algebra with unit, and $f \mapsto \widetilde{f}$ is a homomorphism of differential algebras. Furthermore, $\mathcal{A}(\Gamma) \to \mathbb{C}_z$ is **surjective** (Borel-Ritt). Those in the kernel are called **flat**. A famous example is $e^{-1/z}$.

In this language the remark we made at the beginning of this section can be rephrased as follows: even if two elements $A_1, A_2 \in \mathfrak{gl}_n(\mathbb{C}_{z,\mathrm{cgt}})$ are gauge-equivalent under $\mathrm{GL}_n(\mathbb{C}_z)$, they are not necessarily gauge-equivalent under $\mathrm{GL}_n(\mathbb{C}_{z,\mathrm{cgt}})$. But from the asymptotic analysis one can reduce to $\mathrm{GL}_n(\mathcal{A}(\Gamma))$..!

Theorem 2.1. *Over a sector Γ, consider the system*

$$z^{m+1}\frac{du_i}{dz} = \delta_i u_i + f_i(u_1, \cdots, u_n)(z), 1 \leq i \leq n$$

where δ_i's are nonzero complex numbers, f_i's are polynomials in the u's with coefficients in $\mathcal{A}(\Gamma)$ and the coefficients of f_i are asymptotically of order ≥ 0; those of the terms of degree ≤ 1 in the u_i have asymptotic order > 0. Consider its formalization

$$z^{m+1}\frac{d\widehat{u}_i}{dz} = \delta_i \widehat{u}_i + \widehat{f}_i(\widehat{u}_1, \cdots, \widehat{u}_n)(z).$$

Suppose that the vertex angle of Γ is $\leq \pi/m$. If $v = (v_i)$, $v_i \in \mathbb{C}[[z]]$, is a solution to the formalzed system, and the order of v_i is > 0 for all i, then we can find a solution to the original system in $\mathcal{A}(\Gamma)$ asymptotic to this.

Theorem 2.2. *Let Γ be any sector, and let $A_1, A_2 \in \mathfrak{gl}_n(\mathcal{A}(\Gamma))$ be such that $\xi[\widehat{A}_1] = \widehat{A}_2$ for some $\xi \in \mathrm{GL}_n(\mathbb{C}_z)$. Let r_1 be the common principal level of $\widehat{A}_i$'s. If the angle of Γ is $\leq \pi/(|r_1| - 1)$, we can find $x \in \mathrm{GL}_n(\mathcal{A}(\Gamma))$ such that $x \sim \xi$ and $x[A_1] = A_2$.*

Now one can systematically study this kind of obstruction.

Definition 2.1. *Let $A_0 \in \mathfrak{gl}_n(\mathbb{C}_{z,\mathrm{cgt}})$. The* ***Stokes sheaf*** *of A_0, $\mathrm{St}(A_0)$, is the sheaf of groups on S^1 whose stalk at any point $\theta \in S^1$ is the group of germs of $n \times n$-matrix-valued analytic functions on Γ_δ for $\delta > 0, \Gamma$ a sector containing θ, such that*

- *$u \sim 1$ in some sector containing θ,*
- *$u[A_0] = A_0$.*

This is just the collection of ambiguity on taking asymptotic lifts. Said differently, we can formulate the following.

- Let (A, ξ) for $A \in \mathfrak{gl}_n(\mathbb{C}_{z,\mathrm{cgt}})$ and $\xi \in \mathrm{GL}_n(\mathbb{C}_z)$ with $\xi[A] = A_0$ be called a **marked pair**. Two marked pairs are equivalent if ξ's are gauge equivalent over $\mathbb{C}_{z,\mathrm{cgt}}$. Let $\mathfrak{M}(A_0)$ be the set of equivalence classes of marked pairs.
- Given a marked pair (A, ξ) we can find a finite open covering U_i of S^1 by arcs, $\delta > 0$ and holomorphic maps $x_i : \Gamma(U_i)_\delta \to \mathrm{GL}_n(\mathbb{C})$ such that $x_i \sim \xi$ on some part of sector containing $\Gamma(U_i)_\delta$ and $x_i[A] = A_0$.
- Then $(x_i x_j^{-1})$ forms a 1-cocycle of $\mathrm{St}(A_0)$. Furthermore, the class of this cocycle does not depend on any choice, so we get a well-defined map

$$\Phi : \mathfrak{M}(A_0) \to H^1(S^1, \mathrm{St}(A_0)).$$

Theorem 2.3 (Malgrange–Sibuya). *Φ is a bijection that sends (A_0, id) to the trivial class.*

One can similarly consider a sheaf of groups $\mathcal{G}$ on S^1 where one only requires $u \sim 1$ (thus it does not depend on A_0). Then the analogous consideration gives a map

$$\Theta : \mathrm{GL}_n(\mathbb{C}[[z]])/\,\mathrm{GL}_n(\mathbb{C}\{z\}) \to H^1(S, \mathcal{G}).$$

This is also a bijection (Malgrange–Sibuya).

- If (A, ξ) is a marked pair for A_0, then $\mathrm{St}(A)$ and $\mathrm{St}(A_0)$ are locally isomorphic, in fact using the same terminology, $\mathrm{St}(A_0)|_{U_i} \sim \mathrm{St}(A)|_{U_i}$, and $\mathrm{St}(A)$ is obtained by gluing the sheaves $\mathrm{St}(A_0)|_{U_i}$ along $U_i \cap U_j$ by the isomorphisms $\mathrm{St}(A_0)|_{U_j}|_{U_i \cap U_j} \xrightarrow{\sim} \mathrm{St}(A_0)|_{U_i}|_{U_i \cap U_j}$ given by $s \mapsto (x_i x_j^{-1}) s (x_i x_j^{-1})^{-1}$. We denote this "twisting" by $\mathrm{St}(A) = \mathrm{St}(A_0)^{(\gamma)}$ where γ indicates the corresponding 1-cocycle. Then $H^1(\mathrm{St}(A_0)) \cong H^1(\mathrm{St}(A))$, where $\gamma^{(\gamma)} = 1$. Thus as far as H^1 of Stokes sheaves is concerned one can use formal reduction.
- At each small open arc there is the notion of which spectrum is bigger than the others, and this shows that certain spectrum could be seen in the formal level whereas certain spectra are not (e.g. $e^{-1/z}$). In any case, the Stokes sheaf over a small open arc has sections **which have a natural structure as a unipotent linear algebraic group**, and restriction maps are morphisms of algebraic groups.
- There is a Lie algebra version of Stokes sheaf, $\mathrm{st}(A_0)$, which could be defined analogously using infinitesimal gauge transformation. Then $H^i(S^1, \mathrm{st}(A_0))$ is not nonabelian cohomology anymore and could be discussed. $\dim H^1(S^1, \mathrm{st}(A_0))$ is called the **irregularity of the connection** $\mathrm{ad}\, A_0$**, denoted** $\mathrm{Irr}(\mathrm{ad}\, A_0)$.
- Using this one can define the notion of **Stokes lines** (or anti-Stokes lines) which are the lines where the largest asymptotic solution gets changed.
- Stokes sheaf can be filtered by normal subsheaves where each successive subquotient is Stokes sheaf for elementary connection, namely a connection whose canonical form has only one level. Furthermore, if A is an elementary connection of level r whose lift to th eplane of $\zeta = z^{1/b}$ is unramified, consider $f : \zeta \mapsto z$ be the covering map $f : S^{1,b} \to S^1$ where $S^{1,b}$ means it is a unit circle with total length $2b\pi$. Then for any arc $I \subset S^{1,b}$ of length $\pi/(|r| - 1)$ whose endpoints are not on Stokes lines for A, we have

$$H^0(I, f^* \mathrm{St}(A)) = H^1(I, f^* \mathrm{St}(A)) = 0.$$

- $H^1(S^1, \mathrm{St}(A))$ can be given a structure of complex affine variety. This you can expect because this basically means you can write formal solutions with parameters where at a small neighborhood it only depends on the parameter and everything is formally the same ("isoformal family"). One shows this by either realizing as cocycle space mod coboundary space, or rather cleverly by realizing $H^1(S^1, \mathrm{St}(A))$ as a functor of points over more general $\mathbb{C}$-algebras. Then it is pretty much a formality that $H^1(S^1, \mathrm{St}(A))$ is an affine scheme of dimension $\mathrm{Irr}(\mathrm{ad}\, A_0)$ (bc $\mathrm{St}(A_0)$ is sheaf of unipotent groups).

3. Examples

- Bessel functions
- Whittaker functions
- Airy functions
- Confluent hypergeometric functions