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SERRE CONJECTURES AND THE P-ADIC LOCAL LANGLANDS PROGRAM

PADOVA, 2019; NOTES BY GYUJIN OH

Contents

Part 1. **Lectures**

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Our topics:

(1) Ordinary family of p-adic modular forms.
(2) Overconvergent modular forms and eigencurve following Emerton's viewpoint; completed cohomology.

1. **Modular forms.**

Let us fix $N \geq 5$ prime to p, and consider $Y_1(N)$, a $\mathbb{Z}_p$-scheme representing the functor

$$S/\mathbb{Z}_p \mapsto \{(E/S, \alpha : \mu_N \hookrightarrow E[N])\}.$$

The following are standard facts about modular curves.

- $Y_1(N)$ is an affine smooth curve over $\mathbb{Z}_p$ with geometrically connected fibers.
- Upon a choice of $\mathbb{Z}_p \hookrightarrow \mathbb{C}$ and $\zeta_n \in \mu_n(\mathbb{C})^{\mathrm{prim}}$, $Y_1(N)(\mathbb{C}) \cong \Gamma_1(N)\backslash\mathbb{H}$.
- One obtains the smooth compactification $X_1(N)$ of $Y_1(N)$ by using the j-invariant map $Y_1(N) \longrightarrow \mathbb{A}^1_{\mathbb{Z}_p}$ which is finite étale of degree $[\Gamma(1) : \Gamma_1(N)]$ outside the divisor $j(j-1728) = 0$. Namely, $X_1(N)$ is taken as the normalization of $Y_1(N)$ inside $\mathbb{A}^1_{\mathbb{Z}_p} \subset \mathbb{P}^1_{\mathbb{Z}_p}$.
- The universal elliptic curve $\pi : E \longrightarrow Y_1(N)$ does not extend to $X_1(N)$. On the other hand, using the Tate curve, one can canonically extend $E[M]$ for every M to a finite flat group scheme over $X_1(N)$ such that on $\widehat{X_1(N)}_D$, $D = X_1(N)\backslash Y_1(N)$, the extension $E[M]$ sits inside

$$0 \longrightarrow \mu_M \longrightarrow E[M] \longrightarrow (\mathbb{Z}/M\mathbb{Z}) \longrightarrow 0.$$

- There is also a "canonical extension" of $\pi_*\Omega^1_{E/Y_1(N)} \cong e^*\Omega^1_{E/Y_1(N)}$ as an invertible sheaf ω over $X_1(N)$, where $e : Y_1(N) \longrightarrow E$ is the zero section, satisfying the following property. For all $M \geq 0$, over $U = \widehat{X_1(N)}_D\backslash D \subset \widehat{X_1(N)}_D$, the following diagram

$$\begin{tikzcd} \omega \arrow[r, hook] \arrow[d] & \omega|_U \arrow[d] \\ \omega|_{\mu_M} \arrow[r, hook] & (\omega|_U)|_{\mu_M} \end{tikzcd}$$

commutes, which comes from $\mu_M \hookrightarrow E|_U$.

We can now define modular forms.

Definition 1.1. *Let $k \in \mathbb{Z}$, A a $\mathbb{Z}_p$-algebra. A modular form of weight k, level N, coefficients in A is an element of $H^0(X_1(N)_A, \omega^{\otimes k}) =: M_k(N, A)$.*

In particular, $M_k(N, \mathbb{C})$ is the usual $\mathbb{C}$-vector space of modular forms.

Proposition 1.1. *If $k \geq 2$, the base change map is an isomorphism, i.e.*

$$M_k(N, B) \cong M_k(N, A) \otimes_A B,$$

for any map $A \longrightarrow B$ over $\mathbb{Z}_p$.

2. p-adic modular forms.

Definition 2.1. *Let $X_1(N)_m = X_1(N) \times_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z}$. Let $X_1(N)_m^0 = X_1(M)_m \backslash \{$supersingular points$\}$.*

Then, $X_1(N)_m^0$ is an affine open of $X_1(N)_m$.

Consider now the finite flat group scheme $E[p^n]/X_1(N)$. The Cartier dual of Frobenius on $E[p^n]$ is called **Verschiebung**. As ordinarity of elliptic curve over $\overline{\mathbb{F}}_p$ can be seen by étaleness of $\ker V$, or any power $\ker V^r$, $\ker V^r$ as a finite flat group scheme is étale over $X_1(N)_1^0$. Also we know that, over each geometric point, $\ker(V^r)$ becomes isomorphic to $\mathbb{Z}/p^r\mathbb{Z}$.

Definition 2.2. *The **Igusa tower** $X_1(Np^r)_1^0$ is the étale covering of $X_1(N)_1^0$ parametrizing isomorphisms $\ker(V^r) \cong (\mathbb{Z}/p^r\mathbb{Z})$. Specifically, it represents the functor*

$$S \longrightarrow X_1(N)_1^0 \mapsto \{\ker(V^r) \times_{X_1(N)_1^0} S \cong \mathbb{Z}/p^r\mathbb{Z}\}.$$

Seeing $X_1(N)_m^0$ as an infinitesimal thickening of $X_1(N)_1^0$, there is an étale covering $X_1(Np^r)_m^0 \longrightarrow X_1(N)_m^0$ which makes the diagram

$$\begin{tikzcd} X_1(Np^r)_1^0 \arrow[r] \arrow[d, hook] & X_1(N)_1^0 \arrow[d, hook] \\ X_1(Np^r)_m^0 \arrow[r] & X_1(N)_m^0 \end{tikzcd}$$

Cartesian.

Definition 2.3. *Let $V_{m,r} = \mathcal{O}(X_1(Np^r)_m^0)$ be a $\mathbb{Z}/p^m\mathbb{Z}$-smooth algebra. Let $V_{m,\infty} = \varinjlim_r V_{m,r} = \mathcal{O}(\varprojlim_r X_1(Np^r)_m^0)$.*

Then the inverse limit of Galois group of $X_1(Np^r)_m^0 \longrightarrow X_1(N)_m^0 = (\mathbb{Z}/p^r\mathbb{Z})^\times$, i.e. $\mathbb{Z}_p^\times$, acts smoothly on $V_{m,\infty}$, and furthermore we have a control theorem

$$V_{m,r} = (V_{m,\infty})^{\ker(\mathbb{Z}_p^\times \to (\mathbb{Z}/p^r\mathbb{Z})^\times)}.$$

As we also have $V_{m+1,\infty} \otimes_{\mathbb{Z}/p^{m+1}\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z} \cong V_{m,\infty}$, the inverse limit

$$V_{\mathbb{Z}_p}(N) = \varprojlim_m V_{m,\infty}$$

is complete torsion-free $\mathbb{Z}_p$-module with continuous $\mathbb{Z}_p$-linear action of $\mathbb{Z}_p^\times$. Moreover, $V_{\mathbb{Z}_p}$ has no p-divisible element. Thus, $V_{\mathbb{Z}_p} \hookrightarrow V_{\mathbb{Q}_p} := V_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is the closed unit ball of a unique p-adic norm on $V_{\mathbb{Q}_p}$. As the action of $\mathbb{Z}_p^\times$ preserves $V_{\mathbb{Z}_p}$, $V_{\mathbb{Q}_p}$ with its norm is a **unitary representation** of $\mathbb{Z}_p^\times$.

Definition 2.4. *If $K/\mathbb{Q}_p$ is a finite extension, $V_K(N) := V_{\mathbb{Q}_p}(N) \otimes_{\mathbb{Q}_p} K$ is called the space of **p-adic modular functions** of tame level N and coefficients in K.*

*A **weight** of $\mathbb{Z}_p^\times$ with values in K is a continuous charcater $\chi : \mathbb{Z}_p^\times \longrightarrow K^\times$. Given a weight, we define*

$$V_K(N)[\chi] = \{f \in V_K(N) \mid \forall a \in \mathbb{Z}_p^\times, af = \chi(a)f\},$$

*and we call this the space of **p-adic modular forms of weight** χ. In particular, for $k \in \mathbb{Z}$, weight k means we use $\chi_k(a) = a^k$.*

Why is this a reasonable definition? We first make two observations.

- First note that the exact sequence

$$0 \longrightarrow \ker(F^r) \longrightarrow E[p^r] \longrightarrow \ker(V^r) \longrightarrow 0$$

on $X_1(N)^0_1$ extends to $X_1(N)^0_m$ as follows. First as $X_1(N)^0_m \supset X_1(N)^0_1$ is an infinitesimal thickening, $\ker(V^r)$, being étale, extends uniquely as a finite étale group scheme over $X_1(N)^0_m$. Let's denote is as H_r^D. Define $H_r := (H_r^D)^D$. Then, on $X_1(N)^0_m$, $E[p^r]$ sits inside

$$0 \longrightarrow H_r \longrightarrow E[p^r] \longrightarrow H_r^D \longrightarrow 0.$$

In this case H_r is the canonical subgroup of $E[p^r]$, if $r \geq m$.

Claim. *The natural maps*

$$\omega \longrightarrow \omega_{E[p^r]} \longrightarrow \omega_{H_r},$$

are all isomorphisms when $r \geq m$, where other ω's are cotangent sheaves of the finite flat groups schemes written in the subscripts.

The most nontrivial part is to check that $\omega_{E[p^r]} \longrightarrow \omega_{H_r}$ is an isomorphism. You can check this after étale base change, so in particular we can check it over $Y_1(Np^r)^0_m$. Over it, $H_r \cong \mu_{p^r}$, and $\omega_{\mu_{p^r}}$ is an invertible sheaf, simply because

$$\Omega^1_{\mu_{p^r}/\mathbb{Z}/p^m\mathbb{Z}} \cong \mathbb{Z}/p^m\mathbb{Z}[x]/(x^{p^r} - 1, (x^{p^r} - 1)')dx,$$

where $'$ means formal derivation. Now as $r \geq m$, the derivation part goes away. Now surjectivitiy is clear, so we get the desired isomorphism.
- There is a Hodge-Tate map

$$\mathrm{HT} : H_r^D \longrightarrow \omega_{H_r},$$

a map of étale sheaves, defined as $\mathrm{HT}(f) = f^*(dx/x)$, using the fact that $H_r^D = \mathrm{Hom}(H_r, \mathbb{G}_m)$.

Claim. $H_r^D \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_{X_1(N)^0_m} \xrightarrow{\mathrm{HT}} \omega_{H_r}$ *is an isomorphism.*

This also amounts to the fact that, after an étale base-change to $X_1(Np^r)_m$, both sides are invertible sheaves ($H_r^D \cong \mathbb{Z}/p^r\mathbb{Z}$), and then the Hodge-Tate map becomes surjective as $\omega_{H_r} \cong \omega_{\mu_{p^r}}$ is generated by dx/x.

From our obsevation above, we get trivialization of ω over $X_1(Np^r)^0_m$,

$$\omega \xrightarrow{\sim} \omega_{H_r} \xleftarrow{\mathrm{HT}} H_r^D \otimes_{\mathbb{Z}/p^r\mathbb{Z}} \mathcal{O}_{X_1(Np^r)^0_m} \cong \mathcal{O}_{X_1(Np^r)^0_m},$$

whenever $r \geq m$. Let us call this map $\gamma_{m,r}$. By construction, these isomorphisms are compatible with each other; these are somewhat canonical trivializations of ω.

Now we can see that classical modular forms are p-adic modular forms, by

$$M_k(N, \mathbb{Z}/p^m\mathbb{Z}) = H^0(X_1(N)_m, \omega^{\otimes k}) \hookrightarrow H^0(X_1(N)^0_m, \omega^{\otimes k}) \hookrightarrow$$

$$H^0(X_1(Np^r)_m, \omega^{\otimes k}) \xrightarrow{\sim, \gamma_{m,r}} \mathcal{O}(X_1(Np^r)^0_m) =: V_{m,r} \subset V_{m,\infty},$$

for any $r \geq m$, and by the compatibility the embedding does not depend on r. Also the compatibility in m means the diagram

$$\begin{tikzcd} M_k(N, \mathbb{Z}/p^{m+1}\mathbb{Z}) \arrow[r, hook] \arrow[d] & V_{m+1,\infty} \arrow[d] \\ M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \arrow[r, hook] & V_{m,\infty} \end{tikzcd}$$

commutes, and thus

$$M_k(N, \mathbb{Z}_p) = \varprojlim_m M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \hookrightarrow V_{\mathbb{Z}_p}(N).$$

Let us denote $\iota_k : M_k(N, \mathbb{Q}_p) \hookrightarrow V_{\mathbb{Q}_p}(N)$.

Proposition 2.1. *The image of ι_k sits inside $V_{\mathbb{Q}_p}(N)[\chi_k]$.*

Proof. Note that there is an action of diamond operators on $Y_1(Np^r)^0_m$. Namely, for $a \in \mathbb{Z}_p^\times$,

$$\langle a \rangle (E/S, \alpha_N : \mu_N \hookrightarrow E[N], \alpha_{p^r} : \mu_{p^r} \hookrightarrow E[p^r]) = (E/S, \alpha_N, \alpha_{p^r} \circ \gamma_a),$$

where $\gamma_a(z) = z^a$. We can then check that the image of ι_k transforms under χ_k by taking $\mathrm{HT}(1) =: \lambda_{\mathrm{can}} \in H^0(X_1(Np^r)^0_m, \omega)$, and observing that $(\alpha_{p^r})^* \lambda_{\mathrm{can}} = dx/x$, and $\langle a \rangle^* \lambda_{\mathrm{can}} = a^{-1} \lambda_{\mathrm{can}}$. $\square$

Not only level N classical modular forms lie in the space of p-adic modular forms, but also all level Np^r classical modular forms are p-adic modular forms of tame level N. We use an integral model $Y_1(Np^r)'_{\mathbb{Z}_p}$ representing the functor

$$S \longmapsto (E/S, \mu_{Np^r} \hookrightarrow E).$$

This reduces mod p^m to $Y_1(Np^r)^0_m$. Let $X_1(Np^r)_{\mathbb{Z}_p}$ be the normalization of $Y_1(Np^r)'_{\mathbb{Z}_p}$ in $\mathbb{P}^1_{\mathbb{Z}_p}$. Then, $X_1(Np^r)_{\mathbb{Z}_p}$ is proper but not smooth for $r \geq 1$. Anyways it has the right generic fiber, so

$$M_k(Np^r, \mathbb{Q}_p) = H^0(X_1(Np^r)_{\mathbb{Q}_p}, \omega^{\otimes k}) \longleftarrow H^0(X_1(Np^r)_{\mathbb{Z}_p}, \omega^{\otimes k}),$$

and this "arithmetically integral modular forms" $H^0(X_1(Np^r)_{\mathbb{Z}_p}, \omega^{\otimes k})$ admits a map into $V_{m,\infty}$, namely

$$H^0(X_1(Np^r)_{\mathbb{Z}_p}, \omega^{\otimes k}) \longrightarrow H^0(X_1(Np^r)_m, \omega^{\otimes k}) \longrightarrow H^0(X_1(Np^r)^0_m, \omega^{\otimes k}) \hookrightarrow V_{m,\infty}.$$

Taking the rationalization, this map, a priori not clear whether it is injective, becomes injective, so that we have an emedding $M_k(Np^r, \mathbb{Q}_p) \hookrightarrow V_{\mathbb{Q}_p}(N)$.

Remark 2.1.
1. $M_k(Np^r, \mathbb{Q}_p)$ does not sit inside $V_{\mathbb{Q}_p}(N)[\chi_k]$, but rather $V_{\mathbb{Q}_p}(N)[\chi_k|_{1+p^r\mathbb{Z}_p}]$. More generally, for a nebentypus $\varepsilon : (\mathbb{Z}/p^r\mathbb{Z})^\times \longrightarrow K^\times$,
$$M_k(Np^r, \varepsilon, K) \hookrightarrow V_{\mathbb{Q}_p}(N)[\chi_k \varepsilon].$$
2. $\bigoplus_{k \geq 0} M_k(N, \mathbb{Q}_p)$ is dense in $V_{\mathbb{Q}_p}(N)$.
3. The first Igusa covering $X_1(Np^r)_1$ is connected, and $\pi_1(X_1(N)^0_1) \twoheadrightarrow (\mathbb{Z}/p^r\mathbb{Z})^\times$. This is because $X_1(Np)^0_1$ represents the functor of taking $(p-1)$-st root of Ha, so
$$X_1(Np)^0_1 = \mathrm{Spec}(\bigoplus_{n \geq 0} \omega^{\otimes n}/(\mathrm{Ha} - 1)).$$
This is very useful; for example, this can be used to prove that
$$V_{1,\infty}^{1+p\mathbb{Z}_p} = \sum_{k \geq 0} \mathrm{im}(\iota_k),$$

for $\iota_k : M_k(N, \mathbb{F}_p) \longrightarrow V_{1,\infty}$. You can see this as follows: the LHS is naturally $V_{1,1}$, and the diamond operation decomposes $V_{1,1}$ into $V_{1,1} = \oplus_{a=0}^{p-2} V_{1,1}[\chi_a]$. Given $f \in V_{1,1}[\chi_a]$, take a $(p-1)$-st root of Hasse invariant $\lambda \in H^0(X_1(Np)_1^0, \omega)$. Then, $f\lambda^a \in H^0(X_1(Np)_1^0, \omega^{\otimes a})$ is fixed by the diamond operators $(\mathbb{Z}/p\mathbb{Z})^\times$, which is the Galois group of $X_1(Np)_1^0 \longrightarrow X_1(N)_1^0$. Thus, it descends to a section $f\lambda^a \in H^0(X_1(N)_1^0, \omega^{\otimes a})$. Now we can multiply sufficient power of Hasse invariant to cancel poles at the supersingular points to define a genuine modular form of weight $a + n(p-1)$ for some n.

One can say a bit more. As $\omega^{\otimes(p-1)}$ over $X_1(N)_1^0$ is trivial (and generated by the so-called **Hasse invariant**), $M_k(N, \mathbb{F}_p)$ can be seen as a subspace of $M_{k+(p-1)}(N, \mathbb{F}_p)$ by multiplying Hasse invariant. Thus, letting $\cup_{n\geq 0} M_{a+n(p-1)}(N, \mathbb{F}_p) =: M(N, a, \mathbb{F}_p) \subset V_{1,\infty}$, we have

$$V_{1,\infty}^{1+p\mathbb{Z}_p} = \bigoplus_{a=0}^{p-2} M(N, a, \mathbb{F}_p).$$

3. Hecke operators.

3.1. *$T(\ell)$, for $\ell \neq p$.*

Let $Y_1(Np^r, \ell)_m^0$ represent

$$S \longmapsto (E/S, \alpha : \mu_{Np^r} \hookrightarrow E, H \subset E[\ell]),$$

where H is a finite flat group scheme of order ℓ and $H \cap \operatorname{im} \alpha = 0$. Then we have a correspondence diagram

$$\begin{tikzcd}
 & Y_1(Np^r, \ell)_m^0 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
Y_1(Np^r)_m^0 & & Y_1(Np^r)_m^0
\end{tikzcd}$$

where the maps are defined as $\pi_1(E, \alpha, H) = (E, \alpha)$ and $\pi_2(E, \alpha, H) = (E/H, \alpha : \mu_{Np^r} \longrightarrow E \longrightarrow E/H)$. The maps π_1, π_2 are finite étale.

Definition 3.1. *Let $T(\ell) \in \operatorname{End}(V_{m,n})$ be defined as $(\pi_1)_* \pi_2^*$. This is compatible in r and m, so it defines an operator $T(\ell) \in \operatorname{End}(V_{\mathbb{Q}_p}(N))$.*

Remark 3.1. It has norm $|T(\ell)| \leq 1$. It is also compatible with $T(\ell)$ on the classical modular forms.

3.2. *$T(\ell)$, for $\ell = p$ (i.e. U_p).*

. The Frobenius on $Y_1(N)_1^0$ sends

$$(E, \alpha) \longmapsto (E/H_1, \alpha : \mu_N \longrightarrow E \longrightarrow E/H_1),$$

where H_1 is the canonical subgroup of E. Take a lift of Frobenius on $Y_1(N)_m^0$. On the Igusa tower, this reduces p-level by 1, namely it induces a map $Y_1(Np^r)_m^0 \longrightarrow Y_1(Np^{r-1})_m^0$.

Definition 3.2. *Let $U = \frac{1}{p}\operatorname{tr}(F)$, where F is the algebra endomorphism on $V_{\mathbb{Z}_p}(N)$ induced by the Frobenius on Igusa tower, which is finite flat of degree p and is a lift of Frobenius on $V_{\mathbb{Z}_p}(N) \otimes \mathbb{F}_p$.*

Remark 3.2. This is also compatible with U_p-operator on classical modular forms. This is not exactly the same as $T(p)$ on $M_k(N, \mathbb{Z}_p)$, but still is the same **modulo** p (Exercise).

4. **Ordinary p-adic modular forms.**
Let's fix notations first.

- $\Gamma = 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ ($1 + 4\mathbb{Z}_2$ if $p = 2$).
- $\Lambda = \varprojlim_n \mathbb{Z}_p[\Gamma/p^n] = \mathbb{Z}_p[[\Gamma]]$, the Iwasawa algebra. It is isomorphic to $\mathbb{Z}_p[[X]]$ via identifying $X \mapsto 1 + p \in \Gamma$.

The diamond operator action $\mathbb{Z}_p^\times$ on $V_{\mathbb{Q}_p}(N)$ continuously extends to a Λ-module structure on $V_{\mathbb{Q}_p}(N)$.

Definition 4.1. *Let $\mathcal{V}_{\mathbb{Z}_p} = \mathrm{Hom}_{\mathrm{cts}}(V_{\mathbb{Z}_p}, \mathbb{Z}_p)$, and $\mathcal{V}_{\mathbb{Q}_p} = \mathrm{Hom}_{\mathrm{cts}}(V_{\mathbb{Q}_p}, \mathbb{Q}_p)$. We endow weak topology on the modules.*

Remark 4.1. As a Λ-module, $\mathcal{V}_{\mathbb{Z}_p}$ is compact. This is because $\mathcal{V}_{\mathbb{Z}_p} = \varprojlim_m \mathrm{Hom}(V_{\mathbb{Z}_p} \otimes \mathbb{Z}/p^m\mathbb{Z}, \mathbb{Z}/p^m\mathbb{Z})$, and the weak topology on each Hom is compact, as we have discrete topology on $V_{\mathbb{Z}_p}/\otimes \mathbb{Z}/p^m\mathbb{Z}$.

We are interested in this as $\mathcal{V}_{\mathbb{Z}_p}$ is not of finite type as a Λ-module. Namely, $V_{\mathbb{Q}_p}$ is not always an admissible representation of $\mathbb{Z}_p^\times$.

Definition 4.2 (Ordinary projector)**.** *As $\bigoplus_k M_k(Np, \mathbb{Q}_p) \subset V_{\mathbb{Q}_p}(N)$ is dense, and as each $M_k(Np, \mathbb{Q}_p)$ is U-stable, $V_{\mathbb{Z}_p}(N) \otimes \mathbb{Z}/p^m$ is an increasing union of finite submodules stable under U, namely*

$$(\bigoplus_{k=0}^{j} M_k(Np, \mathbb{Q}_p) \cap V_{\mathbb{Z}_p}) \otimes \mathbb{Z}/p^m.$$

Thus, for any $v \in V_{\mathbb{Z}_p}(N)$, $\{U^{n!}v\}_{n\geq 0}$ converges to some vector, denoted as $e_{\mathrm{ord}}v$. This map e_{ord} is a continuous projector in $\mathrm{End}(V_{\mathbb{Z}_p}(N))$, called the ***ordinary projector****. We define $W^{\mathrm{ord}} = e_{\mathrm{ord}}W$ whenever it makes sense to do so.*

For a general Λ-module $V_{\mathbb{Z}_p}$, $(V_{\mathbb{Z}_p} \otimes \mathbb{F}_p)^\Gamma = (V_{\mathbb{Z}_p} \otimes \mathbb{F}_p)[X]$. Then, as $V_{\mathbb{Z}_p} = \mathrm{Hom}(V'_{\mathbb{Z}_p}, \mathbb{Z}_p)$ for $V'_{\mathbb{Z}_p} = \mathrm{Hom}(V_{\mathbb{Z}_p}, \mathbb{Z}_p)$, we get $(V_{\mathbb{Z}_p} \otimes \mathbb{F}_p)^\Gamma = \mathrm{Hom}(V'_{\mathbb{Z}_p} \otimes_\Lambda \mathbb{F}_p, \mathbb{F}_p)$. What this tells us is that, by topological Nakayama, $V'_{\mathbb{Z}_p}$ is a Λ-module of finite type if and only if $\dim_{\mathbb{F}_p}(V_{\mathbb{Z}_p} \otimes_\Lambda \mathbb{F}_p)^\Gamma < \infty$.

We apply this to $V_{\mathbb{Z}_p}(N)$. We note that $(V_{\mathbb{Z}_p}(N) \otimes \mathbb{F}_p)^\Gamma = V_{1,1} = \mathcal{O}(X_1(Np)_1^0)$.

Theorem 4.1 (Hida)**.** *Let $p > 2$. Then $\mathcal{V}_{\mathbb{Z}_p}^{\mathrm{ord}}$ is a finite free Λ-module. Thus, $V_{\mathbb{Q}_p}^{\mathrm{ord}}(N)$ is an admissible $\mathbb{Z}_p^\times$-representation. Furthermore, for $k \geq 3$, $V_{\mathbb{Q}_p}^{\mathrm{ord}}(N)[\chi_k] = M_k(N, \mathbb{Q}_p)^{\mathrm{ord}}$.*

One obtains the analogous result for cuspidal forms, using

$$V_{\mathbb{Q}_p,\mathrm{cusp}}(N) = \{f \in V_{\mathbb{Q}_p}(N) \mid f \text{ vanishes along } X_1(Np^r)^0 \backslash Y_1(Np^r)^0\}.$$

Theorem 4.2 (Hida)**.**
(1) For $k \geq 3$, $\dim_{\mathbb{Q}_p} M_k(N, \mathbb{Q}_p)^{\mathrm{ord}}$ depends only on the class of k modulo $(p-1)$. The same result holds for cuspidal forms.
(2) Let $\varepsilon : (\mathbb{Z}/p^n\mathbb{Z})^\times \longrightarrow K^\times$ be a Dirichlet character of conductor p^n. For $k \geq 2$, $\dim_{\mathbb{Q}_p} S_k(Np^r, \varepsilon, K)$ depends only on the class of k modulo $(p-1)$ and $\varepsilon|_\Delta$, where $\Delta \cong \mathbb{Z}/(p-1)\mathbb{Z}$ is the primary-to-p part of $\mathbb{Z}_p^\times$.

Proof of Theorem 4.1, using Theorem 4.2. To show finite-typeness, we need to prove that $\dim_{\mathbb{F}_p} V_{1,1}^{\mathrm{ord}} < \infty$. Note that

$$V_{1,1}^{\mathrm{ord}} = \bigoplus_{a \in \mathbb{Z}/(p-1)\mathbb{Z}} M(N, a, \mathbb{F}_p)^{\mathrm{ord}},$$

where

$$M(N, a, \mathbb{F}_p)^{\mathrm{ord}} = \bigcup_{n \geq 0} M_{a+n(p-1)}(N, \mathbb{F}_p)^{\mathrm{ord}}.$$

By Theorem 4.2, $\dim_{\mathbb{F}_p} M_k(N, \mathbb{F}_p)^{\mathrm{ord}} = \dim_{\mathbb{Q}_p} M_k(N, \mathbb{Q}_p)^{\mathrm{ord}}$ stays constant as k increases by a multiple of $(p-1)$.

Let us denote $\mathbb{Z}_p^\times \cong \Delta \times \Gamma$ where $\Delta = (\mathbb{Z}/p\mathbb{Z})^\times$. For $a \in \mathbb{Z}/(p-1)\mathbb{Z}$ and $\chi_a : \Delta \longrightarrow \mathbb{Z}_p^\times$, $z \mapsto z^a$, we define $V_{\mathbb{Z}_p,a} = V_{\mathbb{Z}_p}[\chi_a]$. Then $\mathcal{V}^{\mathrm{ord}}_{\mathbb{Z}_p,a} \otimes_\Lambda \mathbb{F}_p \cong \mathbb{F}_p^{r(a)}$, where $r(a) = \dim_{\mathbb{Q}_p} M_k(N, \mathbb{Q}_p)^{\mathrm{ord}}$. Thus, by Nakayama, there is a surjection $\Lambda^{r(a)} \twoheadrightarrow \mathcal{V}^{\mathrm{ord}}_{\mathbb{Z}_p,a}$.

Note that $M_k(N, \mathbb{Z}_p)^{\mathrm{ord}} \subset V^{\mathrm{ord}}_{\mathbb{Z}_p}(N)[\chi_k]$ is cotorsion-free, as its reduction mod p is $M_k(N, \mathbb{F}_p) \hookrightarrow V_{1,\infty}$. Also, $\mathcal{V}^{\mathrm{ord}}_{\mathbb{Z}_p,a} \twoheadrightarrow \mathrm{Hom}(M_k(N, \mathbb{Z}_p)^{\mathrm{ord}}, \mathbb{Z}_p)$ has a kernel containing $P_k = ((1+x) - (1+p)^k) = \ker(\Lambda \xrightarrow{\chi_k} \mathbb{Z}_p)$. Thus, we get a surjection $(\Lambda/P_k)^{r(a)} \twoheadrightarrow \mathrm{Hom}(M_k(N, \mathbb{Z}_p)^{\mathrm{ord}}, \mathbb{Z}_p)$, which is a surjection between finite free $\mathbb{Z}_p$-modules of the same rank $r(a)$, which is an isomorphism.

This implies that the kernel of the original surjection $\Lambda^{r(a)} \twoheadrightarrow \mathcal{V}^{\mathrm{ord}}_{\mathbb{Z}_p,a}$ is contained in $P_k \Lambda^{r(a)}$ for all $k \equiv a (\mathrm{mod}\, p-1)$ for all $k \geq 3$. By Weierstrass preparation theorem, the kernel is zero. □

5. **Hida family.**

Definition 5.1. *The* ***big Hecke algebra*** $\mathscr{H}(N)$ *is the weak closure in* $\mathrm{End}(V_{\mathbb{Z}_p}(N))$ *of the* $\mathbb{Z}_p$*-algebra generated by the operators* $T(\ell), \ell^{-2}\langle \ell \rangle_{Np}$ *for* $\ell \neq p$*, and* U*, where* $\langle \ell \rangle_{Np}$ *on* $V_{m,r}$ *is the endomorphism defined by*

$$\langle \ell \rangle_{Np}(E/S, \alpha_{Np^r} : \mu_{Np^r} \hookrightarrow E) = (E/S, \alpha_{Np^r} \circ \gamma_\ell).$$

There are variants of the big Hecke algebras, namely $\mathscr{H}(N)^{\mathrm{ord}} = e_{\mathrm{ord}}\mathscr{H}(N)$*,* $\mathfrak{h}(N) = \mathrm{im}(\mathscr{H}(N) \longrightarrow \mathrm{End}(V_{\mathrm{cusp}}(N)))$*,* $\mathfrak{h}(N)^{\mathrm{ord}} = e_{\mathrm{ord}}\mathfrak{h}(N)$*,* $\mathscr{H}(N,a) = \mathrm{im}(\mathscr{H}(N) \longrightarrow \mathrm{End}(V_{\mathbb{Z}_p,a}))$ *for* $a \in \mathbb{Z}/(p-1)\mathbb{Z}$*,* $\cdots$*.*

Remark 5.1. We can define $\mathscr{H}^j(N)$ to be the $\mathbb{Z}_p$-algebra generated by the same operators in $\mathrm{End}(\bigoplus_{k=0}^j M_k)$. As the Hecke operators are compatible with finite-level Hecke operators, $\mathscr{H}(N) = \varprojlim_j \mathscr{H}^j(N)$ and $\mathscr{H}(N) \twoheadrightarrow \mathscr{H}_k(Np)$ for all k; $\mathscr{H}(N)$ has the topology of the projective limit of finite $\mathbb{Z}_p$-algebras.

Remark 5.2. Consider $\Lambda \subset \mathrm{End}(V_{\mathbb{Z}_p}(N))$. For $\ell \equiv 1 (\mathrm{mod}\, pN)$, $[\ell] \in \mathbb{Z}_p[\Gamma]$, the element represented by the group element $\ell \in \Gamma$, has the property that $\ell^{-2}[\ell]$, the operator on $V_{\mathbb{Z}_p}(N)$, acts like $\bigoplus T_k(\ell, \ell)$ on $\bigoplus M_k(Np, \mathbb{Z}_p)$. Thus, $[\ell] \in \mathscr{H}(N)$, and by density, $\Lambda \subset \mathscr{H}(N)$.

Theorem 5.1 (Hida)**.** *For a Dirichlet character* ε *of conductor* p^r *and* $k \geq 2$*, let* $P_{k,\varepsilon} = \ker(\Lambda \xrightarrow{\chi_k \varepsilon} \mathbb{Z}_p)$*. Let* ω *be the Teichmüller character. Then* $\mathscr{H}(N)^{\mathrm{ord}}$ *and* $\mathfrak{h}(N)^{\mathrm{ord}}$ *are finite free* Λ*-modules. Moreover,*

$$\mathscr{H}(N,a)^{\mathrm{ord}}/P_k \xrightarrow{\sim} \mathscr{H}_k(Np, \omega^{a-k})^{\mathrm{ord}},$$

for $k \geq 3$*, and*

$$\mathfrak{h}(N,a)^{\mathrm{ord}}/P_{k,\varepsilon} \xrightarrow{\sim} \mathfrak{h}_k(Np^r, \varepsilon\omega^{a-k})^{\mathrm{ord}}.$$

Proof. Using q-developments theory, one proves that $\mathcal{V}_{\mathbb{Z}_p,\mathrm{cusp}} \cong \mathfrak{h}(N)$ as $\mathfrak{h}(N)$-modules. This implies that $\dim_{\mathbb{Q}_p} \mathscr{H}(Np^r, \varepsilon)_{\mathbb{Q}_p} = \dim_{\mathbb{Q}_p} M_k(Np^r, \varepsilon)$, which implies that big ordinary Hecke algebras are finite free over Λ. To prove the control theorem, one uses Theorem 4.2. □

Now we can define Hida family.

Definition 5.2. *As* $\mathfrak{h}(N,a)^{\mathrm{ord}}$ *is a semi-local algebra,* $\mathfrak{h}(N,a)^{\mathrm{ord}} = \bigoplus_{\mathfrak{m}\ maximal} \mathfrak{h}(N,a)^{\mathrm{ord}}_{\mathfrak{m}}$*. We define* $\mathscr{E}_{\mathfrak{m}} = (\mathrm{Spf}\, \mathfrak{h}(N,a)^{\mathrm{ord}}_{\mathfrak{m}})^{\mathrm{rig}}$*, which admits a finite flat map to the weight space* $\mathcal{W} = (\mathrm{Spf}\, \Lambda)^{\mathrm{rig}}$*.*

A closed point of $\mathscr{E}_{\mathfrak{m}}$ *is the same as a morphism* $\mathfrak{h}(N,a)^{\mathrm{ord}} \longrightarrow K$*, which is a system of eigenvalues of* $\mathfrak{h}(N)$ *on* $V_{\mathbb{Q}_p}(N)^{\mathrm{ord}}$*.*

Using Theorem 5.1, one proves the following

Theorem 5.2 (Classicality). *For a classical weight $y = \chi_k\varepsilon \in \mathcal{W}$, $\pi^{-1}(y)$ contains only classical points.*

6. Cohomological method; completed cohomology.

Now our focus is to prove Theorem 4.2. We will use this as an excuse to introduce the notion of completed (co)homology. We work topologically; we use the local system $\mathcal{V}_k$ on $Y_1(Np^r)$, where $Y_1(Np^r) = \Gamma_1(Np^r)\backslash\mathbb{H}$, and $\mathcal{V}_k$ comes from the representation $\mathrm{Sym}^{k-2}(\mathbb{Z}^2)$ of $\mathrm{SL}_2(\mathbb{Z})$.

Definition 6.1. *The* ***parabolic cohomology*** *is defined as*

$$H^1_p(Y_1(Np^r), \mathcal{V}_k) = \mathrm{im}(H^1_c(Y_1(Np^r), \mathcal{V}_k) \longrightarrow H^1(Y_1(Np^r), \mathcal{V}_k)).$$

Theorem 6.1 (Eichler-Shimura isomorphism). *There is an isomorphism*

$$H^1_p(Y_1(Np^r), \mathcal{V}_k) \otimes_{\mathbb{Z}} \mathbb{C} \cong S_k(\Gamma_1(Np^r)) \oplus \overline{S_k(\Gamma_1(Np^r))},$$

which is Hecke-equivariant.

By the Hecke-equivariantce, in particular the Hecke algebra

$$\mathfrak{h}_k(\Gamma_1(Np^r)) \subset \mathrm{End}(S_k(\Gamma_1(Np^r))),$$

naturally injects into $\mathrm{End}(H^1_p(Y_1(Np^r), \mathcal{V}_k))$.

Corollary 6.1. *The $\mathfrak{h}_k(\Gamma_1(Np^r))_{\mathbb{Q}_p}$-module $\mathrm{Hom}(H^1_p(Y_1(Np^r), \mathcal{V}_k \otimes \mathbb{Q}_p), \mathbb{Q}_p)$ is a free of rank 2.*

Thus, to prove Theorem 4.2, we only have to prove that, for $K/\mathbb{Q}_p$ a finite extension,

$$\dim_K H^1_p(Y_1(Np^r), \chi, \mathcal{V}_k \otimes K)^{\mathrm{ord}},$$

depends only on k modulo $(p-1)$ and $\chi|_\Delta$.

Definition 6.2. *Given an open compact subgroup $K \subset \mathrm{GL}_2(\mathbb{A}_f)$, we define $Y_K = \mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_f)/K)$. For a fixed $K^p = K_1(N) \subset \mathrm{GL}_2(\mathbb{A}_f^p)$, and for a compact open $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$, we denote $Y_{K_p} := Y_{K^pK_p}$.*

Example 6.1. In particular, $Y_{K_1(p^r)} = Y_1(Np^r)$.

Definition 6.3. *The* ***completed cohomology*** *with tame level N is defined by*

$$H^1(N)_{\mathbb{Z}_p} = \varinjlim_{K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)} H^1(Y_{K_p}, \mathbb{Z}_p),$$

which is a $\mathbb{Z}_p$-module. By the double quotient action, it admits an action $\mathrm{GL}_2(\mathbb{Q}_p)$, which is smooth. We define

- *$\widehat{H}^1(N)_{\mathbb{Z}_p}$ is the p-adic completion of $H^1(N)_{\mathbb{Z}_p}$.*
- *$\widehat{H}^1(N) = \widehat{H}^1(N)_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, which is a p-adic Banach space.*

We might omit the tame level N because it will never change.

Remark 6.1. (1) Even though $H^1_{\mathbb{Z}_p}$ is a smooth representation, p-adic completion adds a lot of vectors, making $\widehat{H}^1$ only unitary.

(2) The better definition should be

$$\widehat{H}^1_{\mathbb{Z}_p} \cong \widetilde{H}^1_{\mathbb{Z}_p} := \varprojlim_n \varinjlim_{K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)} H^1(Y_{K_p}, \mathbb{Z}/p^n\mathbb{Z}),$$

which is the same as our definition as Y_{K_p}'s are curves.

Theorem 6.2 (Emerton). *The completed cohomology $\widehat{H}^1$ is an admissible continuous representation of $\mathrm{GL}_2(\mathbb{Q}_p)$.*

We want to compare this to $V_{\mathbb{Q}_p}$, the space of p-adic modular forms. Note that we did not use any coefficients in the definition of completed cohomology, but as one can exchange levels and weights, actually it sees every weight.

Let $k \geq 2$. Then, consider

$$H^1(Y_{K_p}, \mathcal{V}_k \otimes \mathbb{Z}_p) \longrightarrow \varprojlim_n H^1(Y_{K_p}, \mathcal{V}_k \otimes \mathbb{Z}/p^n\mathbb{Z}) \longrightarrow \varprojlim_n \varinjlim_{K'_p \subset K_p} H^1(Y_{K'_p}, \mathcal{V}_k \otimes \mathbb{Z}/p^n\mathbb{Z}).$$

If K'_p is small enough, then $\mathcal{V}_k \otimes \mathbb{Z}/p^n\mathbb{Z}$ becomes trivial on $Y_{K'_p}$, so that

$$H^1(Y_{K'_p}, \mathcal{V}_k \otimes \mathbb{Z}/p^n\mathbb{Z}) \cong H^1(Y_{K'_p}, \mathbb{Z}/p^n\mathbb{Z}) \otimes V_k.$$

Thus, the target of the composition is $(V_k \otimes \mathbb{Q}_p) \otimes \widehat{H}^1$ (where $V_k \otimes \mathbb{Q}_p$ is an algebraic finite-dimensional representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, and in particular it lands in the K_p-invariant subspace.

Theorem 6.3 (Emerton). *The natural map is an isomorphism, i.e.*

$$H^1(Y_{K_p}, \mathcal{V}_k \otimes \mathbb{Q}_p) \xrightarrow{\sim} \mathrm{Hom}_{K_p}((V_{k,\mathbb{Q}_p})', \widehat{H}^1),$$

is an isomorphism. The isomorphism is also Hecke-equivariant for operators $\ell \nmid Np$.

Remark 6.2. In general, you get a spectral sequence of form

$$\mathrm{Ext}^a_{K_p}(V'_{k,\mathbb{Q}_p}, \widehat{H}^b) \Rightarrow H^{a+b}(\mathrm{Sh}_{K_p}, \mathcal{V}_{k,\mathbb{Q}_p}).$$

Remark 6.3. The space of classical modular forms

$$\bigoplus_k H^1(Y_1(N), \mathcal{V}_k) \otimes V'_{k,\mathbb{Q}_p} \hookrightarrow \widehat{H}^1,$$

has a dense image. Thus, even if you construct a "big Hecke algebra with completed cohomology," you don't get anything new. More precisely, if we define $\mathbb{T}^{\mathrm{sph}}$ to be the weak completion of $\mathbb{Z}_p[T(\ell), T(\ell,\ell); \ell \nmid Np] \subset \mathrm{End}(\widehat{H}^1)$, this is smaller than the big Hecke algebra $\mathfrak{h}(N) \subset \mathrm{End}(V_{\mathbb{Q}_p}(N))$ we constructed before.

Proposition 6.1. *The big spherical Hecke algebra $\mathbb{T}^{\mathrm{sph}}$ is a semilocal $\mathbb{Z}_p$-algebra. Thus, $\widehat{H}^1 \cong \oplus_{\mathfrak{m}} \widehat{H}^1_{\mathfrak{m}}$.*

Proof. This is because the completed cohomology is admissible, so that $(\widehat{H}^1_{\mathbb{Z}_p} \otimes \mathbb{F}_p)^{I_1}$ is a finite-dimensional space, so it can have nonzero $\mathfrak{m}$-torsion for only finitely many maximal ideals, and for any $\mathfrak{m} \subset \mathbb{T}^{\mathrm{sph}}$, it has to have nonzero vector in there. □

Now fix a **non-Eisenstein** maximal ideal $\mathfrak{m} \subset \mathbb{T}^{\mathrm{sph}}$, which means that $\overline{\rho}_{\mathfrak{m}} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is irreducible. Then, after localizing at $\mathfrak{m}$, we don't have to worry about parabolic/compact support cohomology. Thus we will localize at some non-Eisenstein prime for simplicity.

Theorem 6.4 (Emerton). *The $\mathrm{GL}_2(\mathbb{Z}_p)$-representation $\widehat{H}^1_{\mathbb{Z}_p,\mathfrak{m}}$ is isomorphic to some direct factor of $C(\mathrm{GL}_2(\mathbb{Z}_p), \mathbb{Z}_p)^{\oplus s}$, for some $s \geq 0$.*

We can now prove Theorem 4.2 at least after localizing at a non-Eisenstein maximal ideal.

Proof of Theorem 4.2 for the non-Eisenstein part. Note that the isomorphism

$$H^1(Y_1(Np^r), \mathcal{V}_{k,\mathbb{Q}_p})^{\mathrm{ord}}_{\mathfrak{m}} \xrightarrow{\sim} \mathrm{Hom}_{K_1(p^r)}(V'_{k,\mathbb{Q}_p}, \widehat{H}^{1,\mathrm{ord}}_{\mathfrak{m}}),$$

is U_p-equivariant. Similarly, there are U_p-equivariant maps

$$\mathrm{Hom}_{K_1(p^r)}(V'_{k,\mathbb{Q}_p}, \widehat{H}^1_{\mathfrak{m}}) \hookrightarrow \mathrm{Hom}_{\left(\begin{smallmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & 1+p^r\mathbb{Z}_p \end{smallmatrix}\right)}(V'_{k,\mathbb{Q}_p}, \widehat{H}^1_{\mathfrak{m}}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p^\times \times (1+p^r\mathbb{Z}_p)}(1 \otimes \chi_{k-2}^{-1}, (\widehat{H}^1_{\mathfrak{m}})^{\left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{smallmatrix}\right)}),$$

where $(1 \otimes \chi_{k-2}^{-1})\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = d^{-(k-2)}$, and the second map is justified by the fact that $1 \otimes \chi_{k-2}^{-1} \subset V'_{k,\mathbb{Q}_p}|_{\left(\begin{smallmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & 1+p^r\mathbb{Z}_p \end{smallmatrix}\right)}$. Our claim is that the maps become isomorphisms after taking ordinary parts.

Remark 6.4. To be clear, for a compact $H \subset \mathrm{GL}_2(\mathbb{Q}_p)$, if π, a smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representation, then the U_p-operator on π^H is defined by

$$U_p : \pi^H \xrightarrow{\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)} \pi^{H \cap \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) H \left(\begin{smallmatrix} p^{-1} & \\ & 1 \end{smallmatrix}\right)} \xrightarrow{\mathrm{Tr}} \pi^H.$$

Using

$$N(\mathbb{Z}_p)/\left[N(\mathbb{Z}_p) \cap \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} N(\mathbb{Z}_p) \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right] \xrightarrow{\sim} K_1(p^r)/\left[K_1(p^r) \cap \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_1(p^r) \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right],$$

the U_p operator on $\pi^{N(\mathbb{Z}_p)}$ can be explicitly defined by $U_p = \sum_{i=0}^{p-1} \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)$, as the coset representatives can be taken pretty explicitly.

We then define, for a continuous admissible unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, $\mathrm{Ord}(\pi) = e_{\mathrm{ord}}\pi^{N(\mathbb{Z}_p)}$, where $e_{\mathrm{ord}} = \lim_{n\to\infty} U_p^{n!}$ is the ordinary projector. This coincides with Emerton's ordinary projector defined in a more general setting. By definition, the ordinary part inherits a $T(\mathbb{Z}_p)$-action, for $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$.

We do not try too hard to justify our claim, which involves a few explicit representation-theoretic calculations (which is crucial in Hida theory). We just say a few words about why it is true:

- The first map becomes isomorphism after taking Ord because U_p acts nilpotently on the kernel of $V_k \longrightarrow 1 \otimes \chi_{k-2}$ ("changing the weight").
- The second map becomes isomorphism after taking Ord because, for $I_m = \Gamma_0(p^m) \cap \Gamma_1(p^n)$, $(e_{\mathrm{ord}}\pi^{I_m}) = (e_{\mathrm{ord}}\pi^{N(\mathbb{Z}_p)})^{T(\mathbb{Z}_p)}$ ("changing the level").

If we believe our claim, then we have an isomorphism

$$H^1(Y_1(Np^r), \mathcal{V}_{k,\mathbb{Q}_p})^{\mathrm{ord}}_{\mathfrak{m}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p^\times \times (1+p^r\mathbb{Z}_p)}(1 \otimes \chi_{k-2}^{-1}, \mathrm{Ord}(\widehat{H}^1_{\mathfrak{m}})).$$

We can further use nebentypus $\varepsilon : \mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/p^r\mathbb{Z})^\times \longrightarrow K^\times$ to specialize the isomoprihms into

$$H^1(Y_1(Np^r), \mathcal{V}_k, \varepsilon)^{\mathrm{ord}} \xrightarrow{\sim} \mathrm{Hom}_{T(\mathbb{Z}_p)}(1 \otimes \chi_{k-2}^{-1}\varepsilon^{-1}, \mathrm{Ord}(\widehat{H}^1_{\mathfrak{m}})).$$

Now we want to understand $\dim \mathrm{Ord}(\widehat{H}^1_{\mathfrak{m}})[1 \otimes \chi_{k-2}\varepsilon]$.

Theorem 6.5 (Hida, Emerton)**.** *For* Π *a unitary admissible continuous representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$, $\mathrm{Ord}(\Pi)$ *is an admissible representation of* $T(\mathbb{Z}_p)$. *If* $\Pi|_{\mathrm{GL}_2(\mathbb{Z}_p)}$ *sits as a direct summand of* $C(\mathrm{GL}_2(\mathbb{Z}_p), \mathbb{Z}_p)^{\oplus s}$, *then* $\mathrm{Ord}(\Pi)$ *sits as a direct summand of* $C(T(\mathbb{Z}_p), \mathbb{Z}_p)^{\oplus s}$.

Now we are finally ready to prove Theorem 4.2. Let $\Pi = \widehat{H}^1_{\mathfrak{m}}$. Then,

$$\mathrm{Ord}(\widehat{H}^1_{\mathfrak{m}}) = \bigoplus_{\psi : T(\mathbb{Z}_p) \to \overline{\mathbb{F}}_p^\times} \mathrm{Ord}(\widehat{H}^1_{\mathfrak{m}})_\psi.$$

Now Π is projective because the $W \mapsto \mathrm{Hom}(\Pi, W)$ is an exact functor for algebraic representations W; this functor is basically, after localizing at $\mathfrak{m}$,

$$W \mapsto \left\{ \begin{array}{c} \textbf{total cohomology} \text{ of local} \\ \text{system associated to } W \text{ over the} \\ \text{tower of modular curves} \end{array} \right\},$$

which is because all the interesting Hecke eigenclasses only appear in H^1 of modular curves, and the functor

$$W \mapsto \{\text{local system associated to } W\},$$

and the total cohomology functor are both exact.

As the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$, where $\Gamma = (1 + p\mathbb{Z}_p)^2$, is a regular local ring (a consequence of Γ being a pro-p group), the projectivity of $\mathrm{Ord}(\widehat{H}^1_{\mathfrak{m}})$, coming from Theorem 6.5 and projectivity of $\widehat{H}^1_{\mathfrak{m}}$, implies that $\mathrm{Ord}(\widehat{H}^1_{\mathfrak{m}})$ is free, namely $\mathrm{Ord}(\widehat{H}^1_{\mathfrak{m}})_\psi \cong \mathcal{C}(\Gamma, \mathbb{Z}_p)^{\oplus t_\psi}$ for some t_ψ.

Thus, for any character $\chi : T(\mathbb{Z}_p) \longrightarrow \mathcal{O}_K^\times$, $K/\mathbb{Q}_p$, satisfying $\psi \equiv \chi (\mathrm{mod}\, \varpi_K \mathcal{O}_K)$,

$$\dim \mathrm{Ord}(\widehat{H}^1_{\mathfrak{m}})[\chi] = t_\psi.$$

□

7. **Construction of eigenvariety via the p-adic Jacquet functor.**

Let Π be a locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Let $N_0 = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$, and consider Π^{N_0}, which has an action of $T_+ = \{\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} \in T(\mathbb{Q}_p) \mid v_p(\alpha) \geq v_p(\beta)\}$ via

$$t.v = \sum_{x \in N_0 / N_0 \cap t N_0 t^{-1}} x t v,$$

which is a Hecke action.

Remark 7.1. If we take $t = \begin{pmatrix} p & \\ & 1 \end{pmatrix}$, then the action of t is the U_p-operator $\sum_{i=0}^{p-1} \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}$.

Remark 7.2. The locally analytic distribution algebra $\mathcal{D}(T(\mathbb{Z}_p), \mathbb{Q}_p)$ is isomorphic to $\mathcal{O}^{\mathrm{rig}}(\widehat{T(\mathbb{Z}_p)})$, where $\widehat{T(\mathbb{Z}_p)}$ is the rigid analytic space parametrizing continuous characters of $T(\mathbb{Z}_p)$, isomorphic to $(\widehat{\mathbb{Z}_p^\times})^2$.

As $T(\mathbb{Q}_p) \cong (p^{\mathbb{Z}} \times \mathbb{Z}_p^\times)^2$,

$$\mathscr{T} := \widehat{T(\mathbb{Q}_p)} \cong \widehat{T(\mathbb{Z}_p)} \times (\mathbb{G}_m^{\mathrm{rig}})^2,$$

the space parametrizing locally analytic characters of $T(\mathbb{Q}_p)$. From this expression, we see that $\mathcal{O}^{\mathrm{rig}}(\widehat{T(\mathbb{Q}_p)})$ is a Fréchet-Stein algebra.

Remark 7.3. The distribution algebra over a non-compact locally analytic group $T(\mathbb{Q}_p)$, $\mathcal{D}(T(\mathbb{Q}_p), \mathbb{Q}_p)$, is a dense subspace of $\mathcal{O}^{\mathrm{rig}}(\widehat{T(\mathbb{Q}_p)})$.

We drop rig in the superscript from now on if the context is clear.

Definition 7.1. *The **locally analytic Jacquet functor** is*

$$J_B(\Pi) = \mathscr{L}^b_{T_+}(\mathcal{O}(\mathscr{T}), \Pi^{N_0}),$$

the T_+-equivariant continuous linear functions $\mathcal{O}(\mathscr{T}) \longrightarrow \Pi^{N_0}$. Equivalently, $J_B(\Pi)' = \mathcal{O}(\mathscr{T})\widehat{\otimes}_{\mathbb{Q}_p[T_+]}(\Pi^{N_0})'$.

Theorem 7.1. *If Π is a locally analytic admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, then $J_B(\Pi)$ is a locally analytic representation of $T(\mathbb{Q}_p)$ such that $J_B(\Pi)'$ is a coadmissible $\mathcal{O}(\mathscr{T})$-module.*

As there is an equivalence of categories

$$\{\text{coherent sheaves over } \mathscr{T}\} \xrightarrow{\sim} \{\text{coadmissible } \mathcal{O}(\mathscr{T})\text{-modules}\},$$
$$\mathcal{M} \longmapsto \mathcal{M}(\mathscr{T}),$$

from locally analytic Jacquet functor, we get a coherent sheaf $\mathcal{M}_\Pi$ on $\mathscr{T}$ such that

$$\mathcal{M}_\Pi \otimes k(x) \cong \mathrm{Hom}_{T_+}(\chi_x, \Pi^{N_0})' \cong (\pi^{N_0}[\chi_x])',$$

where χ is the universal character, i.e. for a closed point $x \in \mathscr{T}$, $\chi_x : T(\mathbb{Q}_p) \longrightarrow \overline{\mathbb{Q}}_p^\times$ is the corresponding character.

We now apply this construction to $\Pi = \widehat{H}^1(N)^{\mathrm{an}}_{\mathfrak{m}}$ which is an admissible $\mathrm{GL}_2(\mathbb{Q}_p)$-representation, for a non-Eisenstein maximal ideal $\mathfrak{m}$ (corresponding to an irreducible residual Galois representation $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$), and get a coherent sheaf $\mathcal{M}_{\overline{\rho}}$ on $\mathscr{T}$ such that

$$J_B(\widehat{H}^1_{\mathfrak{m}}(N)^{\mathrm{an}})' \cong \Gamma(\mathscr{T}, \mathcal{M}_{\overline{\rho}}).$$

Theorem 7.2 (Emerton)**.** *For an admissible locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, if $\Pi|_{\mathrm{GL}_2(\mathbb{Z}_p)}$ is a direct summand of $C^{\mathrm{an}}(\mathrm{GL}_2(\mathbb{Z}_p), \mathbb{Q}_p)^{\oplus s}$, then $\kappa : \mathrm{Supp}\,\mathcal{M}_\Pi \hookrightarrow \mathscr{T} \longrightarrow \mathcal{W} := \widehat{T(\mathbb{Z}_p)} \cong (\widehat{\mathbb{Z}_p^\times})^2$ has discrete fibers. Locally on $\mathrm{Supp}\,\mathcal{M}$, $\mathcal{M}$ is a finite free $\mathcal{O}_{\mathcal{W}}$-module.*

We can use similar technique to consider Hecke operators other than diagonals. Let $\mathbb{T}^{\mathrm{sph}}_{\overline{\rho}} \subset \mathrm{End}(\widehat{H}^1_{\mathfrak{m}}(N)^{\mathrm{an}})$ be the spherical Hecke algebra. Then, we have a map

$$\psi : \mathbb{T}^{\mathrm{sph}}_{\overline{\rho}} \longrightarrow \mathrm{End}(\widehat{H}^1_{\mathfrak{m}}(N)^{\mathrm{an}}) \longrightarrow \mathrm{End}(\mathcal{M}_{\overline{\rho}}).$$

Let $\mathscr{A}_{\overline{\rho}}$ be the sub-$\mathcal{O}_W$-subalgebra of $\mathrm{End}(\mathcal{M}_{\overline{\rho}})$ generated by $\mathrm{im}\,\psi$.

Definition 7.2. *Define the **eigenvariety** $\mathscr{E}_{\overline{\rho}}$ to be the relative Sp of $\mathscr{A}_{\overline{\rho}}$ over $\mathrm{Supp}(\mathcal{M}_{\overline{\rho}})$ (or over $\mathscr{T}$).*

Proposition 7.1. *The map $\kappa : \mathscr{E}_{\overline{\rho}} \longrightarrow \mathcal{W}$ is quasi-finite. Locally on $\mathscr{E}_{\overline{\rho}}$, it is finite surjective on every irreducible component.*

A closed point $x \in \mathscr{E}_{\overline{\rho}}$ gives rise to homomorphisms $\lambda : \mathbb{T}^{\mathrm{sph}}_{\overline{\rho}} \longrightarrow \overline{\mathbb{Q}}_p$ and $\delta : T(\mathbb{Q}_p) \longrightarrow \overline{\mathbb{Q}}_p^\times$. Thus,

$$\mathscr{E}_{\overline{\rho}}(\overline{\mathbb{Q}}_p) = \left\{ \begin{matrix} (\lambda, \delta) \in \left(\mathrm{MaxSpec}\,\mathbb{T}^{\mathrm{sph}}[1/p]\right) \times \widehat{T(\mathbb{Q}_p)} \text{ such that} \\ \mathrm{Hom}_{T(\mathbb{Q}_p)}(\delta, J_B(\widehat{H}^1_{\overline{\rho}}(N)^{\mathrm{an}}[\lambda])) \neq 0 \end{matrix} \right\},$$

where $\widehat{H}^1_{\overline{\rho}}[\lambda] \subset \widehat{H}^1_{\overline{\rho}}$ is a $\mathrm{GL}_2(\mathbb{Q}_p)$-subrepresentation.

Definition 7.3. *A point $x = (\lambda, \delta) \in \mathscr{E}_{\overline{\rho}}(\overline{\mathbb{Q}}_p)$ is **classical** if λ is classical, which means that there is some algebraic representation W of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$ open subgroup such that*

$$\mathrm{Hom}_{K_p}(W, \widehat{H}^1_{\overline{\rho}}[\lambda]) \cong H^1(Y_{K_p}, \mathcal{V}_{W^\vee})_{\mathfrak{m}}[\lambda] \neq 0.$$

Remark 7.4. This is a reasonable definition, because λ associated to a classical modular form $f \in S_k(\Gamma_1(Np^r))$ for $k \geq 2$ is classical (with $W = (\mathrm{Sym}^{k-2}\,\mathbb{Q}_p^2)'$).

8. **Classicality result.**

Definition 8.1. *A continuous character $\delta \in \widehat{T(\mathbb{Q}_p)} = \mathrm{Hom}(T(\mathbb{Q}_p), \overline{\mathbb{Q}}_p^\times)$ is **locally algebraic** if $\delta = \delta^{\mathrm{alg}}\delta^{\mathrm{sm}}$, where $\delta^{\mathrm{alg}}\left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right) = a^{k_1}d^{k_2}$ for some $k_1, k_2 \in \mathbb{Z}$, and δ^{sm} is a smooth (=locally constant) character.*

Definition 8.2. *For δ^{alg} an algebraic character, define*

$$M(\delta^{\mathrm{alg}}) = U(\mathfrak{gl}_2) \otimes_{U(\bar{\mathfrak{b}})} \delta^{\mathrm{alg}} \in \mathcal{O}^{\bar{\mathfrak{b}}}_{\mathrm{alg}},$$

an object in the category-$\mathcal{O}$.

Recall from Herzig's lecture that, for $M \in \mathcal{O}^{\bar{\mathfrak{b}}}_{\mathrm{alg}}$ and $\delta^{\mathrm{sm}} : T(\mathbb{Q}_p) \longrightarrow \overline{\mathbb{Q}}_p^\times$ a smooth character, there is a construction of **Orlik-Strauch representation**

$$\mathfrak{F}^{\mathrm{GL}_2}_{\overline{B}}(M, \delta^{\mathrm{sm}}),$$

which is an admissible $\mathrm{GL}_2(\mathbb{Q}_p)$-representation.

Theorem 8.1 (Emerton, Breuil)**.** *For a unitary p-adic Banach representation Π of $\mathrm{GL}_2(\mathbb{Q}_p)$ and a locally algebraic character $\delta = \delta^{\mathrm{alg}}\delta^{\mathrm{sm}} \in \widehat{T(\mathbb{Q}_p)}$,*

$$\mathrm{Hom}_{T(\mathbb{Q}_p)}(\delta, J_B(\Pi^{\mathrm{an}})) \cong \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathfrak{F}(\delta), \Pi^{\mathrm{an}}),$$

where

$$\mathfrak{F}(\delta) = \mathfrak{F}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\overline{B}(\mathbb{Q}_p)}(M((\delta^{\mathrm{alg}})^{-1})^\vee, \delta^{\mathrm{sm}}).$$

Note that the duality functor

$$\mathcal{O}^{\bar{\mathfrak{b}}}_{\mathrm{alg}} \xrightarrow{\sim} \mathcal{O}^{\bar{\mathfrak{b}}}_{\mathrm{alg}},$$
$$M \longmapsto M^\vee,$$

is contravariant, and fixes simple modules.

Remark 8.1. What is this construction $\mathfrak{F}(\delta)$?

- If $k_1 < k_2$, then $M((\delta^{\mathrm{alg}})^{-1})^\vee \cong M((\delta^{\mathrm{alg}})^{-1})$ is simple, and as explained in Herzig's lecture,

  $$\mathfrak{F}(\delta) = \left(\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\overline{B(\mathbb{Q}_p)}}\ \delta\right)^{\mathrm{an}},$$

  which is topologically irreducible.
- If $k_1 \geq k_2$, then there is a short exact sequence

  $$0 \longrightarrow M((\delta')^{-1}) \longrightarrow M((\delta^{\mathrm{alg}})^{-1}) \longrightarrow L((\delta^{\mathrm{alg}})^{-1}) \longrightarrow 0,$$

  where if $\delta^{\mathrm{alg}} = \delta_{k_1,k_2} : \left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right) \mapsto a^{k_1}d^{k_2}$, then $\delta' = \delta_{k_2-1,k_1+1}$. Thus, there is a short exact sequence

  $$0 \longrightarrow (\mathrm{Ind}^G_{\overline{B}}\ \delta'\delta^{\mathrm{sm}})^{\mathrm{sm}} \longrightarrow \mathfrak{F}(\delta) \longrightarrow L(\delta^{\mathrm{alg}}) \otimes (\mathrm{Ind}^{\mathrm{GL}_2}_{\overline{B}}\ \delta^{\mathrm{sm}})^{\mathrm{sm}} \longrightarrow 0.$$

  - Recall also that $L((\delta^{\mathrm{alg}})^{-1})$ is finite-dimensional. In particular in more familiar terms $L(\delta^{\mathrm{alg}}) = (\mathrm{Sym}^{k_1-k_2}\,\mathbb{Q}_p^2 \otimes \det{}^{k_2})$.
  - This is the opposite direction to how $\mathrm{Ind}^{\mathrm{GL}_2}_{\overline{B}}(\delta)^{\mathrm{an}}$ is filtered, i.e.

  $$0 \longrightarrow L(\delta^{\mathrm{alg}}) \otimes (\mathrm{Ind}^{\mathrm{GL}_2}_{\overline{B}}\ \delta^{\mathrm{sm}})^{\mathrm{sm}} \longrightarrow \mathrm{Ind}^{\mathrm{GL}_2}_{\overline{B}}(\delta)^{\mathrm{an}} \longrightarrow (\mathrm{Ind}^G_{\overline{B}}\ \delta'\delta^{\mathrm{sm}})^{\mathrm{sm}} \longrightarrow 0.$$

Theorem 8.2 (Classicality; Coleman, Emerton)**.** *For $x = (\lambda, \delta) \in \mathscr{E}_{\overline{\rho}}(\overline{\mathbb{Q}}_p)$, if $\delta = \delta^{\mathrm{alg}}\delta^{\mathrm{sm}}$ with $\delta^{\mathrm{alg}} = \delta_{k_1,k_2}$, $k_1 \geq k_2$, then*

$$v_p\left(\delta^{\mathrm{sm}}\begin{pmatrix} p & \\ & 1\end{pmatrix}\right) < -k_2 + 1 \Longrightarrow \lambda \text{ is classical.}$$

Proof. As $\mathrm{Hom}(\delta, J_B(\widehat{H}^1_{\overline{\rho}}(N)^{\mathrm{an}}[\lambda])) \neq 0$, there is a nonzero map $\mathfrak{F}(\delta) \longrightarrow \widehat{H}^1_{\overline{\rho}}[\lambda]$. We want to show that this factors through the quotient map

$$\mathfrak{F}(\delta) \twoheadrightarrow L(\delta^{\mathrm{alg}}) \otimes (\mathrm{Ind}_{\overline{B}}^{\mathrm{GL}_2}\delta^{\mathrm{sm}})^{\mathrm{sm}},$$

which will show that there is a nonzero map $L(\delta^{\mathrm{alg}}) \otimes (\mathrm{Ind}_{\overline{B}}^{\mathrm{GL}_2}\delta^{\mathrm{sm}})^{\mathrm{sm}} \longrightarrow \widehat{H}^1_{\overline{\rho}}[\lambda]$, which will imply classicality. As the kernel of $\mathfrak{F}(\delta) \twoheadrightarrow L(\delta^{\mathrm{alg}}) \otimes (\mathrm{Ind}_{\overline{B}}^{\mathrm{GL}_2}\delta^{\mathrm{sm}})^{\mathrm{sm}}$ is $(\mathrm{Ind}_{\overline{B}}^{\mathrm{GL}_2}\delta'\delta^{\mathrm{sm}})^{\mathrm{an}}$, it will suffice to show that there is no nonzero map $(\mathrm{Ind}_{\overline{B}}^{\mathrm{GL}_2}(\delta'\delta^{\mathrm{sm}}))^{\mathrm{an}} \longrightarrow \widehat{H}^1_{\overline{\rho}}$. As $(\mathrm{Ind}_{\overline{B}}^{\mathrm{GL}_2}(\delta'\delta^{\mathrm{sm}}))^{\mathrm{an}}$ is topologically irreducible, it will suffice to show that there is no embedding $(\mathrm{Ind}_{\overline{B}}^{\mathrm{GL}_2}(\delta'\delta^{\mathrm{sm}}))^{\mathrm{an}} \hookrightarrow \widehat{H}^1_{\overline{\rho}}$.

As $\widehat{H}^1_{\overline{\rho}}$ is a unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, it will be sufficient to prove that there is no $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant norm on $\Pi := \mathrm{Ind}_{\overline{B}}^{\mathrm{GL}_2}(\delta_{k_2-1,k_1+1}\delta^{\mathrm{sm}})^{\mathrm{an}}$. Suppose that there is an invariant norm $\|-\|$. Then, $\|U_p\| \leq 1$. We define a function $f \in \Pi$, via

- $f\left(b\begin{pmatrix} & 1\\ 1 & \end{pmatrix}u\right) = 0$,
- $f\left(b\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right) = \tilde{\delta}(b)$,

where $b \in \overline{B}(\mathbb{Q}_p)$, $u \in I = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times\end{pmatrix}$, $x \in \mathbb{Q}_p$. These formulae really define a function because of the Iwasawa decomposition $\mathrm{GL}_2(\mathbb{Q}_p) = \overline{B}I \coprod \overline{B}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}I$ (so that f is supported on $\overline{B}I$).

One also checks by hand that $U_p(f) = \tilde{\delta}\begin{pmatrix} p & \\ & 1\end{pmatrix}f$. Thus, $|\tilde{\delta}\begin{pmatrix} p & \\ & 1\end{pmatrix}| \leq 1$. As $\tilde{\delta} = \delta_{k_2-1,k_1+1}\delta^{\mathrm{sm}}$, this means $v_p(\delta^{\mathrm{sm}}\begin{pmatrix} p & \\ & 1\end{pmatrix}) \geq -k_2 + 1$, a contradiction. □

Example 8.1. This recovers Coleman's classicality theorem, as follows.

Let $f \in S_k(N)$ be an eigenform with $p \nmid N$ and $\overline{\rho}_f \cong \overline{\rho}$. Let α be a root of Hecke polynomial of f at p, namely $\alpha^2 - a_p\alpha + p^{k-1} = 0$. Then, by local-global compatibility, there is $(\lambda_f, \delta) \in \mathscr{E}_{\overline{\rho}}$, $\delta = \delta_{0,k-2}(\mathrm{un}_\alpha \otimes \mathrm{un}_{\alpha^{-1}}|-|^{2-k})$.

Conversely, if $(\lambda, \delta) \in \mathscr{E}_{\overline{\rho}}$ is a closed point, if $\delta = \delta_{0,2-k}\delta^{\mathrm{sm}}$, $\delta^{\mathrm{sm}} = \mathrm{un}_\alpha \otimes \mathrm{un}_{\alpha^{-1}}|-|^{2-k}$, then $v_p(\alpha) < k-1$ implies that λ is classical, and $\lambda = \lambda_f$ for some modular eigenform f.

Corollary 8.1. *The classical points are Zariski dense in $\mathscr{E}_{\overline{\rho}}$. Furthermore, $\mathscr{E}_{\overline{\rho}} \hookrightarrow (\mathrm{Spf}\,\mathbb{T}_{\overline{\rho}}^{\mathrm{sph}})^{\mathrm{rig}} \times \widehat{T(\mathbb{Q}_p)}$ is the Zariski closure of pairs (λ, δ) with λ classical, δ locally algebraic with smooth part unramified at p.*

Thus, $\mathscr{E}_{\overline{\rho}}$ can be thought as systems of Hecke eigenvalues interpolating classical systems.

9. **Galois representations.**

Let $x = (\lambda, \delta) \in \mathscr{E}_{\overline{\rho}}$ be a classical point. Then, there exists a unique $\rho_x : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ unramified outside of Np such that, for all $\ell \nmid Np$, $\mathrm{tr}(\rho_x(\mathrm{Frob}_\ell)) = \lambda(T(\ell))$.

Using the density result and techniques of pseudo-representations, one can attach Galois representations $\rho_x : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ to all closed points $x \in \mathscr{E}_{\overline{\rho}}$. It turns out that ρ_x is determined by λ, where $x = (\lambda, \delta) \in \mathscr{E}_{\overline{\rho}}$.

Question. Can we read δ on ρ_x?

Example 9.1. Given $(\lambda, \delta) \in \mathscr{E}_{\overline{\rho}}$ with λ classical, $\delta = \delta_{k_1,k_2}\delta^{\mathrm{sm}}$ with δ^{sm} unramified, by the local-global compatibility, we know that ρ_x is semistable, thus **trianguline**; here trianguline means that $D_{\mathrm{rig}}(\rho_x|_{G_{\mathbb{Q}_p}})$, a (φ, Γ)-module over the Robba ring R, is upper triangular. In particular, if

$$D_{\mathrm{rig}}(\rho_x|_{G_{\mathbb{Q}_p}}) \cong \begin{pmatrix} R(\delta_1) & * \\ 0 & R(\delta_2(x|x|)^{-1}) \end{pmatrix},$$

then $\delta = (\delta_1, \delta_2)$; indeed, a 1-dimensional (φ, Γ)-module over R is the same as a $\overline{\mathbb{Q}}_p$-valued character of $\mathbb{Q}_p^\times$.

It turns out that every Galois representation associated to a closed point on $\mathscr{E}_{\overline{\rho}}$ is trianguline.

Theorem 9.1 (Kisin)**.** *For a closed point $x = (\lambda, \delta) \in \mathscr{E}_{\overline{\rho}}$, $\rho_x|_{G_{\mathbb{Q}_p}}$ is trianguline, and, for $\delta = (\delta_1, \delta_2)$, there is a nonzero map $R(\delta_1) \hookrightarrow D_{\mathrm{rig}}(\rho_x)$.*

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We will talk about theory of local and global automorphic representation theory, local-global compatibility, local Langlands and do Langlands-Kottwitz for $\mathrm{GL}_{2,\mathbb{Q}}$.

1. **Local representation theory.**

Let $F/\mathbb{Q}_p$ be a finite extension and $G = \mathrm{GL}_n(F)$. Its p-adic topology is given by the basis $\{1 + \varpi_F^n M_n(\mathcal{O}_F)\}$. We will use $\mathbb{C}$ equipped with discrete topology for coefficients.

Definition 1.1. *A complex representation (π, V) of G is called* ***smooth*** *if for every $v \in V$, $\mathrm{Stab}_G(v)$ is an open subgroup of G. A smooth representation is* ***admissible*** *if V^K is finite-dimensional for every open compact subgroup $K \subset G$.*

Remark 1.1. Irreducible smooth reprsentations are admissible.

Remark 1.2. Given any representation (π, V), we can take the subrepresentation of smooth vectors $(\pi^{\mathrm{sm}}, V^{\mathrm{sm}})$, where V^{sm} is the space of vectors with open stabilizers.

For an irreducible smooth representation π, there is a central character ω_π by Schur's lemma.

Definition 1.2. *Given (π, V) an irreducible smooth admissible representation, $V^\vee = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})^{\mathrm{sm}}$ and $\pi^\vee(g)(f)(v) = f(\pi(g^{-1})v)$ gives the* ***contragredient*** *$(\pi^\vee, V^\vee)$.*

There is double duality, $(\pi^\vee)^\vee \cong \pi$, and $\omega_{\pi^\vee} = \omega_\pi^{-1}$.

Definition 1.3. *For $\underline{n} = (n_1, \cdots, n_r)$ such that $\sum_{i=1}^r n_i = n$, we define $P_{\underline{n}}$ to be the upper-triangular parabolic subgroup corresponding to the partition $\underline{n}$. Its Levi decomposition is denoted as $P_{\underline{n}} = M_{\underline{n}} N_{\underline{n}}$.*

The modulus character $\delta_{\underline{n}} : M_{\underline{n}} \longrightarrow \mathbb{C}^\times$ is defined as $\delta_{\underline{n}}(m) = \left|\det(\mathrm{Ad}(m \mid \mathrm{Lie}\, N_{\underline{n}}))\right|_F$.

For example, for GL_2 and partition $2 = 1 + 1$, it is the famous charcater $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto |ad^{-1}|$.

Definition 1.4. *For a representation $\pi_{\underline{n}}$ of $M_{\underline{n}}$, the* ***normalized parabolic induction*** *is the representation*

$$\mathrm{Ind}_{P_{\underline{n}}}^G(\pi_{\underline{n}} \otimes \delta_{\underline{n}}^{1/2})^{\mathrm{sm}}$$

of G. We will denote it as n–ind.

Remark 1.3. This is the point where working with $\mathbb{C}$-coefficients is useful, because taking square-root is quite canonical. Otherwise you have to choose square roots.

Remark 1.4. The functor n–ind is reasonable. For example, it is an exact functor, has explicity left and right adjoints (i.e. Jacquet functor), preserves admissibility and finite-length-ness.

Remark 1.5. The introduction of modulus character is nice because the normalized induction preserves unitarizability and is "Weyl-symmetric", i.e. semisimplification of normalized induction does not change by changing orders.

Definition 1.5. *An irreducible smooth representation is* ***(essentially) square-integrable (=discrete series)*** *if its matrix coefficients are square-integrable modulo center up to character twist. It is* ***supercuspidal*** *if its matrix coefficients are compactly supported modulo center.*

We have the following hierarchy.

$$\left\{\begin{array}{c}\text{Smooth irreducible}\\ \text{representations}\end{array}\right\} \supset \left\{\begin{array}{c}\text{Square-integrable}\\ \text{representations}\end{array}\right\} \supset \left\{\begin{array}{c}\text{Supercuspidal}\\ \text{representations}\end{array}\right\}.$$

Supercuspidal representations are "building blocks" of constructing admissible representations, because supercuspidals are precisely those that do not arise as subquotient of parabolic induction.

Definition 1.6. *We fix a normalized Haar measure on G such that* $\mathrm{vol}(\mathrm{GL}_n(\mathcal{O}_F)) = 1$. *Then the* ***local Hecke algebra*** *is $\mathscr{H}(G) := C_c^\infty(G)$, which is an infinite-dimensional noncommutative non-unital $\mathbb{C}$-algebra. It is identified with*

$$\bigcup_{K \text{ open compact subgroups}} C_c^\infty(K\backslash G/K).$$

Then smooth G-representations are just smooth $\mathscr{H}(G)$-modules, where smooth means $\mathscr{H}(G)V = V$. Given smooth G-representation (π, V), we define the action of $f \in \mathscr{H}(G)$ on $v \in V$ as

$$\pi(f)v = \int_G f(g)\pi(g)v\,dg.$$

If $f \in C_c^\infty(K\backslash G/K)$, then $\mathrm{im}(\pi(f)) \subset V^K$. This yields a 1-1 correspondence

$$\left\{\begin{array}{c}\text{Irreducible smooth}\\ G\text{-representation}\\ \text{with } \pi^K \neq 0\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{Irreducible}\\ C_c^\infty(K\backslash G/K)\text{-}\\ \text{module}\end{array}\right\}$$

via $\pi \mapsto \pi^K$. This is a useful viewpoint because admissibility tells you that $\mathrm{tr}\,\pi(f)$ is well-defined. For example, $\mathrm{tr}\,\pi(g)$ does not a priori make sense (but actually it makes sense for g regular semisimple by Harish-Chandra).

2. Bernstein-Zelevinsky classification.

Example 2.1. For $G = \mathrm{GL}_2(F)$, the unnormalized induction of trivial representation from the upper Borel has trivial representation as a subrepresentation, and its quotient is irreducible, which is usually called **Steinberg representation**. It is square-integrable.

This is a general phenomenon, namely one can obtain all square-integrable representations from supercuspidals. One might want to call this as **generalized Steinberg representation**. Namely, given π_0 a supercuspidal representation of $\mathrm{GL}_r(F)$,

$$\mathrm{St}_m(\pi_0) = \mathrm{n\text{-}ind}(\pi_0 \boxtimes \pi_0|\det| \boxtimes \cdots \boxtimes \pi_0|\det|^{m-1})$$

is a square-integrable irreducible representation of G_{mr}. This construction exhausts all irreducible square-integrable representations.

From square-integrable representations, we can get all irreducible admissible representations via **Langlands quotients**. Namely, an irreducible smooth admissible representation π of G is uniquely expressable as

$$\pi = \boxplus_{i=1}^r \mathrm{St}_{m_i}(\pi_i),$$

for π_i supercuspidal representations of G_{r_i}, where $\boxplus_{i=1}^r \mathrm{St}_{m_i}(\pi_i)$ is a distinguished irreducible subquotient of $\mathrm{n\text{-}ind}(\boxtimes_{i=1}^r \mathrm{St}_{m_i}(\pi_i))$.

3. **Satake isomorphism.**

Let us consider $K_0 = \mathrm{GL}_n(\mathcal{O}_F)$, which is a maximal compact subgroup of G.

Definition 3.1. *An irreducible smooth admissible representation π of G is* ***unramified*** *if $\pi^{K_0} \neq 0$.*

If $n = 1$, this corresponds to unramified characters via local class field theory.

We denote $\mathscr{H}^{\mathrm{ur}}(G) := \mathscr{H}(G, K_0)$. From general theory, we know that irreducible unramified representations of G corresponds to irreducible $\mathscr{H}^{\mathrm{ur}}(G)$-modules. Thus, to describe unramified representations, it is desirable to describe the spherical Hecke algebra $\mathscr{H}^{\mathrm{ur}}(G)$.

To proceed, we fix the standard $T \subset B \subset G$, namely T is the group of diagonal matrices, and B is the group of upper triangular matrices. It corresponds to the partition $\underline{n} = (1, \cdots, 1)$. The modulus character $\delta : B \twoheadrightarrow T \longrightarrow \mathbb{R}^\times_{>0}$ is given by $\mathrm{diag}(t_1, \cdots, t_n) \mapsto |t_1^{n-1} t_2^{n-3} \cdots t_n^{1-n}|$. After a choice of uniformizer, we have a very explicit description of $\mathscr{H}^{\mathrm{ur}}(T)$, namely

$$\begin{aligned} \mathscr{H}^{\mathrm{ur}}(T) &\xrightarrow{\sim} \mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}], \\ \mathbb{1}_{\mathrm{diag}(\varpi^{a_1}, \cdots, \varpi^{a_n})} &\mapsto t_1^{a_1} \cdots t_n^{a_n}. \end{aligned}$$

It is then possible to understand $\mathscr{H}^{\mathrm{ur}}(G)$ in terms of $\mathscr{H}^{\mathrm{ur}}(T)$.

Theorem 3.1 (Satake isomorphism)**.** *The map $\mathcal{S} : \mathscr{H}^{\mathrm{ur}}(G) \longrightarrow \mathscr{H}^{\mathrm{ur}}(T)$, defined by*

$$\mathcal{S}(f)(t) = \delta^{1/2}(t) \int_N f(tn) dn,$$

for the normalized Haar measure on N (i.e. $\mathrm{vol}(N(\mathcal{O}_F)) = 1$), induces an isomorphism

$$\mathscr{H}^{\mathrm{ur}}(G) \longrightarrow \mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]^{S_n} \subset \mathscr{H}^{\mathrm{ur}}(T),$$

where the S_n-action permutes $t_1, \cdots, t_n$.

Proof. One checks that it is an algebra homomorphism and that the image lands in the Weyl-invariant subspace. And then you check the bijectivitiy by exhibiting explicit bases on both sides. Namely, $\mathscr{H}^{\mathrm{ur}}(G)$ has basis

$$\{\mathbb{1}_{K_0\, \mathrm{diag}(\varpi^{a_1}, \cdots, \varpi^{a_n}) K_0} \mid a_1 \geq \cdots \geq a_n\},$$

which comes from the Cartan decomposition, and $\mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]^{S_n}$ has basis

$$\sum_{w \in S_n} t_1^{a_{w(1)}} \cdots t_n^{a_{w(n)}}.$$

Then, one can show that $\mathcal{S}$ is given by an "upper-triangular matrix with nonzero diagonal." Of course, both sides are infinite dimensional vector spaces, so one has to be careful, but one can make sense. $\square$

Corollary 3.1. *(1) $\mathscr{H}^{\mathrm{ur}}(G)$ is commutative, so that its irreducible modules are 1-dimensional.*

(2) One gets

$$\left\{ \begin{array}{c} \textit{Unramified} \\ \textit{irreducible} \\ \textit{representations of } G \end{array} \right\} = \{\mathscr{H}^{\mathrm{ur}}(G)\textit{-modules}\} = (\mathbb{C}^\times)^n / S_n,$$

where the composition is $\pi \mapsto \mathrm{Sat}(\pi) \in (\mathbb{C}^\times)^n / S_n$, the ***Satake parameters*** *of π, and $\mathrm{Sat}(\pi)$ is (unordered) tuple of t_i-eigenvalues of π. What we really mean by taking t_i-eigenvalues is to take the roots of $x^n - (t_1 + \cdots + t_n)x^{n-1} + \cdots + (-1)^n t_1 \cdots t_n = 0$.*

Furthermore, its inverse can be given as follows. Given $(s_1, \cdots, s_n) \in (\mathbb{C}^\times)^n / S_n$,

$$\mathrm{Sat}(\chi_{s_1} \boxplus \cdots \boxplus \chi_{s_n}) = (s_1, \cdots, s_n),$$

where $\chi_{s_i} : F^\times \longrightarrow \mathbb{C}^\times$ is the unramified charcater given by $\chi_{s_i} = s_i^{v(a)}$, with v being the normalized valuation $v : F^\times \twoheadrightarrow \mathbb{Z}$.

The set $(\mathbb{C}^\times)^n/S_n$ is better expressed as

- $(\mathbb{G}_m^n/S_n)(\mathbb{C})$, as $\mathscr{H}^{\mathrm{ur}}(G) = \mathcal{O}(\mathbb{G}_m^n/S_n)$,
- or even better, the set of semisimple conjugacy classes of $\mathrm{GL}_n(\mathbb{C})$.

Such viewpoints are more apt for generalizations to other groups.

4. Basics on Galois representations.

Let Γ be a topological group and k be a topological field. Then, given a continuous representation $\rho : \Gamma \longrightarrow \mathrm{GL}_k(V)$ for a finite dimensional vector space V over k, the **semisimplification** of ρ, denoted ρ^{ss}, is defined as

$$\rho^{\mathrm{ss}} = \bigoplus_i V_i/V_{i-1},$$

where $0 \subset V_1 \subset \cdots \subset V_r = V$ is a decomposition series.

Proposition 4.1 (Brauer-Nesbitt)**.** *If $\rho_1, \rho_2 : \Gamma \longrightarrow \mathrm{GL}_n(k)$ are two continuous representations such that* $\mathrm{char.poly}(\rho_1(\gamma)) = \mathrm{char.poly}(\rho_2(\gamma))$ *for γ in a dense subset of Γ, then $\rho_1^{\mathrm{ss}} \cong \rho_2^{\mathrm{ss}}$.*

Remark 4.1. If the characteristic of k is zero, one is enough to require that the traces are equal.

Now let Γ be furthermore compact (e.g. Galois group).

Proposition 4.2.

(1) A continuous representation $\rho : \Gamma \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ has image in $\mathrm{GL}_n(E)$ for some finite extension $E/\mathbb{Q}_\ell$.

(2) A continuous representation $\rho : \Gamma \longrightarrow \mathrm{GL}_n(E)$, for a finite extension $E/\mathbb{Q}_\ell$, has image in $\mathrm{GL}_n(\mathcal{O}_E)$ up to conjugation.

It now makes sense to define "mod ℓ reduction."

Definition 4.1. *Given a continuous representation $\rho : \Gamma \longrightarrow \mathrm{GL}_n(E)$, we define $\overline{\rho} : \Gamma \longrightarrow \mathrm{GL}_n(k_E)$, where k_E is the residue field of $\mathcal{O}_E$, to be the semisimplification of the reduction of a conjugation of ρ whose image is inside $\mathrm{GL}_n(\mathcal{O}_E)$. The definition is well-defined by the Brauer-Nesbitt theorem.*

Now one can put absolute Galois groups for Γ; if F is a number field and if v is a place, upon choosing algebraic closures $\overline{F}$, $\overline{F}_v$ and embedding $i_v : \overline{F} \hookrightarrow \overline{F}_v$, we get $\Gamma_v \hookrightarrow \Gamma$ via restriction. Recall also that, if v is a nonarchimedean place, one has a structure theorem

$$1 \longrightarrow I_v \longrightarrow \Gamma_v \longrightarrow \mathrm{Gal}(\overline{k}_v/k_v) \longrightarrow 1,$$

where k_v is the residue field of F_v. The $\mathrm{Gal}(\overline{k}_v/k_v)$ is identified with $\widehat{\mathbb{Z}}$, namely one can choose a topological generator. There are two conventions, either the **arithmetic Frobenius** $x \mapsto x^{\#k_v}$, or the **geometric Frobenius**, the inverse of the arithmetic Frobenius. We will denote the (conjugacy class of) **geometric Frobenius** as Frob_v.

Definition 4.2. *The **local Weil group** W_{F_v} is defined by the pullback of $\mathbb{Z}$ via $\Gamma_v \twoheadrightarrow \mathrm{Gal}(\overline{k}_v/k_v) \cong \widehat{\mathbb{Z}}$, with topology that makes I_v an open subgroup with the same topology as the subspace topology on I_v from the profinite topology of Γ_v.*

The **local class field theory** identifies, via the **Artin isomorphism**,

$$\mathrm{Art}_{F_v} : F_v^\times \xrightarrow{\sim} W_{F_v}^{\mathrm{ab}}.$$

Definition 4.3. *A local Galois representation $\rho : \Gamma_v \longrightarrow \mathrm{GL}_n(k)$ is **unramified** if $\rho|_{I_v} = 1$.*

In this case, $\rho(\mathrm{Frob}_v)$ is well-defined.

Definition 4.4. *A global Galois representation $\rho : \Gamma \longrightarrow \mathrm{GL}_n(k)$ is* ***unramified*** *if $\rho|_{I_v} = 1$.*

This is well-defined, because every $i_v : \Gamma_v \hookrightarrow \Gamma$ is well-defined up to conjugation by element in Γ.

Proposition 4.3. *If X/F is a smooth projective variety, then $H^i_{\text{ét}}(X_{\overline{F}}, \overline{\mathbb{Q}}_\ell)$, as a Γ-representation, is unramified almost everywhere.*

Proposition 4.4. *If a Galois representation $\rho : \Gamma \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is unramified outside S, then ρ^{ss} is determined by $\rho(\mathrm{Frob}_v)$ for $v \notin S$.*

This is just Cebotarev density theorem.

5. **Weil-Deligne representations.**

We would like to use Weil(-Deligne) reprsentations because we want to somehow turn Galois representations into representations "without topology."

- This is better for comparison with automorphic side (local Langlands correspondence viewpoint); recall that smooth representations "do not care much about topology."
- For the formulation of the notion of compatible system, it is important to "forget topology."

Recall that the local Weil group sits inside the following diagram,

$$\begin{tikzcd} 1 \arrow[r] & I_{F_v} \arrow[r] \arrow[d, equal] & W_{F_v} \arrow[r] \arrow[d, hook] & \mathbb{Z} \arrow[r] \arrow[d, hook] & 0 \\ 1 \arrow[r] & I_{F_v} \arrow[r] & \Gamma_{F_v} \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] & 0 \end{tikzcd}$$

The local class field theory equips a valuation $|\cdot| : W_{F_v} \longrightarrow \mathbb{R}^\times_{>0}$ via

$$W_{F_v} \twoheadrightarrow W^{ab}_{F_v} \xrightarrow{\sim} F_v^\times \xrightarrow{|\cdot|} \mathbb{R}^\times_{>0},$$

where the geometric Frobenius goes to the inverse of the size of the residue field.

Definition 5.1. *A* ***Weil-Deligne representation*** *of W_{F_v}, over a characteristic zero field k, is a triple (V, r, N) where*

- *V is a finite-dimensional vector space over k,*
- *$r : W_{F_v} \longrightarrow \mathrm{GL}(V)$ is a continuous homomorphism, with respect to the discrete topology on $\mathrm{GL}(V)$, i.e. $r|_{I_{F_v}}$ has open kernel,*
- *and $N \in \mathrm{End}_k(V)$ is a nilpotent operator such that $r(w)Nr(w)^{-1} = |w|N$ for all $w \in W_{F_v}$.*

The definition makes sense, as $|w| \in \mathbb{Q}$.

Remark 5.1. The condition $r(w)Nr(w)^{-1} = |w|N$ for all $w \in W_{F_v}$ automatically implies that N is nilpotent.

Definition 5.2. *A Weil-Deligne representation (V, r, N) is* ***Frobenius-semisimple****, if $r(w)$ is semisimple for all $w \in W_{F_v}$, or equivalently for some w with $|w| \neq 1$.*

We say (V, r, N) is ***semisimple*** *if it is Frobenius-semisimple and $N = 0$.*

We say (V, r, N) is ***unramified*** *if $r(I_{F_v}) = 1$ and $N = 0$.*

Remark 5.2. (1) The equivalence of two conditions on Frobenius-semisimplicity is because the semisimplicity of $r(w)$ for $w \in I_{F_v}$ is automatic; I_{F_v} acts via finite quotient and the base field is of characteristic 0.

(2) The unramifiedness condition requires $N = 0$ because the N-action arises from tame Galois action when the Weil-Deligne representation is made out of local Galois representation.

Example 5.1. A Weil-Deligne representation is Frobenius-semisimple and unramified if and only if $N = 0$ and r is a direct sum of unramified characters of W_{F_v}, namely characters factoring through $W_{F_v} \twoheadrightarrow W_{F_v}/I_{F_v} \cong \mathbb{Z}$. Thus, n-dimensional Frobenius-semisimle unramified Weil-Deligne representations are in natural bijection with $(k^\times)^n/S_n$.

Definition 5.3. *Given a Weil-Deligne representation $\sigma = (V, r, N)$, the* ***Frobenius-semisimplification*** *$\sigma^{\mathrm{Fss}} = (V, r^{\mathrm{ss}}, N)$, which is defined by taking the semisimple part of r elementwise in terms of Jordan decomposition.*

Taking the semisimple part elementwise is a reasonable operation, i.e. it yields a representation.

Example 5.2. Given $r : W_{F_v} \twoheadrightarrow W_{F_v}/I_{F_v} \longrightarrow \mathrm{GL}_2(k)$, $1 \mapsto \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, $r^{\mathrm{ss}}(1) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

We have the following hierarchy.

$$\left\{\begin{matrix}\text{Frobenius-semisimple} \\ \text{Weil-Deligne representations}\end{matrix}\right\} \supset \left\{\begin{matrix}\text{Indecomposable} \\ \text{Frobenius-semisimple} \\ \text{Weil-Deligne representations}\end{matrix}\right\} \supset \left\{\begin{matrix}\text{Irreducible} \\ \text{Weil-Deligne} \\ \text{representations}\end{matrix}\right\}.$$

This turns out to correspond to the hierarchy of smooth admissible representations on the automorphic side we saw before via the local Langlands correspondence. Then, at least we should know how to build Weil-Deligne representations out of irreducible Weil-Deligne representations, in the analogy with automorphic side.

Exercise 5.1. Show that $N = 0$ if $\sigma = (V, r, N)$ is irreducible.

Building Frobenius-semisimple representations out of indecomposable Frobenius-semisimple representations is simple, by just taking direct sums. To build indecomposable Frobenius-semisimple Weil-Deligne representations out of irreducible Weil-Deligne representations, we need an analogue of generalized Steinberg representations.

Definition 5.4. *Let $\sigma = (V, r, N)$ be an irreducible Weil-Deligne representation. Then, the generalized Steinberg representation $\mathrm{Sp}_m(\sigma)$ is defined by*

$$\mathrm{Sp}_m(\sigma) = (V^{\oplus m}, r \oplus r|\cdot| \oplus \cdots \oplus r|\cdot|^{m-1}, N_m),$$

where $N_m(v_1, \cdots, v_m) = (0, v_1, \cdots, v_{m-1})$.

Exercise 5.2. Check that this is a Weil-Deligne representation.

Exercise 5.3. Check that generalized Steinberg representations exhaust all Frobenius-semisimple indecomposable Weil-Deligne representations, i.e. there is a bijection

$$\left\{\begin{matrix}\text{Frobenius-semisimple} \\ \text{indecomposable} \\ \text{Weil-Deligne representations}\end{matrix}\right\} \leftrightarrow \{(m, \sigma_0) \mid m \geq 1, \sigma_0 \text{ is an irreducible Weil-Deligne representation}\},$$

$$\mathrm{Sp}_m(\sigma_0) \longleftarrow\!\!\dashv (m, \sigma_0).$$

Also check that direct sums exhaust all Frobenius-semisimple Weil-Deligne representations.

Remark 5.3. Anything beyond Frobenius-semisimplicity is not captured by automorphic side. On the other hand, it is expected as Frobenius should act semisimply in the ℓ-adic context (if $\ell \neq p$) under the Tate conjecture.

6. **Weil-Deligne representations and local Galois representations.**

Let $k = \overline{\mathbb{Q}}_\ell$, and $F/\mathbb{Q}_p$ be a finite extension.

If $\ell \neq p$, there is a fully faithful functor

$$\begin{aligned} \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\Gamma_F) &\longrightarrow \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{WD}_F), \\ (V, \rho) &\longmapsto (V, r, N). \end{aligned}$$

We will not discuss the precise definition. Instead, we will just see some special cases of it.

Example 6.1. If $\rho(I_F)$ is finite, then $r = \rho|_{W_F}$, and $N = 0$.

In general, N remembers some infinite unipotent action of ℓ-part of tame inertia. To prove the well-definedness one needs to use Grothendieck's ℓ-adic monodromy theorem.

Remark 6.1. This is not an equivalence of categories; any eigenvalue of ℓ-adic Γ_F-representation is an ℓ-adic unit, as the representation can be conjugated into $\mathrm{GL}_n(\overline{\mathbb{Z}}_\ell)$.

If $\ell = p$, there is a functor, not fully faithful,

$$\mathrm{Rep}^{\mathrm{dR}}_{\overline{\mathbb{Q}}_p}(\Gamma_F) \longrightarrow \mathrm{Rep}_{\overline{\mathbb{Q}}_p}(\mathrm{WD}_F).$$

7. **Local Langlands correspondence for GL_n.**

The local Langlands correspondence for GL_n is proven by Harris-Taylor, Henniart, Scholze (and the function field case proven by Laumon-Rapoport-Stuhler). All known proofs are global.

Theorem 7.1. *There is a unique bijection*

$$\mathrm{LL}_n : \left\{ \begin{array}{c} \textit{Smooth irreducible} \\ \textit{representations of} \\ \mathrm{GL}_n(F) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} n\textit{-dimensional} \\ \textit{Frobenius-semisimple} \\ \textit{representations of}\ \mathrm{WD}_F \\ \textit{over}\ \mathbb{C} \end{array} \right\},$$

such that

(1) the $n = 1$ case is given by the local class field theory, namely

$$\left\{ \begin{array}{c} \textit{Complex characters} \\ \textit{of}\ F^\times \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Continuous characters} \\ W_F \longrightarrow \mathbb{C}^\times \end{array} \right\},$$

(2) if $\sigma = \mathrm{LL}_n(\pi)$, then
- $\mathrm{LL}_1(\omega_\pi) = \det(\sigma)$,
- $\mathrm{LL}_n(\pi \otimes (\chi \circ \det)) = \sigma \otimes \chi$, *for any character χ,*
- $\mathrm{LL}_n(\pi^\vee) = \sigma^\vee$,

(3)
- *square-integrable representations correspond to indecomposable Weil-Deligne representations,*
- *supercuspidal representations correspond to irreducible Weil-Deligne representations,*
- *unramified representations correspond to unramified Weil-Deligne representations,*
- *and $\mathrm{LL}_n\left(\boxplus_i \mathrm{St}_{m_i}(\pi_i)\right) = \bigoplus_i \mathrm{Sp}_{m_i}(\sigma_i)$, for supercuspidal representations π_i, and $\sigma_i = \mathrm{LL}(\pi_i)$,*

(4) and if $\mathrm{LL}(\pi_i) = \sigma_i$ for $i = 1, 2$, then the L-factors and ε-factors of $\pi_1 \times \pi_2$ and $\sigma_1 \otimes \sigma_2$ match.

Remark 7.1. The L and ε-factors of $\pi_1 \times \pi_2$ do not mean you take L and ε-factors of some representation $\pi_1 \times \pi_2$, but you have some way of defining such factors for a pair of representations.

Remark 7.2. There is another normalization that respects automorphism of the coefficient field, namely $\mathrm{LL}_n \otimes |\cdot|^{\frac{1-n}{2}}$.

Remark 7.3. There are operations on the Weil-Deligne side like base-change, induction and tensor product. Thus, for example the L and ε-factors of $\pi_1 \times \pi_2$ are actually L and ε-factor of some smooth irreducible representation of $\mathrm{GL}_{m_1+m_2}(F)$, when π_i is a smooth irreducible representation of $\mathrm{GL}_{m_i}(F)$.

Remark 7.4. Henniart proved (1) and (4) for ε-factors.

Remark 7.5. The local class field theory is enough for the uniqueness of local Langlands correspondence.

8. **Automorphic representations.**

Let now F be a number field, and $G = \mathrm{GL}_{n,F}$, $Z = Z(\mathrm{GL}_n)$, $\mathbb{A}_F = \mathbb{A} \otimes_{\mathbb{Q}} F$, and $\omega : \mathbb{A}_F^\times / F^\times \longrightarrow \mathbb{C}^\times$ be a continuous character, where $\mathbb{C}^\times$ has the complex topology.

Consider $L^2(G(F)\backslash G(\mathbb{A}_F), \omega)$, which means the space of square-integrable functions on $G(F)\backslash G(\mathbb{A}_F)$ with central character ω. To be more precise, if $|w| = 1$, then $f \in L^2$ means

$$\int_{G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} |f(g)|^2 dg < \infty.$$

Definition 8.1. *The subspace $L^2_{\mathrm{cusp}} \subset L^2$ is defined by $f \in L^2_{\mathrm{cusp}}$ if*

$$\int_{N_{\underline{n}}(F)\backslash N_{\underline{n}}(\mathbb{A}_F)} f(ng)dn = 0,$$

for any nontrivial partition $\underline{n}$ and almost every $g \in G$.

*An irreducible representation π of $G(\mathbb{A}_F)$ is **cuspidal automorphic** if π is a closed sub-$G(\mathbb{A}_F)$-module of $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}_F), \omega)$.*

This definition captures cusp forms from modular forms when $G = \mathrm{GL}_{2,\mathbb{Q}}$.

Theorem 8.1 (Flath decomposition)**.** *If π is a cuspidal automorphic representation (or more generally irreducible admissible representation), then $\pi = \widehat{\otimes}'_v \pi_v$, where v runs over all places of F and π_v is an irreducible representation of $G(F_v)$.*

This comes from $G(\mathbb{A}_F) = \prod'_v G(F_v)$.

Remark 8.1. At a finite place v, π_v is not necessarily smooth, but you can take smooth vectors π_v^{sm}. Basically you can go back and forth (take unitary completion) with ease.

At an infinite place v, again there is a dense subspace $\pi_v \supset \pi_v^{\mathrm{sm}}$ which is the space of K_v-finite vectors. It turns out that π_v^{sm} is a (Lie $G(F_v), K_v$)-module.

Theorem 8.2 (Strong multiplicity one)**.** *If π, π' are cuspidal automorphic representations such that $\pi_v \cong \pi'_v$ for almost every v, then $\pi = \pi'$ as subrepresentations of L^2_{cusp} (the same subspace!).*

There is a similar notion of "parameters" at infinite places. Namely, given an "irreducible representation of $\mathrm{GL}_n(F_v)$" (better is a $(\mathfrak{g}_v, K_v)$-module) for $v \mid \infty$, one can attach a parameter, **Harish-Chandra parameters**, or **infinitesimal character**, $\inf(\pi)$, which is an element of $(\mathbb{C}^n/S_n)^{[F_v:\mathbb{R}]}$. Roughly speaking, $Z(\mathfrak{g}_v)$ acts on π via scalar, and the Harish-Chandra isomorphism says

$$Z(\mathfrak{g}_v) \cong (\mathbb{C}[t_1, \cdots, t_n]^{S_n})^{\otimes [F_v:\mathbb{R}]}.$$

Then, $\inf(\pi)$ will consist of "t_i-eigenvalues" of this action.

Definition 8.2 (Buzzard-Gee)**.** *A cuspidal automorphic representation π of $G(\mathbb{A}_F)$ is L-algebraic if* $\inf(\pi_v) \in (\mathbb{Z}^n/S_n)^{[F_v:\mathbb{R}]}$ *for all $v \mid \infty$, and C-algebraic if* $\inf(\pi_v) \in ((\mathbb{Z} + \frac{n-1}{2})^n/S_n)^{[F_v:\mathbb{R}]}$ *for all $v \mid \infty$.*

Example 8.1. For $G = \mathrm{GL}_{2,\mathbb{Q}}$, the cuspidal automorphic representation π_f corresponding to a classical cuspform f of weight $k \geq 2$ has infinitesimal character $\inf(\pi_f, \infty) = \left(k - \frac{3}{2}, -\frac{1}{2}\right)$, which is C-algebraic. This is the lecturer's normalization, and it can easily change up to character twist. More specifically, twisting by the a-th power of $|\det| : \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}) \longrightarrow \mathbb{R}^\times_{>0}$ adds $(-a, -a)$ to the infinitesimal character.

9. **Global Langlands correspondence for** GL_n.

Fix ℓ and $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$.

Conjecture 9.1. *There is a unique bijection*

$$\left\{\begin{array}{c} \textit{Cuspidal automorphic} \\ C\textit{-algebraic representations } \pi \\ \textit{of } \mathrm{GL}_n(\mathbb{A}_F) \end{array}\right\} = \left\{\begin{array}{c} \textit{Irreducible continuous} \\ \rho : \Gamma_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \\ \textit{unramified almost everywhere} \\ \textit{and } \rho_v \textit{ is de Rham for all } v \mid \ell \end{array}\right\},$$

such that $\pi \leftrightarrow \rho$ if and only if $\mathrm{LL}(\pi_v) = \left(\mathrm{WD}(\rho_v)^{\mathrm{Fss}} \otimes_{\overline{\mathbb{Q}}_\ell,\iota} \mathbb{C}\right) \otimes |-|^{\frac{1-n}{2}}$ *for all $v \nmid \infty$ (**local-global compatibility**,* LGC_v*).*

Remark 9.1. The RHS depends on ℓ and ι, but the LHS does not. This suggests that the compatible system of ℓ-adic Galois representations come from something living over $\overline{\mathbb{Q}}$.

Remark 9.2. Some of the known instances of Conjecture 9.1 are as follows.

- "Construction of Galois representations", namely from LHS to RHS:
  - Cohomology of Shimura varieties (+ others).
  - Done if F is CM or totally real and π is regular (=$\inf(\pi)$ consists of distinct parameters) by Harris-Lan-Taylor-Thorne, Scholze.
- "Automorphy of Galois representations", namely from RHS to LHS:
  - Modularity lifting techniques (Taylor-Wiles-Kisin, Calegari-Geraghty).
  - The 10-author paper, the 4-author paper.

Our goal now is to illustrate an instance of "construction of Galois representations" for elliptic cusp forms.

Theorem 9.1. *Let $G = \mathrm{GL}_{2,\mathbb{Q}}$, and π is a cuspidal automorphic regular C-algebraic representation, unramified outside S. Then, there is a unique $\rho_\pi : \Gamma_\mathbb{Q} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ such that LGC_v holds at all $v \notin S \cup \{\ell\}$.*

Explicitly, LGC_v *means $p^{1/2}\,\mathrm{Sat}(\pi_v)$ are eigenvalues of $\rho(\mathrm{Frob}_v)$ via ι.*

Remark 9.3. Being a cuspidal regular C-algebraic automorphic representation in the case of $G = \mathrm{GL}_{2,\mathbb{Q}}$ just means that the infinity part is a discrete series of weight $\pm k$ for $k \geq 2$, or more concretely, it comes from a cuspidal newform of weight $k \geq 2$ (or its complex conjugate).

Remark 9.4. The normalizing factor $p^{1/2}$ is somehow expected, as you want to tranform $\mathrm{Sat}(\pi_v)$ into a "weight 1" Weil number.

Remark 9.5. The theorem itself is proven by Eichler-Shimura and Deligne. Local-global compatibility at S is due to Carayol, and at ℓ is due to T. Saito.

We will try to use a very general Langlands-Kottwitz method that is suitable for generalization (not the way originally used by Eichler-Shimura, Deligne). We will still try to realize π-part in cohomology of modular curves.

10. **Langlands-Kottwitz method.**

We would like to prove Theorem 9.1, using Langlands-Kottwitz method. Recall what were modular curves:

Definition 10.1. *Let $N \geq 3$. The modular curve of full level N, M_N, is the scheme over $\mathbb{Z}[1/N]$ representing the functor*

$$(\mathrm{Sch}/\mathbb{Z}[1/N]) \longrightarrow \mathrm{Sets}$$
$$S \mapsto \{(E, \alpha_N) : E \text{ is an elliptic curve over } S, \text{ and } \alpha_N : (\mathbb{Z}/N\mathbb{Z})^2_S \xrightarrow{\sim} E[N]\}.$$

Remark 10.1. Note that the complex points of M_N is not the complex modular curve of full level N one might guess, or rather a disjoint union of such modular curves, i.e.

$$M_N(\mathbb{C}) = \coprod_{(\mathbb{Z}/N\mathbb{Z})^\times} \Gamma(N)\backslash\mathfrak{h},$$

as one connected component is only definable over $\mathbb{Q}(\zeta_N)$. By putting every Galois orbit into one scheme, one can demand that M_N is defined over $\mathbb{Q}$ (or $\mathbb{Z}[1/N]$).

Consider the inverse system

$$M = \varprojlim_N M_N,$$

which is defined over $\mathbb{Q}$, and has a Hecke action by $\mathrm{GL}_2(\mathbb{A}^\infty)$. It also has a compatible system of universal elliptic curve $p : \mathcal{E}^{\mathrm{univ}} \longrightarrow M_N$.

Definition 10.2. *Define*

$$\mathcal{L}_k := \mathrm{Sym}^{k-2}(R^1 p_* \mathcal{E}^{\mathrm{univ}}),$$

and

$$H^1_{c,k,\text{ét}}(M) := \varinjlim_N H^1_{c,\text{ét}}(M_{N,\overline{\mathbb{Q}}}, \mathcal{L}_k).$$

For simplicity one can just think of $k = 2$ case, i.e. $\mathcal{L}_k = \overline{\mathbb{Q}}_\ell$.

As Hecke actions are geometric correspondences defined over $\mathbb{Q}$, the $\mathrm{GL}_2(\mathbb{A}^\infty)$-Hecke action on $H^1_{c,k,\text{ét}}(M)$ commutes with the Galois action $\Gamma_\mathbb{Q}$ on $H^1_{c,k,\text{ét}}(M)$. Thus we have a candidate for the correspondence of Theorem 9.1:

$$\rho_\pi = \mathrm{Hom}_{G(\mathbb{A}^\infty)}((\pi^\infty)^\vee, H^1_{c,k,\text{ét}}(M)).$$

We haven't done anything so far though, ρ_π can be just zero! One can directly show that $\dim \rho_\pi = 2$ using various methods, for example Eichler-Shimura isomorphism, or Matsushima formula. On the other hand, our main problem is to show local-global compatibility.

Proof that ρ_π satisfies the local-global compatibility at $p \notin S \cup \{\ell\}$. We instead consider the inverse system

$$M^p := \varprojlim_{(N,p)=1} M_N,$$

which can be defined over $\mathbb{Z}_{(p)}$, so that we can take mod-p reduction. Then similarly define

$$H^1_{c,k,\text{ét}}(M^p) = \varinjlim_{(N,p)=1} H^1_{c,\text{ét}}(M_{N,\overline{\mathbb{Q}}}, \mathcal{L}_k).$$

One can then check that

$$\rho_\pi \cong \mathrm{Hom}_{G(\mathbb{A}^{\infty,p})}((\pi^{\infty,p})^\vee, H^1_{c,k,\text{ét}}(M^p)),$$

as Galois representations. This is because of $\dim \pi_p^{G(\mathbb{Z}_p)} = 1$ and strong multiplicity one.

Now one remains to compute the action of $\Gamma_\mathbb{Q} \times \mathrm{GL}_2(\mathbb{A}^{\infty,p})$ on $H^1_{c,k,\text{ét}}(M^p)$. As $H^0_{c,k,\text{ét}}$ and $H^2_{c,k,\text{ét}}$ are very easy, we can try to consider the action on the Euler characteristic $H^*_{c,k,\text{ét}}(M^p)$ (just an alternating sum as a virtual representation). Also, by the unramifiedness at p, we want to study the action of $\mathrm{Frob}_p \times \mathrm{GL}_2(\mathbb{A}^{\infty,p})$ on $H^*_{c,k,\text{ét}}(M^p)$. For notational simplicity let us denote Frob_p as Φ from now on.

We explain the outline of the rest of the proof, as we have not enough time to describe everything.

1. **Step 1.** Describe the action of $\Phi \times \mathrm{GL}_2(\mathbb{A}^{\infty,p})$ on $M^p(\overline{\mathbb{F}}_p)$ in terms of "linear algebraic data."
2. **Step 2.** Obtain a trace formula computing the action by applying a suitable fixed point formula.
3. **Step 3.** Massage the formula to make it resemble the Selberg trace formula. In particular, the fixed point formula gives an information about $\mathrm{Frob}_p \times G(\mathbb{A}^{\infty,p})$-action, and the Selberg trace formula gives an information about $G(\mathbb{A}^\infty)$-action.
4. **Step 4.** Compare the massaged fixed point trace formula to the Selberg trace formula. If all goes well, then for all $j \geq 1$,

   $$\mathrm{tr}(f_p^{(j)}|_{\pi_p}) = \mathrm{tr}(\mathrm{Frob}_p^j|_{\rho_\pi}),$$

   where $f_p^{(j)} \in \mathscr{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))$ are explicit functions (base-change transfers of some explicit functions living on degree j unramified extension of $\mathbb{Q}_p$). In this way, the informations on Frob_p-action and $G(\mathbb{A}_p)$-action are "matched," and the local-global compatibility at p is proved.

$\square$

11. **Langlands-Rapoport for modular curves.**

We focus on explaining the first step, namely describing the $\overline{\mathbb{F}}_p$-points of M^p.

Remark 11.1. One can try to describe just $\mathbb{F}_{p^j}$-points as Scholze does in one of the references, but it is more convenient to describe $\overline{\mathbb{F}}_p$-points if one is interested in Hecke action. Kottwitz's approach is a hybrid of the two.

Definition 11.1. *Let*

$$\widehat{\mathbb{Z}}^p = \varprojlim_{(N,p)=1} \mathbb{Z}/N\mathbb{Z},$$

$$\mathscr{E}^0 = \{\textit{isogeny classes of elliptic curves over } \overline{\mathbb{F}}_p\}.$$

For an elliptic curve E over $\overline{\mathbb{F}}_p$, define

$$T^pE = \varprojlim_{(N,p)=1} E[N],$$

$$V^pE = T^pE \otimes_\mathbb{Z} \mathbb{Q}.$$

Then T^pE is a free $\widehat{\mathbb{Z}}^p$-module of rank 2, which is a $\widehat{\mathbb{Z}}^p$-lattice in V^pE.

Definition 11.2. *Let $\check{T}_p E$ be the covariant Dieudonné module of $E[p^\infty]$, where it is a free rank 2 module of $\check{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$, which is equipped with F^{-1}, V^{-1}-actions, with $F^{-1}V^{-1} = V^{-1}F^{-1} = p$. It is a $\check{\mathbb{Z}}_p$-lattice in $\check{V}_p E = \check{T}_p E \otimes_{\mathbb{Z}} \mathbb{Q}$, a $\check{\mathbb{Q}}_p$-vector space of dimension 2, over which F, V are σ-semilinear, where σ is the Frobenius on $\check{\mathbb{Q}}_p$.*

Remark 11.2. The tragedic use of F^{-1}, V^{-1} is because we are trying to use homological convention, so really F^{-1}, V^{-1} are duals of F, V in the contravariant Dieudonné module. One could have just taken étale cohomology and crystalline cohomology which will make notations slightly better.

We now want to describe $M^p(\overline{\mathbb{F}}_p) = \varprojlim_{(N,p)=1} M_N(\overline{\mathbb{F}}_p)$, where we know that it is the same as

$$\{(E, \alpha) \mid E \text{ is elliptic curve over } \overline{\mathbb{F}}_p,\ \alpha : (\widehat{\mathbb{Z}}^p)^2 \xrightarrow{\sim} T^p E\},$$

via the moduli description. Partitioning into isogeny classes, we have

$$\coprod_{E_0 \in \mathscr{E}^0} \{(E, \alpha) \mid \cdots, \text{ there exists an isogeny } f : E \longrightarrow E_0\}$$

$$= \coprod_{E_0 \in \mathscr{E}^0} \mathrm{Aut}^0(E_0) \backslash \{(L^p, \phi^p, L_p) \mid L^p \subset V^p E_0\ \widehat{\mathbb{Z}}^p\text{-lattice},$$

$$\phi^p : (\widehat{\mathbb{Z}}^p)^2 \xrightarrow{\sim} L^p,\ L_p \subset \check{V}_p E_0 \text{ is } F^{-1}, V^{-1} \text{ invariant } \check{\mathbb{Z}}_p\text{-lattice}\}/\cong,$$

where $\mathrm{Aut}^0(E_0)$ is the automorphism group of E_0 in the isogeny category (i.e. group of self-quasiisogenies). The identification is done by the following recipe: given (E, α) with an isogeny $f : E \longrightarrow E_0$, then

$$L^p = f(T^p E),$$
$$\phi^p : (\widehat{\mathbb{Z}}^p)^2 \xrightarrow{\alpha} T^p E \xrightarrow{f} f(T^p E) = L^p,$$
$$L_p = f(\check{T}_p E),$$

and quotienting out by $\mathrm{Aut}^0(E_0)$ is to forget the choice of f. Morally, L^p, L_p remember the information of E, and ϕ^p remembers the information of α.

Then we define $X^p(E_0) = \{(L^p, \phi^p)\}$, which has a Hecke action of $\mathrm{GL}_2(\mathbb{A}^{\infty,p})$; if $g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}^p)$, then the action is just $\phi^p \mapsto \phi^p \circ g$, and if g is not integral, one has to change lattices. This is a consequence of defining $X^p(E_0)$ using lattices; one can equivalently try to define the same set with rationalized data and then the Hecke action only acts on the trivialization.

The p-part $X_p(E_0)$ can be defined as F^{-1}, V^{-1}-invariant lattices in $\check{V}_p E_0$ which has an action of Φ by F. Then we have the formula

$$M^p(\overline{\mathbb{F}}_p) = \coprod_{E_0 \in \mathscr{E}^0} I(E_0) \backslash X^p(E_0) \times X_p(E_0),$$

where $I(E_0) = \mathrm{Aut}^0(E_0)$. The identification is $\Phi^{\mathbb{Z}} \times G(\mathbb{A}^{\infty,p})$-equivariant.

We want more group-theoretic description of X_p and X_p:

- $X^p(E_0)$ is a $G(\mathbb{A}^{\infty,p})$-torsor.
- For $X_p(E_0)$, first fix $\phi_{0,p} : (\check{\mathbb{Z}}_p)^2 \xrightarrow{\sim} \check{T}_p E_0$. Using this, we can alter the description into $\check{X}_p(E_0)$, which consists of $L_p \subset (\check{\mathbb{Q}}_p)^2$ which is F^{-1}, V^{-1}-invariant such that

$$L_p \subset F(L_p) \subset p^{-1} L_p,$$

and has dimension conditions

$$\dim_{\overline{\mathbb{F}}_p}(F(L_p)/L_p) = 1, \dim_{\overline{\mathbb{F}}_p}(p^{-1}L_p/F(L_p)) = 1.$$

Rewriting $F = b\sigma$ for $b \in G(\check{\mathbb{Q}}_p)$ and $L_p = g_p(\check{\mathbb{Z}}_p^2)$ for $g_p \in G(\check{\mathbb{Q}}_p)/G(\check{\mathbb{Z}}_p)$, the dimension condition (or condition on relative position) becomes

$$g_p^{-1} b\sigma(g_p) \in \check{K}_p \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \check{K}_p,$$

where $\check{K}_p = G(\check{\mathbb{Z}}_p)$.

1. p-adic groups.

For simplicity, we set $G = \mathrm{GL}_n(\mathbb{Q}_p)$. Denote $K = \mathrm{GL}_n(\mathbb{Z}_p)$, and it has basis $K(r) = 1 + p^r M_n(\mathbb{Z}_p)$ of compact open subgroups. This tells us that G is totally disconnected.

Remark 1.1. If H is a topological group, any open subgroup is closed, and any closed subgroup of finite index is open.

Definition 1.1. *A **profinite group** (resp. a pro-p group) is a topological group that is compact Hausdorff and has a fundamental system of neighborhoods of 1 consisting of open normal subgroup of finite (resp. p-power) index.*

Example 1.1. K is profinite, and $K(r)$ is pro-p; $K(r)/K(r+1) \cong M_n(\mathbb{F}_p)$. This shows that G has no $\overline{\mathbb{F}}_p$-valued Haar measure, as we cannot divide by p.

Consider the standard notations for subgroups of G, B, T, U, P, M, N, $\overline{P}$, $\overline{N}$.

Proposition 1.1 (Iwasawa decomposition). *$G = BK$, which implies $G = PK$.*

Proposition 1.2 (Cartan decomposition). $G = \coprod_{a_1 \geq \cdots \geq a_n \text{ integers}} K \begin{pmatrix} p^{a_1} & & \\ & \cdots & \\ & & p^{a_n} \end{pmatrix} K.$

Definition 1.2. *For H a Hausdorff topological group having a fundamental system of neighborhoods consisting of compact open subgroups, and for C any field, a representation π of H over C is **smooth** if any of the following equivalent conditions is satisfied.*

(1) For all $x \in \pi$, stabilizer of x is open in H.
(2) $\pi = \bigcup_{U \text{ compact open}} \pi^U$.
(3) The action map $H \times \pi \longrightarrow \pi$ is continuous, where π carries the discrete topology.

A map of smooth H-representations is any H-linear map between them. These form an abelian category of smooth representations of H.

Example 1.2. For $H = \mathbb{Q}_p^\times$, giving a smooth character $\chi : \mathbb{Q}_p^\times \longrightarrow C^\times$ is the same as giving $\chi(p) \in C^\times$ and a character $\mathbb{Z}_p^\times/(1 + p^r\mathbb{Z}_p^\times) \longrightarrow C^\times$ for some r.

Definition 1.3. *If $H' \subset H$ is a closed subgroup and σ is a smooth representation of H', then*

$\mathrm{Ind}_{H'}^H(\sigma) = \{f : H \longrightarrow \sigma \mid f(hg) = hf(g), \text{ and there is compact open } U \text{ such that } f(ga) = f(g) \text{ for } a \in U\}.$

The G-action is defined by $(gf)(h) = f(hg)$, and this is a smooth G-representation.

Remark 1.2. For $f \in \mathrm{Ind}_{H'}^H(\sigma)$, the support of f in $H' \backslash H$ is open and closed.

Definition 1.4. *We define the **compact induction** by*

$$\mathrm{ind}_{H'}^H \sigma = \{f \in \mathrm{Ind}_{H'}^H \sigma \mid \mathrm{supp}(f) \text{ is compact}\}.$$

It is a subrepresentation of $\mathrm{Ind}_{H'}^H \sigma$.

Proposition 1.3 (Frobenius reciprocity). *(1) For a smooth representation π of H and σ is a smooth representation of $H' \leq H$,*

$$\mathrm{Hom}_H(\pi, \mathrm{Ind}_{H'}^H \sigma) = \mathrm{Hom}_{H'}(\pi|_{H'}, \sigma).$$

(2) *If H' is furthermore open, then*

$$\mathrm{Hom}_H(\mathrm{ind}_{H'}^H \sigma, \pi) = \mathrm{Hom}_{H'}(\sigma, \pi|_{H'}).$$

The functor $\mathrm{ind}_{H'}^H$ *is exact.*

Proofs are straightforward, you "evaluate at 1."

Proposition 1.4. *If P is a parabolic subgroup of G, Ind_P^G is exact.*

Proof. There is a continuous section $P\backslash G \longrightarrow G$, which comes from the direct product decomposition $G = P\overline{N}$. Then, $\mathrm{Ind}_P^G(\sigma) = C^\infty(P\backslash G, \sigma)$. □

Definition 1.5. *If σ is a smooth M-representation, we inflate it to a smooth P-representation, and $\mathrm{Ind}_P^G(\sigma)$ is called* ***parabolic induction****. The operation is transitive.*

2. mod p representation of $\mathrm{GL}_n(\mathbb{Q}_p)$.

Now let's suppose $C = \overline{C}$, and char $C = p$.

Lemma 2.1 ("p-group lemma")**.** *Any smooth representation τ of a pro-p group H has a fixed vector.*

Proof. WLOG $C = \mathbb{F}_p$. Pick a nonzero vector x. Then, there is an open normal subgroup $U \le H$ such that x is fixed by U. Thus, H/U is a finite p-group acting on τ^U. Now it is a standard problem in the first course in group theory. □

Corollary 2.1. (1) *A smooth nonzero G-representation has a $K(1)$-fixed vector.*
(2) *Any irreducible smooth K-representation V is trivial on $K(1)$. Thus, irreducible smooth K-representations are in bijection with irreducible $\mathrm{GL}_n(\mathbb{F}_p) =: G(\mathbb{F}_p)$-representations.*

Definition 2.1. *An irreducible smooth K-representaiton is called a* ***weight****.*

Corollary 2.2. *Any nonzero smooth G-representation π contains a weight V, i.e. $V \subset \pi|_K$.*

Example 2.1. If $n = 2$, the weights are

$$V_{a,b} = \mathrm{Sym}^{a-b} C^2 \otimes \det{}^b,$$

where $(a, b) \in \mathbb{Z}^2$ with $0 \le a - b \le p - 1$ and $0 \le b < p - 1$. One can see $\mathrm{Sym}^{a-b} C^2$ as a module of homogeneous polynomials.

Theorem 2.1. *For a weight V of G, the induced map*

$$V^{\overline{N}(\mathbb{F}_p)} \longrightarrow V \longrightarrow V_{N(\mathbb{F}_p)},$$

is an isomorphism of irreducible representations of $M(\mathbb{F}_p)$. In particular, if $P = B$, $V^{\overline{U}(\mathbb{F}_p)} \cong V_{U(\mathbb{F}_p)}$ is one-dimensional.

Theorem 2.2 (Curtis)**.** *We have a bijection*

$$\{\textit{Weights of } G\} = \left\{ \begin{array}{c} \textit{Pairs } (\psi, P) \textit{ such that } \psi : T(\mathbb{F}_p) \longrightarrow C^\times \\ \textit{is a charcater and } P \textit{ is a parabolic such} \\ \textit{that } \psi \textit{ can be extended to } P(\mathbb{F}_p) \end{array} \right\}.$$

The map $V \longmapsto (\psi_V, P_V)$ such that $\psi_V = V_{U(\mathbb{F}_p)}$ and P_V is the largest $P = MN$ such that $V_{N(\mathbb{F}_p)}$ is 1-dimensional.

Example 2.2. The vector $Y^{a-b} \in V_{a,b}$ is $\overline{U}(\mathbb{F}_p)$-stable, so that $\psi_{V_{a,b}} : T_{\mathbb{F}_p} \longrightarrow C^\times$ is given by $\operatorname{diag}(x, y) \mapsto x^b y^a$. On the other hand, $P_{V_{a,b}} = \begin{cases} G & \text{if } a = b \\ \mathrm{B} & \text{otherwise} \end{cases}$.

Remark 2.1. There is a Steinberg parametrization of weights, using representations of algebraic group $\mathrm{GL}_{n,\mathbb{F}_p}$.

3. **Hecke algebras.**

If π is an irreducible smooth G-representation, then $V \subset \pi|_K$ gives $\operatorname{ind}_K^G V \twoheadrightarrow \pi$.

Definition 3.1. *The Hecke algebra of weight V is $\mathscr{H}_G(V) = \operatorname{End}_G(\operatorname{ind}_K^G V)$.*

Lemma 3.1. *The Hecke algebra is identified with*

$$\mathscr{H}_G(V) \cong \{\varphi : G \longrightarrow \operatorname{End}_C(V) \mid \operatorname{supp} \varphi \textit{ is compact in } G,\ \varphi(k_1 g k_2) = k_1 \circ \varphi(g) \circ k_2\},$$

where the RHS is a ring under convolution product, namely

$$(\varphi_1 * \varphi_2)(g) = \sum_{\gamma \in G/K} \varphi_1(g\gamma)\varphi_2(\gamma^{-1}),$$

which is a finite sum because of the compact support condition.

At least one can construct an element of the RHS from an element of $\mathscr{H}_G(V)$ from Frobenius reciprocity for compact induction.

Remark 3.1. If π is a smooth G-representation, then $\operatorname{Hom}_K(V, \pi|_K) \cong \operatorname{Hom}_G(\operatorname{ind}_K^G V, \pi)$, which admits a natural right action by $\mathscr{H}_G(V)$. Explicitly, if $f : V \longrightarrow \pi|_K$ and $\varphi \in \mathscr{H}_G(V)$, then $(f * \varphi)(x) = \sum_{g \in K\backslash G} g^{-1} f(\varphi(g)x)$.

Example 3.1. If $V = \mathbb{1}$, then $\mathscr{H}_G(V) = C_c(K\backslash G/K, C)$. It acts on $\operatorname{Hom}_K(\mathbb{1}, \pi|_K) = \pi^K$ in the usual way: $1_{KgK} : \pi^K \longrightarrow \pi^K$ sends x to $\sum_i g_i^{-1} x$, where $KgK = \coprod_i K g_i$.

4. **mod p Satake isomorphism.**

Our goal now is to construct an injective algebraic homomorphism $\mathscr{H}_G(V) \hookrightarrow \mathscr{H}_T(V_{U(\mathbb{F}_p)})$, and determine its image. More generally, this will become $\mathscr{H}_G(V) \hookrightarrow \mathscr{H}_M(V_{N(\mathbb{F}_p)})$.

Lemma 4.1. *There is a natural isomorphism*

$$\operatorname{Hom}_G(\operatorname{ind}_K^G V, \operatorname{Ind}_P^G(-)) \cong \operatorname{Hom}_M(\operatorname{ind}_{M\cap K}^M(V_{N(\mathbb{F}_p)}), -),$$
$$f \longmapsto f_M,$$

as functors

$$\{\textit{Smooth } M\textit{-representations}\} \longrightarrow \{C\textit{-vector spaces}\}.$$

Proof. By Frobenius reciprocity,

$\operatorname{Hom}_G(\operatorname{ind}_K^G V, \operatorname{Ind}_P^G(-)) = \operatorname{Hom}_K(V, \operatorname{Ind}_P^G(-)|_K) = \operatorname{Hom}_K(V, \operatorname{Ind}_P^G(-|_{P\cap K})) = \operatorname{Hom}_K(V, \operatorname{Ind}_{P\cap K}^K(-|_{P\cap K}))$,

by Iwasawa decomposition $PK = G$, and again by Frobenius reciprocity, this is equal to

$$\operatorname{Hom}_{P\cap K}(V|_{P\cap K}, -|_{P\cap K}).$$

As $N \cap K$ acts trivially on $(-)|_{P\cap K}$, this is equal to

$$\operatorname{Hom}_{M\cap K}(V_{N\cap K}, -|_{M\cap K}) = \operatorname{Hom}_M(\operatorname{ind}_{M\cap K}^M V_{N(\mathbb{F}_p)}, -).$$

$\square$

From this, by Yoneda's lemma, any $\varphi \in \mathscr{H}_G(V)$, which gives a natural transformation of $\mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{Ind}_P^G(-))$, induces a natural transformation on $\mathrm{Hom}_M(\mathrm{ind}_{M\cap K}^M(V_{N(\mathbb{F}_p)}), -)$, and induces a unique M-endomorphism $S_M^G(\varphi)$ of $\mathrm{ind}_{M\cap K}^M(V_{N(\mathbb{F}_p)}) = \mathscr{H}_M(V_{N(\mathbb{F}_p)})$, such that $(f\circ\varphi)_M = f_M \circ S_M^G(\varphi)$. This yields a C-algebra homomorphism

$$\mathcal{S}_M^G : \mathscr{H}_G(V) \longrightarrow \mathscr{H}_M(V_{N(\mathbb{F}_p)}),$$

the **mod p Satake homomorphism**.

Proposition 4.1. *Explicitly, we have*

$$\mathcal{S}_M^G(\varphi) : M \longrightarrow \mathrm{End}_C(V_{N(\mathbb{F}_p)}),$$
$$m \longmapsto \sum_{n\in N\cap K\backslash N} p_N \circ \varphi(nm),$$

where $p_N : V \twoheadrightarrow V_{N(\mathbb{F}_p)}$.

A priori what we have written down is a map $V \longrightarrow V_{N(\mathbb{F}_p)}$, but it factors through p_N so that it yields an endomorphism of $V_{N(\mathbb{F}_p)}$.

Proof. Take f in the Yoneda lemma construction so that $f_M = \mathrm{id}$ of $\mathrm{ind}_{M\cap K}^M V_{N(\mathbb{F}_p)}$. $\square$

Definition 4.1. *Let $T^+ = \{\mathrm{diag}(t_1, \cdots, t_n) \mid \mathrm{val}(t_1) \geq \cdots \geq \mathrm{val}(t_n)\}$, and $\mathscr{H}_T^+(V_{U(\mathbb{F}_p)}) = \{\psi \in \mathscr{H}_T(V_{U(\mathbb{F}_p)}) \mid \mathrm{supp}\, \psi \subset T^+\}$.*

Theorem 4.1 (Herzig, Henniart, Vigneras, $\cdots$). *The map*

$$\mathcal{S}_T^G : \mathscr{H}_G(V) \longrightarrow \mathscr{H}_T(V_{U(\mathbb{F}_p)}),$$

is injective with image $\mathscr{H}_T^+(V_{U(\mathbb{F}_p)})$.

Corollary 4.1.

$$\mathscr{H}_G(V) \cong C[\Lambda^+],$$

where $\Lambda^+ = T^+/(T\cap K)$ is a monoid. Through valuation, $\Lambda^+ \cong \mathbb{Z}_+^n = \{(\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}$, so that $\mathscr{H}_G(V)$ is commutative finite type C-algebra.

Remarks on the proof.

1. Find nice bases of $\mathscr{H}_G, \mathscr{H}_T$: for $\lambda \in \mathbb{Z}_+^n$, let $t_\lambda = \mathrm{diag}(p^{\lambda_1}, \cdots, p^{\lambda_n}) \in T^+$. Then, there is a unique $T_\lambda \in \mathscr{H}_G$, such that
   (a) $\mathrm{supp}\, T_\lambda = K t_\lambda K$,
   (b) $T_\lambda(t_\lambda) \in \mathrm{End}_C(V)$ is a linear projection.
   We can see the uniqueness as follows. The reduction of $K \cap t_\lambda^{-1} K t_\lambda$ is $P_\lambda(\mathbb{F}_p)$ for a standard parabolic P_λ, and note that $T_\lambda(t_\lambda)$ by K-biequivariance has to factor $V \twoheadrightarrow V_{N_\lambda(\mathbb{F}_p)} \dashrightarrow V^{\overline{N}_\lambda(\mathbb{F}_p)} \hookrightarrow V$ which is $M_\lambda(\mathbb{F}_p)$-linear. Thus, as $V_{N_\lambda(\mathbb{F}_p)}$ and $V^{\overline{N}(\mathbb{F}_p)}$ are isomorphic, we get uniqueness.
   By Cartan decomposition, we deduce that $(T_\lambda)_{\lambda\in\mathbb{Z}_+^n}$ forms a C-basis of $\mathscr{H}_G$. Similarly, one takes a basis $(\tau_\lambda)_{\lambda\in\mathbb{Z}^n}$ of $\mathscr{H}_T$ (normalized).
2. One proves that $\mathcal{S}_T^G(T_\lambda) = \tau_\lambda + \sum_{\mu<\lambda} a_\mu \tau_\mu$.
3. $\mathrm{im}(\mathcal{S}_T^G) \subset \mathscr{H}_T^+$.
4. Triangular argument.

$\square$

Remark 4.1. By the same formalism, one has a Satake transform for maps of compact inductions of two **different weights**, which no longer has algebra structure but is a Hecke bimodule.

Proof of Corollary 4.1. As T is commutative, $\tau_\lambda \tau_\mu = \tau_{\lambda\mu}$, so $\mathscr{H}_T^+ \cong C[\mathbb{Z}_+^n] \cong C[x_1, \cdots, x_n, x_n^{-1}]$. □

Proposition 4.2. *We have a commutative diagram*

$$\begin{tikzcd} \mathscr{H}_G(V) \arrow[r, "S_M^G"] \arrow[dr, hook, "S_T^G"'] & \mathscr{H}_M(V_{N(\mathbb{F}_p)}) \arrow[d, hook, "S_T^M"] \\ & \mathscr{H}_T(V_{U(\mathbb{F}_p)}) \end{tikzcd}$$

which implies that S_M^G is injective. Moreover, there is $\varphi \in \mathscr{H}_G(V)$ such that $\mathscr{H}_G(V)[\varphi^{-1}] \xrightarrow{\sim} \mathscr{H}_M(V_{N(\mathbb{F}_p)})$.

Proof. We only prove the last part. Note that $\mathrm{im}(S_T^G) = C[\Lambda^+]$ and $\mathrm{im}(S_T^M) \cong C[\Lambda^{+,M}]$, where $\Lambda^{+,M}$ is the dominance with respect to M. Thus, S_M^G is identified with the inclusion

$$C[\Lambda^+] \hookrightarrow C[\Lambda^{+,M}],$$

and one only needs to localize at $\lambda \in \Lambda^+ \cong \mathbb{Z}_+^n$ such that $\lambda_1 = \cdots = \lambda_{n_1} > \lambda_{n_1+1} = \cdots = \lambda_{n_2} > \lambda_{n_2+1} = \cdots$. □

5. Admissible representations and supersingular representations.

Definition 5.1. *A smooth G-representation is admissible if $\dim_C \pi^W < \infty$ for all compact open subgroups W.*

Remark 5.1. The notion is stable under taking subrepresentations (obvious) and quotients (not obvious).

Lemma 5.1. *A smooth G-representation π is admissible if and only if there is $W \leq G$ a compact open pro-p subgroup such that $\dim_C \pi^W < \infty$.*

Proof. Suppose W' is any compact open subgroup. One may shrink W' so that WLOG $W' \subset W$. Then,

$$\pi^{W'} = \mathrm{Hom}_{W'}(\mathbb{1}, \pi) \cong \mathrm{Hom}_W(\mathrm{ind}_{W'}^W \mathbb{1}, \pi).$$

It is then sufficient to prove the following, as $\mathrm{ind}_{W'}^W \mathbb{1}$ is finite-dimensional:

Claim. $\mathrm{Hom}_W(\sigma, \pi)$ is finite-dimensional, for every finite dimensional smooth σ.

If σ is irreducible, then by p-group lemma, $\sigma = \mathbb{1}$, so the finite-dimensionality is exactly our assumption. A general situation follows from this by devissage. □

Lemma 5.2. *If π is admissible, then it contains an irreducible subrepresentation.*

Proof. Fix W an open pro-p subgroup. For all $0 \neq \tau \subset \pi$, $0 \neq \tau^W \subset \pi^W$ by p-group lemma. We can choose τ such that $\dim(\tau^W)$ is minimal. Then, the G-representation generated by τ^W is irreducible. □

Exercise 5.1. (1) If π is smooth, then π is admissible if and only if $\mathrm{Hom}_K(V, \pi)$ is finite-dimensional for all weights V.
(2) If π is irreducible and admissible, then π has a central character.
(3) Show that $\mathrm{Ind}_P^G(-)$ preserves admissibility.

Remark 5.2. It can be possible that an irreducible smooth representation is **not smooth** (Daniel Le, 2018).

Definition 5.2. *If π is an admissible G-representation, and if V is a weight, then $\mathrm{Hom}_K(V,\pi)$ is a finite-dimensional vector space with an action of a commutative algebra $\mathscr{H}_G(V)$. Then we define*

$$\mathrm{Eval}_G(V,\pi) = \{\textit{algebra homomorphisms } \mathscr{H}_G(V) \longrightarrow C \textit{ that occurs as eigenvalues on } \mathrm{Hom}_K(V,\pi)\}.$$

Recall $\mathscr{H}_T^+$ has basis $\{\tau_\lambda\}_{\lambda\in\mathbb{Z}_+^n}$. Note that $\tau_\lambda \in (\mathscr{H}_T^+)^\times$ if and only if $\lambda \in \mathbb{Z}_0^n := \mathbb{Z}_+^n \cap (-\mathbb{Z}_+^n)$.

Lemma 5.3. *For an irreducible admissible G-representation π and V a weight, the following are equivalent.*

1. *(1) For all $\chi \in \mathrm{Eval}_G(V,\pi)$, $\chi(\tau_\lambda) = 0$ for all $\lambda \in \mathbb{Z}_+^n \backslash \mathbb{Z}_0^n$.*
2. *(2) For all $\chi \in \mathrm{Eval}_G(V,\pi)$, χ does not factor through $\mathcal{S}_M^G : \mathscr{H}_G(V) \longrightarrow \mathscr{H}_M(V_{N(\mathbb{F}_p)})$ for all $M \neq G$.*

Proof. Use that $\mathscr{H}_G[\tau_\lambda^{-1}] \cong \mathscr{H}_{M_\lambda}$, where M_λ is the centralizer of t_λ. □

Definition 5.3 (Barthel-Livné, Breuil, $\cdots$)**.** *If the above equivalent conditions are satisfied, then we call π a **supersingular representation**.*

Remark 5.3. (1) This condition is really about some Hecke operator being zero mod p.
(2) From this, one has to check for $n-1$ Hecke operators. From classification of irreducible admissible representations, it will turn out that it is sufficient to check for only one weight and one Hecke operator.
(3) There is also an equivalent condition that uses Iwahori-Hecke algebra.

When $n = 1$, everything is supersingular. When $n = 2$, Breuil showed the following

Theorem 5.1 (Breuil)**.** *Irreducible supersingular representaitons of $\mathrm{GL}_2(\mathbb{Q}_p)$ are of form*

$$\frac{\mathrm{ind}_K^G V}{(\tau_{(1,0)}, \tau_{(1,1)} - \alpha)\,\mathrm{ind}_K^G V},$$

for $\alpha \in C^\times$ and weight V.

It makes sense, α specifies the central character and $\tau_{(1,0)}$-eigenvalue is killed.

6. Classification in terms of supersingular representations.

Definition 6.1 (Steinberg representations)**.** *If Q is a standard parabolic subgroup, then*

$$\mathrm{St}_Q := \frac{\mathrm{Ind}_Q^G(\mathbb{1})}{\sum_{Q'\supsetneq Q \text{ parabolic}} \mathrm{Ind}_{Q'}^G(\mathbb{1})}$$

*is a **generalized Steinberg representation**.*

It is the usual Steinberg representation if $Q = B$ and is trivial is $Q = G$.

Theorem 6.1 (Grosse-Klönne, Herzig, Ly)**.** *The representations St_Q are irreducible admissible and pairwise non-isomorphic. The irreducible constituents of $\mathrm{Ind}_Q^G(\mathbb{1})$ are the $\mathrm{St}_{Q'}$, $Q' \supset Q$, with multiplicity 1.*

Proposition 6.1. *Suppose σ is an (irreducible/admissible/smooth) M-representation, then there exists a unique largest parabolic $P(\sigma)$ containing P such that σ, considered as a P-representation, extends to a $P(\sigma)$-representation $\widetilde{\sigma}$ which is unique (and is irreducible/admissible/smooth).*

The extension $\tilde{\sigma}$ is automatically trivial on $N(\sigma) \subset N$.

Example 6.1. Say $M = \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix} \subset \mathrm{GL}_3$, and $\sigma = \tau \boxtimes \chi$ where τ and χ are irreducible admissible representations of GL_2 and GL_1, respectively. If $P(\sigma) = G$, then $\tilde{\sigma}$ is trivial on the normal subgroup generated by $N = \begin{pmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{pmatrix}$, which contains $N(\mathbb{Q}_p)$ and $\overline{N}(\mathbb{Q}_p) = \mathrm{SL}_3(\mathbb{Q}_p)$. In general, it is the image of simply connected universal cover of the derived subgroup. To have $P(\sigma) = G$, then necessarily $\tau = \chi \circ \det$ (so that $\tilde{\sigma} = \chi \circ \det$).

Definition 6.2. *Suppose* (P, σ, Q) *consists of the following data:*

- *a standard parabolic* $P = MN$*,*
- *an irreducible admissible supersingular* M*-representation* σ*,*
- *and a parabolic* $P \subset Q \subset P(\sigma)$*.*

Then,

$$I(P, \sigma, Q) := \mathrm{Ind}_{P(\sigma)}^{G}(\tilde{\sigma} \otimes \mathrm{St}_Q^{P(\sigma)}),$$

where

$$\mathrm{St}_Q^{P(\sigma)} = \frac{\mathrm{Ind}_Q^{P(\sigma)}(\mathbb{1})}{\sum_{Q \subsetneq Q' \subset P(\sigma)} \mathrm{Ind}_{Q'}^{P(\sigma)}(\mathbb{1})}.$$

Remark 6.1. As $N \trianglelefteq P$ and $N \leq Q$, N acts trivially on St_Q^P, and

$$\mathrm{St}_Q^P|_M \cong \frac{\mathrm{Ind}_{Q \cap M}^{M}(\mathbb{1})}{\sum_{Q \subsetneq Q' \subset P} \mathrm{Ind}_{Q' \cap M}^{P \cap M}(\mathbb{1})},$$

which is a general Steinberg representation of M. In particular, $\tilde{\sigma} \otimes \mathrm{St}_Q^{P(\sigma)}$ is trivial on $N(\sigma)$.

Theorem 6.2 (Abe-Henniart-Herzig-Vigneras)**.** *The map*

$$\left\{ \begin{array}{c} \textit{Triples } (P, \sigma, Q) \textit{ as} \\ \textit{above} \end{array} \right\} / \cong \longrightarrow \left\{ \begin{array}{c} \textit{Irreducible} \\ \textit{admissible} \\ G\textit{-representations} \end{array} \right\} / \cong,$$

$$(P, \sigma, Q) \mapsto I(P, \sigma, Q),$$

is a bijection. Here, an isomorphism between two (P, σ, Q)*'s is just an isomorphism of* σ*'s.*

The above theorem works for any p-adic reductive group over any p-adic local field (equal or mixed characteristic).

Idea of proof. In showing that $\mathrm{Ind}_P^G(\sigma)$ is irreducible, one needs to show that a nonzero subrepresentation $\tau \subset \mathrm{Ind}_P^G(\sigma)$ is everything. Pick a weight $V \hookrightarrow \tau|_K$. By Frobenius reciprocity, $\mathrm{ind}_K^G(V) \longrightarrow \tau \hookrightarrow \mathrm{Ind}_P^G \sigma$. For some $\chi : \mathscr{H}_G(V) \longrightarrow C$, one then considers $C \otimes_{\mathscr{H}_G(V), \chi} \mathrm{ind}_K^G V \longrightarrow \mathrm{Ind}_P^G \sigma$. If $P_V \subset P$, where P_V is the parabolic coming from Curtis parametrization of weights, then one proves that $C \otimes_{\mathscr{H}_G(V), \chi} \mathrm{ind}_K^G V \cong \mathrm{Ind}_P^G(C \otimes_{\mathscr{H}_M} \mathrm{ind}_{M \cap K}^M V_{N(\mathbb{F}_p)})$ and that it surjects into $\mathrm{Ind}_P^G \sigma$, which implies that τ is everything.

However, even though almost every weight V has $P_V = B$, there are certain weights V where P_V is not contained in P. Thus one needs to "change weights", according to σ.

To show exhaustion one needs to use the "left adjoint" to Ind_P^G, used by Emerton ("ordinary parts" paper). $\square$

Concretely, the recipe on $\mathrm{GL}_n(\mathbb{Q}_p)$ is as follows. Suppose P has blocks of size $n_1, \cdots, n_r$, and let $\sigma = \sigma_1 \boxtimes \cdots \boxtimes \sigma_r$, where σ_i is an irreducible admissible supersingular representation of $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$. Then, $P(\sigma)$ is the standard parabolic subgroup which is formed after combining consecutive 1-blocks with the same characters. Thus, any irreducible admissible representation is of the form $\mathrm{Ind}_{P'}^G(\tau)$ where $\tau = \tau_1 \boxtimes \cdots \boxtimes \tau_s$, and each τ_i is either supersingular or $\tau_i \cong \mathrm{St}_{Q_i}^{\mathrm{GL}_{n_i'}} \otimes (\eta_i \circ \det)$ where $\eta_i : \mathbb{Q}_p^\times \longrightarrow C^\times$'s are different characters.

Remark 6.2. One should not confuse with the complex case where one can change the order of blocks. In this case permuting blocks changes everything.

Example 6.2. Let's say $n = 2$. Then we have four cases.

- $P = B, \sigma = \chi_1 \boxtimes \chi_2, Q = B$ for $\chi_1 \neq \chi_2$. Then, $P(\sigma) = B$, and we get $\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)$.
- $P = B, \sigma = \chi \boxtimes \chi, Q = B$. Then, we get $\mathrm{St} \otimes (\chi \circ \det)$.
- $P = B, \sigma = \chi \boxtimes \chi, Q = G$. Then, we get $\chi \circ \det$.
- $P = G, Q = G$, and σ is a supersingular representation, then we get supersingular representations.

7. **Consequences of classification.**

We can compute Jordan-Holder constituents of parabolically induced representations.

Lemma 7.1. *If σ is an irreducible admissible supersingular M-representation, then $\mathrm{Ind}_P^G(\sigma)$ is of finite length. Its irreducible constituents are $I(P, \sigma, Q)$, where $P \subset Q \subset P(\sigma)$, with multiplicity one.*

Definition 7.1. *An irreducible admissible G-representation π is **supercuspidal** if it is not a subquotient of $\mathrm{Ind}_P^G \sigma$ for all $P \neq G$ and σ irreducible admissible.*

Corollary 7.1. *If π is irreducible admissible, then π is supersingular if and only if π is supercuspidal.*

Proof. If π is supercuspidal, by Theorem 6.2, $\pi = I(P, \sigma, Q)$. By the above lemma, π is a subquotient of $\mathrm{Ind}_P^G(\sigma)$. By the supercuspidality, $P = G = Q$, and $\pi = \sigma$.

If π is supersingular and if it occurse in $\mathrm{Ind}_Q^G(\tau)$ for an irreducible admissible representation τ of Q, then the lemma for τ plus the transitivity implies that π occurs in $\mathrm{Ind}_P^G(\sigma)$ for $P \subset Q$ and σ supersingular. Thus, $\pi \cong I(P, \sigma, Q')$ for some Q'. By Theorem 6.2, $P = Q = G$. $\square$

8. p-**adic functional analysis.**

Now we switch gears to p-adic representations. Let $E/\mathbb{Q}_p$ be a finite extension, which will be our coefficient. Let V be an E-vector space, and $\mathcal{O} = \mathcal{O}_E$.

A good reference is Schneider's book.

Definition 8.1. *A nonarchimedean **seminorm** is a function $|-| : V \longrightarrow \mathbb{R}_{\geq 0}$ such that $|x + y| \leq \max(|x|, |y|)$, and $|\lambda x| = |\lambda|_E |x|$ for all $\lambda \in E$, $x, y \in V$. It is called a **norm** if $|x| = 0$ if and only if $x = 0$.*

Definition 8.2. *A **lattice** in V is an $\mathcal{O}$-submodule $\Lambda \subset V$ that spans V as E-vectorspace.*

Notice that this definition is weaker than one might expect, e.g. V is a lattice of V.

Definition 8.3. *A **locally convex vector space**, or **lcv**, is a vector space V equipped with a topology defined by seminorms $\{|-|_i\}_{i\in I}$. Namely, basic opens can be chosen as*

$$\{x_0 + \{|x|_{i_1} \le \varepsilon, \cdots, |x|_{i_n} \le \varepsilon\} \mid i_j \in I, \varepsilon > 0, x_0 \in V\}.$$

Equivalently, its topology is given by a basis of form $x_0 + \Lambda_j$, $j \in J$, where the Λ_j are a family of lattices such that

1. *for all $\alpha \in E^\times$ and $j \in J$, there is $k \in J$ such that $\alpha\Lambda_j \supset \Lambda_k$,*
2. *for all $i, j \in J$, $\Lambda_i \cap \Lambda_j \supset \Lambda_k$ for some $k \in J$.*

The equivalence is seen as follows: if $|-|$ is a seminorm, then $\{|x| \le \varepsilon\}$ is a lattice; on the other hand, if Λ is a lattice, then $|x|_\Lambda := \inf_{x\in\lambda\Lambda} |\lambda|_E$ is a seminorm.

Remark 8.1. Every lcv will be Hausdorff in this course, i.e. $\bigcap_{\Lambda \text{ open lattice}} \Lambda = \{0\}$.

Example 8.1. If V is an lcv and if $W \subset V$ is a subspace, then the subspace topology on W and the quotient topology on V/W will give lcv's.

Remark 8.2. If $W \subset V$ is closed, then V/W is Hausdorff.

Example 8.2. If $\{V_i\}_{i\in I}$ is a family of lcv's, then so is $\prod_{i\in I} V_i$, with the product topology. Similarly, $\varprojlim_i V_i$ is an lcv. On $\bigoplus_{i\in I} V_i$, take the finest locally convex topology such that each $V_j \longrightarrow \bigoplus_{i\in I} V_i$ is continuous. Similarly, $\varinjlim_i V_i$.

Example 8.3. If V is an lcv, then so is its strong dual $V_b' := \mathrm{Hom}_E^{\mathrm{cts}}(V, E)$ with topology defined by lattices $\{f : |f(B)| \le \varepsilon\}$ for all bounded B and $\varepsilon > 0$. Here, $B \subset V$ is bounded if, for all $\Lambda \subset V$ is an open lattice, there is $\alpha \in E$ such that $B \subset \alpha\Lambda$.

Definition 8.4. *An lcv V is **Banach** (resp. **Fréchet**) if its topology can be defined by a single norm (resp.a countable family of seminorms/lattices) such that it is (sequentially) complete.*

Remark 8.3. There is an implication

$$\text{Banach} \Longrightarrow \text{Fréchet} \Longrightarrow \text{metrisable}.$$

Remark 8.4. A Banach space does not carry a fixed norm; one just remembers topology.

Proposition 8.1. *A finite-dimensional vector space carries a unique Hausdorff locally convex topology. If $V = E^n$, one can define it by $\|a\| := \max_{1\le i\le n} |a_i|$.*

Example 8.4. If I is a set, then

$$\ell^\infty(I) = \{\text{bounded functions } f : I \longrightarrow E\}, \text{ with supremum norm,}$$

$$c_0(I) = \{f \mid \forall\varepsilon > 0, \{|f| > \varepsilon\} \text{ is finite}\},$$

are Banach spaces.

If X is a compact topological space, then $C^0(X, E)$ with supremum norm is Banach.

Remark 8.5. For Fréchet spaces, one has Open Mapping Theorem and Closed Graph Theorem as usual.

Definition 8.5. *A map $f : V \longrightarrow W$ of Banach spaces is **compact** if for any/some unit ball $V^\circ \subset V$, $\overline{f(V^\circ)}$ is compact.*

Definition 8.6. *An lcv V is **of compact type** if $V \cong \varinjlim_{n\ge1} V_n$ where V_n is Banach and $V_n \longrightarrow V_{n+1}$ are injective and compact.*

Example 8.5. If $\dim V$ is countable, equip it with the finest locally convex topology, and it becomes of compact type. This means that $V = \bigcup_{n\geq 1} V_n$, where $V_1 \subset V_2 \subset \cdots$ are finite-dimensional vector spaces, and V equipped with the direct limit topology $\varinjlim V_n$ is of compact type.

Proposition 8.2.
- *If V is of compact type and W is a closed subspace, then W and V/W are compact type.*
- *The strong dual induces equivalence of categories*

$$\{\textit{Compact-type spaces}\} \Longleftrightarrow \{\textit{“Nuclear” Fréchet spaces}\},$$

$$\varinjlim_n V_n \mapsto \varprojlim_n (V_n)'_b.$$

9. **Locally analytic and Banach representations.**

Definition 9.1. *If $a \in \mathbb{Q}_p^d$ and $r > 0$, then*

$$B_r(a) = \{x \in \mathbb{Q}_p^d \mid \|x - a\| \leq r\},$$

is called a ***closed ball*** *(even though it is both open and closed).*

Definition 9.2. *A ($\mathbb{Q}_p$–)**locally analytic manifold** of dimension d is a paracompact Hausdorff topological space which is locally modeled upon closed balls $B_r(a)$ and transition maps are locally analytic. To be more precise, it is a paracompact Hausdorff topological space M and a maximal atlas of charts (U, φ_U), where*

- *$U \subset M$ is an open subset,*
- *all U's cover M,*
- *$\varphi_U : U \xrightarrow{\sim} B_U \subset \mathbb{Q}_p^d$ is a homeomorphism, where B_U is a closed ball,*
- *$\varphi_U \circ \varphi_{U'}^{-1} : \varphi_{U'}(U \cap U') \xrightarrow{\sim} \varphi_U(U \cap U')$ is a locally analytic function from an open subset of $\mathbb{Q}_p^d$ to $\mathbb{Q}_p^d$, i.e. locally it is given by a convergent power series.*

These form a category of locally analytic manifolds.

Definition 9.3. *A **locally analytic group** is a group object in this category, e.g. $\mathrm{GL}_n(K)$ for $K/\mathbb{Q}_p$ finite extension.*

Definition 9.4. *If $B = B_r(a) \subset \mathbb{Q}_p^d$ and V is a Banach space with defining norm $\| - \|$, we define*

$$C^{\mathrm{rig}}(B, V) := \{f = \sum_{\underline{i}\in\mathbb{N}^d} v_{\underline{i}}(x_1 - a_1)^{i_1} \cdots (x_d - a_d)^{i_d} \mid v_{\underline{i}} \in V, \|v_{\underline{i}}\| r^{\sum i_j} \longrightarrow 0 \text{ as } |\underline{i}| = \sum i_j \longrightarrow \infty\}.$$

Given $f \in C^{\mathrm{rig}}(B, V)$, let $\|f\|_B := \max_{\underline{i}} \|v_{\underline{i}}\| r^{|\underline{i}|} \in \mathbb{R}_{\geq 0}$.

Lemma 9.1. *(1) $\| - \|_B$ is independent of choice of a.*
(2) $C^{\mathrm{rig}}(B, V)$ equipeed with $\| - \|_B$ is complete, i.e. $C^{\mathrm{rig}}(B, V)$ is a Banach space.

Remark 9.1. We have a continuous injective evaluation map $C^{\mathrm{rig}}(B, V) \hookrightarrow C^0(B, V)$.

Definition 9.5. *If $B_1, B_2 = B_r(a)$ are closed balls in $\mathbb{Q}_p^d$, let*

$$C^{\mathrm{rig}}(B_1, B_2) = \{f + a \mid f \in C^{\mathrm{rig}}(B_1, \mathbb{Q}_p^d), \|f\|_{B_1} \leq r\},$$

which is independent of choice of a.

Definition 9.6. *Suppose M is a locally analytic manifold, V is a Banach space, then let the space of **locally analytic functions** be defined as*

$$C^{\mathrm{an}}(M,V) = \varinjlim_{M=\coprod_{i\in I} U_i,\ charts\ \varphi_i\ :\ U_i \xrightarrow{\sim} B_i\ ball} \prod_{i\in I} C^{\mathrm{rig}}(B_i, V),$$

*where transition maps in the limit are given by **refinements**. Namely, we say $(U_i,\varphi_i)_{i\in I} \le (W_j,\psi_j)_{j\in J}$ if, for all $j \in J$, there uniquely exists $i(j) \in I$ such that $W_j \subset U_{i(j)}$, which is given by a rigid analytic function $C^{\mathrm{rig}}(B_j, B_{i(j)})$ after sending $W_j \subset U_{i(j)}$ to $B_j \longrightarrow B_{i(j)}$ via ψ_j and $\varphi_{i(j)}$, and then the transition map $\prod_{i\in I} C^{\mathrm{rig}}(U_i,V) \longrightarrow \prod_{j\in J} C^{\mathrm{rig}}(W_j,V)$ is given by the natural maps $C^{\mathrm{rig}}(U_{i(j)},V) \longrightarrow C^{\mathrm{rig}}(W_j,V)$.*

Remark 9.2. The transition maps are compatible with compositions, and any two indices admit a common refinement. From this, one deduces that $C^{\mathrm{an}}(M,V)$ is an lcv, and the evaluation map $C^{\mathrm{an}}(M,V) \longrightarrow C^0(M,V)$ is continuous.

Definition 9.7. *If M is a locally analytic manifold, and if V is a lcv, then*

$$C^{\mathrm{an}}(M,V) := \varinjlim_{(U_i,\varphi_i,V_i),\ V_i\ Banach,\ V_i \hookrightarrow V\ cts.\ inj.} \prod_{i\in I} C^{\mathrm{rig}}(U_i, V_i).$$

Exercise 9.1. If $M = \mathbb{Z}_p \subset \mathbb{Q}_p$, then the set

$$\{(a + p^n\mathbb{Z}_p, \mathrm{id}) \ :\ a \in \mathbb{Z}/p^n\mathbb{Z}\}_{n\ge 0},$$

is cofinal among indices. Thus, $C^{\mathrm{an}}(\mathbb{Z}_p, V) = \varinjlim_{n\ge 0} \prod_{a\in\mathbb{Z}/p^n\mathbb{Z}} C^{\mathrm{rig}}(a + p^n\mathbb{Z}_p, V)$, where each space in the limit is a Banach space. As the transition maps are compact, it follows that $C^{\mathrm{an}}(\mathbb{Z}_p, E)$ is of compact type.

Proposition 9.1. *If M is a compact locally analytic manifold, and $V = E$ (or more generally of compact type), then $C^{\mathrm{an}}(M,V)$ is of compact type.*

Proposition 9.2. *If $M = \coprod_{i\in I} M_i$, then $C^{\mathrm{an}}(M,V) \cong \prod_{i\in I} C^{\mathrm{an}}(M_i,V)$.*

Definition 9.8. *Let G be a locally analytic group. Then, a **Banach space representation** of G is a Banach space V with a continuous linear action $G \times V \longrightarrow V$. It is **unitary** if there exists a G-invariant norm defining the topology.*

Remark 9.3. Imposing continuity is the same as imposing separate continuity, i.e. $G \times V \longrightarrow V$ is continuous after fixing one coordinate, by Steinhaus theorem.

Remark 9.4. Any subrepresentation or a quotient representation of a Banach space representation is still a Banach space representation.

Example 9.1.
(1) A finite-dimensional continuous representation with its unique Hausdorff topology is a Banach space representation.
(2) If $H \le G$ is a closed subgroup such that $H\backslash G$ is compact, and if W is a Banach space representation of H, then

$$(\mathrm{Ind}_H^G W)^{C^0} = \{f : G \longrightarrow W \text{ cts.} \mid f(hg) = hf(g) \text{ for } h \in H\} \cong C^0(H\backslash G, W),$$

is a Banach space representation of G. Here one uses a section of $G \longrightarrow H\backslash G$. This shows that for example a parabolic induction is a Banach space representation in this setting.
(3) If G is compact, then any Banach space representation is unitary.

Definition 9.9. *A **locally analytic representation** of G is a compact type space V equipped with a continuous linear action $G \times V \longrightarrow V$ such that the orbit maps $o_v : G \longrightarrow V$, $g \mapsto gv$, are locally analytic.*

Remark 9.5. Again, any subrepresentation or a quotient representation of a locally analytic representation is still locally analytic.

Example 9.2.
(1) Again, a finite-dimensional continuous representation with its unique Hausdorff topology is a Banach space representation.
(2) If $H \leq G$ is a closed subgroup with $H\backslash G$ compact, then
$$(\mathrm{Ind}_H^G W)^{\mathrm{an}} = \{f : G \longrightarrow W \text{ loc. an.} \mid f(hg) = hf(g) \text{ for } h \in H\} \cong C^{\mathrm{an}}(H\backslash G, W),$$
is a locally analytic representation of G. Here one uses a section of $G \longrightarrow H\backslash G$.
(3) V_{sm}, a smooth representation of countable dimension, is locally analytic (o_v's are locally constant).
(4) If $G = \underline{G}(\mathbb{Q}_p)$ for an algebraic group $\underline{G}$ over $\mathbb{Q}_p$, then an algebraic representation V_{alg} of G is locally analytic (o_v's are locally polynomial). Thus $V_{\mathrm{alg}} \otimes V_{\mathrm{sm}}$, a **locally algebraic representation**, is locally analytic.

Remark 9.6. The relevant categories are not abelian! (Schneider) This is why introduce duality theory below.

10. Duality of smooth representations with mod p coefficients.

Let G be a compact locally analytic group.

Definition 10.1. *Let $C/\mathbb{F}_p$ be a finite extension. Then, the **ring of distributions** is defined as*
$$D^\infty(G) := C^\infty(G, C)' = \left(\varinjlim_{U \trianglelefteq G \text{ open}} C(G/U, C) \right)' = \varprojlim_U C[G/U] =: C[[G]],$$
*which is a profinite ring and is **noetherian** (Lazard).*

Note that if V is a smooth G-representation over C, then $V = \varinjlim_U V^U$ has an action of $C[[G]]$, which gives an action of $C[[G]]$ on V'.

Proposition 10.1 (Pontryagin duality)**.** *There is an anti-equivalence of categories*
$$\left\{ \begin{array}{c} \textit{smooth} \\ G\textit{-representations} \\ \textit{over } C \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} D^\infty(G)\textit{-modules with} \\ \textit{profinite topology such} \\ \textit{that action is cts} \end{array} \right\},$$
$$V \mapsto V'.$$

Remark 10.1. By Nakayama's lemma, V is admissible if and only if V' is finitely generated $D^\infty(G)$-module.

Remark 10.2. Any finitely generated $D^\infty(G)$-module carries a unique profinite topology such that the action of $D^\infty(G)$ is continuous. Thus, when we restrict the Pontryagin duality to admissible G-representations, then we can forget topology,
$$\left\{ \begin{array}{c} \text{admissible} \\ G\text{-representations} \\ \text{over } C \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{finitely generated} \\ D^\infty(G)\text{-modules} \end{array} \right\}.$$
In particular, as $D^\infty(G)$ is noetherian, the **RHS category is abelian**! In particular, the LHS is closed under quotients.

11. **Duality of Banach space representations.**

This is the theory of Schneider-Teitelbaum. Again, let G be a compact locally analytic group.

Definition 11.1. *Let $D^c(G) := C^0(G, E)' \cong \mathcal{O}[[G]][1/p]$, where $\mathcal{O}[[G]] = \varprojlim_{n,U}(\mathcal{O}/\varpi^n)[G/U]$. This is again profinite and noetherian.*

Similarly we have the following

Proposition 11.1. *If V is a Banach space representation, then there is a G-invariant lattice $V^\circ \subset V$ defining the topology, which has natural $\mathcal{O}[[G]]$-action. Thus, V, V' are $D^c(G)$-modules.*

This is because

- G is compact, so V is unitary, so the existence of V° is clear,
- and we have $V^\circ = \varinjlim_{n\geq 0} V^\circ/\varpi^n V^\circ$, where each $V^\circ/\varpi^n V^\circ$ is smooth, and so has a continuous action of $(\mathcal{O}/\varpi^n)[[G]]$.

Definition 11.2. *We say a Banach space representation V is **admissible** if V' is finitely generated as a $D^c(G)$-module.*

This is equivalent to $V^\circ/\varpi V^\circ$ being admissible in the previous sense.

Theorem 11.1 (Schneider-Teitelbaum). *There is an anti-equivalence of categories*

$$\left\{\begin{array}{c}\textit{admissible Banach space}\\ \textit{representations}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\textit{finitely generated}\\ D^c(G)\textit{-modules}\end{array}\right\},$$

$$V \mapsto V',$$

and the RHS is an abelian category.

Example 11.1. Under the above correspondence, $C^0(G, E)$ goes to $D^c(G)$.

Corollary 11.1. *(1) Any map $f : V \longrightarrow W$ of admissible Banach space representations is **strict**, i.e. $V/\ker f \xrightarrow{\sim} \operatorname{im} f$ is a topological isomorphism.*
(2) Any closed subrepresentation $W \subset V$ and its quotient V/W of an admissible Banach space representation V are all admissible.
(3) One has the usual notion of kernel and cokernel within the category of admissible Banach space representations.

Remark 11.1. For a general locally analytic group G, one can use any compact open subgroup as the notion of admissibility does not depend on a choice of compact open subgroup (difference is only finite-dimensional). It is more subtle to get an equivalence.

12. **Duality of locally analytic representations.**

Definition 12.1. *Let the **ring of locally analytic distributions** be defined as*

$$D^{\mathrm{an}}(G) := C^{\mathrm{an}}(G, E)'_b,$$

which is a nuclear Fréchet space.

Unlike the previous cases, it is not obvious that $D^{\mathrm{an}}(G)$ is a ring.

Theorem 12.1 (de Lacroix)**.** *The* ***Dirac delta distributions*** *δ_g, $g \in G$, defined in an obvious way, span a dense subspace $D^{\mathrm{an}}(G)$. There is a unique continuous multiplication on $D^{\mathrm{an}}(G)$ extending the multiplication on the Dirac delta distributions $\delta_g * \delta_h = \delta_{gh}$. More concretely, the multiplication is given by*

$$(\delta_1 * \delta_2)(f) = \delta_1(g_1 \mapsto \delta_2(g_2 \mapsto f(g_1 g_2))),$$

for $\delta_1, \delta_2 \in D^{\mathrm{an}}(G)$.

If V is a locally analytic representation, then there is a unique separably continuous action $D^{\mathrm{an}}(G) \times V \longrightarrow V$ such that $\delta_g v = gv$. The same statement holds for V_b'.

We cannot just mindlessly take finitely generated $D^{\mathrm{an}}(G)$-modules to define admissible representations, as $D^{\mathrm{an}}(G)$ is **not noetherian**.

Theorem 12.2 (Schneider-Teitelbaum)**.** *There is an anti-equivalence of categories*

$$\left\{ \begin{array}{c} \textit{locally analytic} \\ \textit{representations, on} \\ \textit{compact type spaces} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{separably continuous} \\ D^{\mathrm{an}}(G)\textit{-modules on nuclear} \\ \textit{Fréchet spaces} \end{array} \right\},$$

$$V \mapsto V_b'.$$

Remark 12.1. Let $\mathfrak{g} = \operatorname{Lie} G$, then the map $\mathfrak{g} \longrightarrow D^{\mathrm{an}}(G)$ defined by

$$X \mapsto (f \mapsto \frac{d}{dt}|_{t=0} f(e^{tX})),$$

is a Lie algebra homomorphism, where we use $\exp : U \longrightarrow G$ for an open neighborhood U of 0 in $\mathfrak{g}$.

Remark 12.2. We have $D^c(G)$ sitting inside $D^{\mathrm{an}}(G)$ as a subring, but as we have mentioned above, $D^{\mathrm{an}}(G)$ is not noetherian.

Example 12.1. Take the simplest example of $G = \mathbb{Z}_p$.

- Mahler proved that, for E a p-adic field, $C^0(\mathbb{Z}_p, E)$ can be expressed as

  $$\{\sum_{n \geq 0} a_n \binom{x}{n} \mid a_n \in E, a_n \longrightarrow 0\},$$

  where it means a_n's have bounded denominators and the numerators converge to zero p-adically.
- Inside this, we have locally analytic functions on $\mathbb{Z}_p$,

  $$C^{\mathrm{an}}(\mathbb{Z}_p, E) = \{\sum_{n \geq 0} a_n \binom{x}{n} \mid |a_n| r^n \longrightarrow 0 \text{ for some } r > 1\}.$$

- The **Amice transform** says that there is an algebra isomorphism

  $$D^{\mathrm{an}}(\mathbb{Z}_p) \xrightarrow{\sim} C^{\mathrm{rig}}(X_{<1}),$$

  $$\delta \mapsto \text{“}\delta((1+T)^x)\text{”} = \sum_{n \geq 0} \delta(\binom{x}{n}) T^n,$$

  where $C^{\mathrm{rig}}(X_{<1})$ is the ring of rigid analytic functions on **open unit disc**. From this we can check that $D^{\mathrm{an}}(\mathbb{Z}_p)$ is not noetherian.

Remark 12.3. Even so, as the Amice transform suggests that $D^{\mathrm{an}}(\mathbb{Z}_p)$ has an interpretation using rigid analytic geometry, in particular $C^{\mathrm{rig}}(X_{<1}) \cong \varprojlim_{r<1, r\in p^{\mathbb{Q}}} C^{\mathrm{rig}}(X_{\leq r})$, where now $X_{\leq r}$ is an affinoid for $r \in p^{\mathbb{Q}}$. Thus, $C^{\mathrm{rig}}(X_{<1})$ is an inverse limit of countable collections of Tate algebras, which are very nice (in particular noetherian). As there is a notion of coherent sheaves on rigid analytic spaces, what Schneider-Teitelbaum did was to emulate the theory of coherent sheaves on the represenation theoretic side (e.g. Fréchet-Stein algebra, coadmissible modules, $\cdots$), using the situation of increasing union of affinoids covering a non-affinoid rigid analytic space.

Definition 12.2. *A Fréchet algebra A is* ***Fréchet-Stein*** *if there exist seminorms $q_1 \leq q_2 \leq \cdots$, defining the topology of A, such that*

1. *the multiplication $A \times A \longrightarrow A$ is continuous with respect to q_n for all n (so that $A \cong \varprojlim_{n\geq 1} A_{q_n}$, even topologically),*
2. *the completion A_{q_n} is left-noetherian,*
3. *A_{q_n} is flat as a right $A_{q_{n+1}}$-module.*

Definition 12.3. *If A is a Fréchet-Stein algebra, an A-module M is* ***coadmissible*** *if*

1. *$M_n := A_{q_n} \otimes_A M$ is finitely generated for all n,*
2. *and $M \longrightarrow \varprojlim_n M_n$ is a bijection.*

Proposition 12.1 (Schneider-Teitelbaum)**.** *Let G be a compact locally analytic group.*

1. *$D^{\mathrm{an}}(G)$ is a Fréchet-Stein algebra.*
2. *Coadmissible modules are the same as compatible systems of finitely generated A_{q_n}-modules.*
3. *The category of coadmissible modules is an abelian subcategory.*
4. *Any finitely presented A-module is coadmissible.*

Remark 12.4. Any coadmissible module M carries a **canonical topology** by definition:

- M_n carries a unique Banach topology as $A_{q_n} \otimes_A M$;
- M, as $\varprojlim_n M_n$, carries a unique Fréchet topology.

Under this topology, any map between coadmissible modules is continuous and strict.

Proof that $D^{\mathrm{an}}(G)$ is Fréchet-Stein. One can pass to a small compact open subgroup that is "uniform pro-p," so that in particular it is isomorphic to $\mathbb{Z}_p^d$. Now one uses the theory of Mahler expansions, with an additional consideration of multiplcation structure, which we need to use Lazard's results. ☐

Now we can finally define admissibility for locally analytic representations.

Definition 12.4. *A locally analytic representation V is* ***admissible*** *if V_b' is isomorphic to a coadmissible module with its canonical topology. Thus, there is an anti-equivalence of categories*

$$\left\{ \begin{array}{c} \textit{Admissible locally} \\ \textit{analytic} \\ \textit{representations} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{Coadmissible modules} \\ \textit{over } D^{\mathrm{an}}(G) \end{array} \right\},$$
$$V \mapsto V_b'.$$

Corollary 12.1. *(1) The category of admissible locally analytic representations is abelian.*

(2) Any map of admissible G-representations is strict and has closed image.

(3) Closed submodules and Hausdorff quotients of an admissible module are admissible.

(4) There is a usual notion of kernel and cokernel.

Example 12.2. If V is an admissible smooth representaiton, then it is an admissible locally analytic representation.

Definition 12.5. *Let V bs an admissible Banach space representation of G. Then the space of* ***locally analytic vectors*** *V_{an} is defined by*

$$V_{\mathrm{an}} = \{v \in V \mid o_v \in C^{\mathrm{an}}(G, V)\} \hookrightarrow C^{\mathrm{an}}(G, V),$$

with the subspace topology.

Theorem 12.3 (Schneider-Teitelbaum)**.** *Let V be an admissible Banach space representation.*

(1) *V_{an} is compact type, and dense in V.*
(2) *V_{an} is an admissible locally analytic representation, and $(V_{\mathrm{an}})' = D^{\mathrm{an}}(G) \otimes_{D^c(G)} V'$.*
(3) *$V \mapsto V_{\mathrm{an}}$ is an exact functor.*

13. **Orlik-Strauch representations.**

Let $G = \mathrm{GL}_n(\mathbb{Q}_p)$, $P = MN$ be a parabolic subgroup, $\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{p} = \mathrm{Lie}\, P$, $\mathfrak{m} = \mathrm{Lie}\, M$, $\mathfrak{n} = \mathrm{Lie}\, \mathfrak{n}$. Denote $U(\mathfrak{g})_E = U(\mathfrak{g}) \otimes_{\mathbb{Q}_p} E$.

For an application of locally analytic representation theory, Orlik-Strauch developed a theory of "Category $\mathcal{O}$", "blocks", "Verma modules", $\cdots$, that is useful to study locally analytic principal series representations.

Definition 13.1. *A finite dimensional irreducible representation of $\mathfrak{m}$ over E is* ***algebraic*** *if it integrates to an irreducible algebraic representation of M.*

Example 13.1. If $P = B$ is the standard upper triangular Borel, then a character

$$\begin{aligned} \mathfrak{t} = \mathbb{Q}_p^d &\longrightarrow E, \\ \underline{x} &\mapsto \sum_i \lambda_i x_i, \end{aligned}$$

is algebraic if and only if $\lambda_i \in \mathbb{Z}$ for all i, so that it integrates into

$$\begin{aligned} T &\longrightarrow E^\times, \\ \mathrm{diag}(t_1, \cdots, t_n) &\mapsto \prod_i t_i^{\lambda_i}. \end{aligned}$$

Definition 13.2. *The* ***category*** *$\mathcal{O}_{\mathfrak{p}}^{\mathrm{alg}}$ is defined as follows.*

- *Finitely generated $U(\mathfrak{g})_E$-modules L, such that $L|_{\mathfrak{m}}$ is a (direct) sum of irreducible algebraic representations of $\mathfrak{m}$, and for all $x \in L$, $U(\mathfrak{n})_E x$ is finite-dimensional, are the objects of $\mathcal{O}_{\mathfrak{p}}^{\mathrm{alg}}$.*
- *Morphisms are just $U(\mathfrak{g})_E$-linear maps.*

Example 13.2. (1) If $\mathfrak{p} = \mathfrak{g}$, then $\mathcal{O}_{\mathfrak{g}}^{\mathrm{alg}}$ is the category of finite dimensional algebraic $\mathfrak{g}$-representaitons.

(2) If W is an irreducible algebraic $\mathfrak{m}$-representations, consider it as $U(\mathfrak{p})_E$-module via the quotient $U(\mathfrak{p})_E \twoheadrightarrow U(\mathfrak{m})_E$. Then, the **generalized Verma module**

$$M(W) := U(\mathfrak{g})_E \otimes_{U(\mathfrak{p})_E} W,$$

is in $\mathcal{O}_{\mathfrak{p}}^{\mathrm{alg}}$. This is an analogue of parabolic induction. This can be checked using

$$M(W) \cong U(\overline{\mathfrak{n}})_E \otimes_E W,$$

which is a consequence of Poincaré-Birkhoff-Witt theorem.

Proposition 13.1. *The category $\mathcal{O}^{\mathrm{alg}}_{\mathfrak{p}}$ satisfies the following properties.*

- *It is an abelian category.*
- *It is closed under sub, quotient, and direct sum.*
- *Every object has finite length.*
- *If $P \subset Q$ are two parabolic subgroups, then $\mathcal{O}^{\mathrm{alg}}_{\mathfrak{q}} \subset \mathcal{O}^{\mathrm{alg}}_{\mathfrak{p}}$.*

Similar to the construction of generalized Verma module, Orlik-Strauch defines the following construction.

Definition 13.3 (Orlik-Strauch representations)**.** *Let $L \in \mathcal{O}^{\mathrm{alg}}_{\mathfrak{p}}$, and π_M be an admissible smooth M-representation. From the properties demanded for objects of $\mathcal{O}^{\mathrm{alg}}_{\mathfrak{p}}$, one can take a $U(\mathfrak{p})_E$-generated subspace $W \subset L$, which is a finite-dimensional $\mathfrak{p}$-subrepresentation, that generates L as $U(\mathfrak{g})_E$-module. Then, define the $U(\mathfrak{g})_E$-submodule $\partial \subset U(\mathfrak{g})_E \otimes_{U(\mathfrak{p})_E} W$ as the kernel of the surjective map*

$$U(\mathfrak{g})_E \otimes_{U(\mathfrak{p})_E} W \twoheadrightarrow L.$$

Note that as the $\mathfrak{m}$-action on W integrates into an algebraic M-action, the $\mathfrak{p}$-action integrates into an algebraic P-action too, as on unipotent part exponential is algebraic. Consider $C^{\mathrm{an}}(G, W' \otimes \pi_M)$, which has an action by $\mathfrak{g} \times G$ where $\mathfrak{g}$ acts by the differentiation of left translation and G acts by the right translation. Consider the pairing

$$(U(\mathfrak{g})_E \otimes_E W) \times \mathrm{Ind}_P^G(W' \otimes \pi_M)^{\mathrm{an}} \longrightarrow C^{\mathrm{an}}(G, \pi_M),$$

$$((X \otimes w), f)\,(g) := \langle (Xf)(g), w \rangle,$$

which makes sense as $W' \otimes \pi_M$ is locally analytic as a P-representation. As f is left P-equivariant, this pairing factors through

$$(U(\mathfrak{g})_E \otimes_{U(\mathfrak{p})_E} W) \times \mathrm{Ind}_P^G(W' \otimes \pi_M)^{\mathrm{an}} \longrightarrow C^{\mathrm{an}}(G, \pi_M).$$

Then, the ***Orlik-Strauch representation*** *$\mathfrak{F}_P^G(L, \pi_M)$ is the annihilator of ∂ in $\mathrm{Ind}_P^G(W' \otimes \pi_M)$, which is a closed G-subrepresentation.*

Example 13.3. For $L = U(\mathfrak{g})_E \otimes_{U(\mathfrak{p})_E} W$ a generalized Verma module, as $\partial = 0$,

$$\mathfrak{F}_P^G(U(\mathfrak{g})_E \otimes_{U(\mathfrak{p})_E} W, \pi_M) = \mathrm{Ind}_P^G(W' \otimes \pi_M)^{\mathrm{an}}.$$

Theorem 13.1 (Orlik-Strauch)**.** *(1) $\mathfrak{F}_P^G$ is independent of choice of W.*

(2) $\mathfrak{F}_P^G(L, \pi_M)$ is admissible, functorial and exact in both arguments.

(3) If $Q \supset P$ and $L \in \mathcal{O}^{\mathrm{alg}}_{\mathfrak{q}}$, then

$$\mathfrak{F}_P^G(L, \pi_M) \cong \mathfrak{F}_Q^G(L, (\mathrm{Ind}_{P \cap M_Q}^{M_Q} \pi_M)^{\mathrm{sm}}),$$

where in the RHS we use smooth induction.

(4) If L and π_M are irreducible and P is maximal for L (i.e. L is not an object of $\mathcal{O}^{\mathrm{alg}}_{\mathfrak{q}}$ for all Q containing P), then $\mathfrak{F}_P^G(L, \pi_M)$ is ***topologically irreducible****.*

Corollary 13.1. *If π_M is of finite length, then $\mathfrak{F}_P^G(L, \pi_M)$ is topologically of finite length.*

Example 13.4. Let $n = 2$. Let $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2_+ \subset \mathfrak{t}'$, where $\mathbb{Z}^2_+$ means $\lambda_1 \geq \lambda_2$. The Verma module $M(\lambda) = U(\mathfrak{g})_E \otimes_{U(\mathfrak{b})_E} \lambda$ has a filtration

$$0 \longrightarrow L(\lambda') \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0,$$

where L is the unique irreducible quotient of the corresponding Verma module (the usual highest weight representation), and $\lambda' = (1\ 2) \cdot \lambda = (\lambda_2 - 1, \lambda_1 + 1)$. Note that $L(\lambda)$ is finite-dimensional as λ is dominant, while $L(\lambda')$ is not, so in particular $L(\lambda') \notin \mathcal{O}_{\mathfrak{g}}^{\mathrm{alg}}$. In fact, $L(\lambda') = M(\lambda')$.

We apply the Orlik-Strauch construction, for $\chi = \chi_1 \otimes \chi_2 : T \longrightarrow E^\times$ a smooth character, and we have an exact sequence

$$0 \longrightarrow \mathfrak{F}_B^G(L(\lambda), \chi) \longrightarrow \mathfrak{F}_B^G(M(\lambda), \chi) \longrightarrow \mathfrak{F}_B^G(L(\lambda'), \chi) \longrightarrow 0.$$

Let $\chi_\lambda : T \longrightarrow E^\times$ be the algebraic character satisfying $d\chi_\lambda = \lambda$. Then, the last two terms are locally analytic parabolic inductions,

$$0 \longrightarrow \mathfrak{F}_B^G(L(\lambda), \chi) \longrightarrow \mathrm{Ind}_B^G(\chi_\lambda^{-1} \otimes \chi)^{\mathrm{an}} \longrightarrow \mathrm{Ind}_B^G(\chi_{\lambda'}^{-1} \otimes \chi)^{\mathrm{an}} \longrightarrow 0.$$

The last term is irreducible, as B is maximal for $L(\lambda') = M(\lambda')$. Also, by one of the properties of Orlik-Strauch construction,

$$\mathfrak{F}_B^G(L(\lambda), \chi) = \mathfrak{F}_G^G(L(\lambda), (\mathrm{Ind}_B^G \chi)^{\mathrm{sm}}) = L(\lambda)' \otimes \mathrm{Ind}_B^G(\chi)^{\mathrm{sm}}.$$

This is locally algebraic, and $(\mathrm{Ind}_B^G \chi)^{\mathrm{sm}}$ is irreducible if and only if $\chi_1\chi_2^{-1} \neq 1$ or $| - |^2$ (Bernstein-Zelevinsky). Thus, this gives a fairly explicit filtration of principal series locally analytic representations.

1. Fontaine's period ring formalism.

We want to study continuous representations of $G_K = \mathrm{Gal}(K^s/K)$ for a p-adic local field K (mainly a finite extension of $\mathbb{Q}_p$, but also can be a local function field) on a finitely generated $\mathbb{F}_p$-vector space/$\mathbb{Z}_p$-module/$\mathbb{Q}_p$-vector space.

A general strategy of Fontaine is to construct an interesting ring with interesting structures and Galois action and we can study the representation by base-changing to the ring. More precisely, let's say G is a group, L is a field, and we want to study representations of G on a finite-dimensional L-vector space. Then what we generally want to do is to find an L-algebra B, which is a domain, with a L-linear G-action. Then, we can endow the diagonal G-action on $V \otimes_L B$, and define $D_B(V) = (V \otimes_L B)^G$, which is a B^G-module. From this, we have a G-equivariant map

$$D_B(V) \otimes_{B^G} B \longrightarrow V \otimes_L B.$$

We want good situations where this map is isomorphism. Then, we can hope to use structures of B (and $D_B(V)$) to study V.

Definition 1.1. *The L-algebra B is called G-**regular** if*

- $B^G = (\mathrm{Frac}\, B)^G$,
- *if* $0 \neq b \in B$ *and* $L \cdot b$ *is a G-stable line, then* $b \in B^\times$.

Obviously, if B is G-regular, then $E = B^G$ is a field.

Proposition 1.1. *If B is G-regular,* $\dim_E D_B(V) \leq \dim_L V$, *and the map* $D_B(V) \otimes_{B^G} B \longrightarrow V \otimes_L B$ *is injective.*

Definition 1.2. *For a G-regular ring B, if* $\dim_E D_B(V) = \dim_L V$ *(or equivalently the map* $D_B(V) \otimes_{B^G} B \longrightarrow V \otimes_L B$ *is an isomorphism), then we say V is B-**admissible**.*

Remark 1.1. (1) Let ρ denote the G-representation on V, and let $n = \dim_L V$. Then, upon choosing a basis of V, we can see ρ as a cocycle $\rho \in H^1(G, \mathrm{GL}_n(L))$, where $\mathrm{GL}_n(L)$ has a trivial G-action. Then, saying (ρ, V) is B-admissible is equivalent to saying that

$$\rho \in \ker(H^1(G, \mathrm{GL}_n(L)) \longrightarrow H^1(G, \mathrm{GL}_n(B))).$$

(2) If G is a topological group and B is a topological L-algebra with continuous G-action, the above remark holds for continuous representations (ρ, V) and continuous H^1.
(3) In practice, B comes with additional structures (e.g. endomorphism, filtration, $\cdots$) commuting with G-action. Given V, these induce additional structures on $D_B(V)$. Our general hope is that, by putting enough additional structures, we can recover B-admissible V from $D_B(V)$.

2. φ-modules.

We study one of the easiest examples of the above situation. Let F be a local field of characteristic p, of form $\mathbb{F}_q((t))$, $G = G_F$ and $L = \mathbb{F}_p$.

In this case, let $B = F^s$. There is obviously an action of G_F, and as it is a field, it is obviously G_F-regular, and $B^{G_F} = F$.

Lemma 2.1. *Every continuous G_F-representation on a finite-dimensional $\mathbb{F}_p$-vector space is F^s-admissible.*

Proof. This is immediate from Hilbert 90 (so that $H^1(G_F, \mathrm{GL}_n(F^s)) = 0$) and the cohomological way of seeing B-admissibility. $\square$

Now a nice thing is that we have a Frobenius endomorphism $\varphi : F^s \longrightarrow F^s$, $x \mapsto x^p$, which commutes with G_F-action. Thus, given V, we get an induced map $\phi : D(V) \xrightarrow{1 \otimes \varphi} D(V)$ where $D(V) = (V \otimes_{\mathbb{F}_p} F^s)^{G_F}$.

Definition 2.1. *A φ-**module** over a ring A with an endomorphism $\varphi : A \longrightarrow A$ is a finitely generated A-module D, together with a φ-semilinear map $\phi : D \longrightarrow D$, such that the A-linear map $\varphi^* D = D \otimes_{A,\varphi} A \xrightarrow{d \otimes 1 \mapsto \varphi(d)} D$ is an isomorphism.*

Thus, $D(V)$ is a φ-module over F, because φ is injective so that ϕ on $D(V)$ is injective and thus the linearization of ϕ is injective too.

Corollary 2.1. *The functor*

$$D : \left\{ \begin{array}{c} \text{continuous representations of} \\ G_F \text{ on finite-dimensional} \\ \mathbb{F}_p\text{-vector spaces} \end{array} \right\} \longrightarrow \{\varphi\text{-modules over } F\},$$

is fully faithful.

Theorem 2.1 (Fontaine)**.** *The above functor D is an equivalence of categories, with quasi-inverse*

$$D \mapsto \mathbf{V}(D) := (D \otimes_F F^s)^{\varphi = \mathrm{id}},$$

where $\varphi : D \otimes_F F^s \longrightarrow D \otimes_F F^s$ is φ diagonally.

Proof. It is fully faithful, as this gives a quasi-inverse to the essential image: given a representation V,

$$D(V) \otimes_F F^s \xrightarrow{\sim} V \otimes_{\mathbb{F}_p} F^s,$$

is not only Galois equivariant but φ-equivariant, where φ on the RHS is just φ on the second factor, and $(F^s)^{\varphi=\mathrm{id}} = \mathbb{F}_p$.

For essential surjectivity, it reduces to show that we don't lose dimension via $\mathbf{V}$. Namely, given a φ-module F, we have to show $\dim_{\mathbb{F}_p} \mathbf{V}(D) = \dim_F D$. One uses F^s is separably closed and counts roots. $\square$

Remark 2.1. It might be hard to construct Galois representations as the structure of Galois group is mysterious. Thus, this theory can be thought as an easy explicit way of producing Galois representations of G_F.

Now we want to "lift" the situation to torsion coefficients and $\mathbb{Q}_p$-coefficients. To do this, we "lift" F^s to characteristic 0.

Definition 2.2. *Let $(\mathcal{O}_{\mathcal{E}}, \varphi)$ be a Cohen ring for F, i.e. $\mathcal{O}_{\mathcal{E}}$ is a complete dvr with uniformizer p with residue field F, and φ is a lift of Frobenius on F.*

Example 2.1.
(1) If F is perfect, then $(W(F), \varphi)$ is the unique possible Cohen ring, where φ is the unique lift of Frobenius.
(2) In our case of $F = \mathbb{F}_q((t))$, we can take $\mathcal{O}_{\mathcal{E}} = W(\mathbb{F}_q)((t))^\wedge$, where $\wedge$ means we take p-adic completion. One can take φ to be the lift of Frobenius on $W(\mathbb{F}_q)$ (which we don't have any choice) and $t \mapsto t^p$ (which we have freedom to choose; we might as well take $t \mapsto (1+t)^p - 1$, which turns out to be more useful).

Definition 2.3. *We define as follows.*

- $\mathcal{E} = \mathcal{O}_{\mathcal{E}}[1/p]$.
- $\mathcal{E}^{\mathrm{un}}$ *is the maximal unramified extension of* $\mathcal{E}$.
- $\mathcal{O}_{\mathcal{E}^{\mathrm{un}}}$ *is the ring of integers of* $\mathcal{E}^{\mathrm{un}}$.
- $\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}$ *is the* p*-adic completion of* $\mathcal{O}_{\mathcal{E}^{\mathrm{un}}}$.
- $\widehat{\mathcal{E}}^{\mathrm{un}} = \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}[1/p]$.

The extension $\mathcal{E}^{\mathrm{un}}/\mathcal{E}$ *is Galois with Galois group* G_F*, and we have* φ *and* G_F*-actions on all the above rings.*

As we have seen before, it might be useful to calculate G_F and φ-invariants of the above rings.

Lemma 2.2. *(1)* $\mathcal{O}_{\mathcal{E}} \xrightarrow{\sim} (\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})^{G_F}$.

(2) $\mathbb{Z}_p = (\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})^{\varphi=\mathrm{id}}$.

(3) $H^1(G_F, \mathrm{GL}_n(\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})) = 1$.

This allows us to lift to characteristic zero.

Proof. We use successive approximation, by noticing that we can filter $\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}$ ($\mathrm{GL}_n(\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})$, resp.) such that graded pieces are F^s (either $\mathrm{GL}_n(F^s)$ or $M_n(F^s)$, resp.). Note that for GL_n we use obvious congruence subgroups. And we know analogous calculations for graded pieces. □

Corollary 2.2. *(1) Let* Λ *be a finitely generated* $\mathbb{Z}_p$*-module with continuous* $\mathbb{Z}_p$*-linear* G_F*-action. Then,*

$$D(\Lambda) := (\Lambda \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})^{G_F},$$

is a φ*-module over* $\mathcal{O}_{\mathcal{E}}$*, and*

$$D(\Lambda) \otimes_{\mathcal{O}_{\mathcal{E}}} \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}} \longrightarrow \Lambda \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}},$$

is a (G_F, φ)*-equivariant isomorphism.*

(2) This functor gives an equivalence of categories

$$\left\{\begin{matrix} \textit{continuous } G_F\textit{-representations} \\ \textit{on finitely generated} \\ \mathbb{Z}_p\textit{-modules} \end{matrix}\right\} \longrightarrow \{\varphi\textit{-modules over } \mathcal{O}_{\mathcal{E}}\},$$

with quasi-inverse

$$\mathbf{V}(D) := (D \otimes_{\mathcal{O}_{\mathcal{E}}} \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})^{\varphi=\mathrm{id}}.$$

(3) Now let V *be a finite-dimensional* G_F*-representation over* $\mathbb{Q}_p$*. Then,*

$$D(V) := (V \otimes_{\mathbb{Q}_p} \widehat{\mathcal{E}}^{\mathrm{un}})^{G_F},$$

is a fully faithful functor

$$\left\{\begin{matrix} \textit{continuous } G_F\textit{-representations} \\ \textit{on finite dimensional} \\ \mathbb{Q}_p\textit{-vector spaces} \end{matrix}\right\} \longrightarrow \{\varphi\textit{-modules over } \mathcal{E}\},$$

with essential image being the subcategory of ***étale*** φ***-modules****. Here,* (D, φ)*, a* φ*-module over* $\mathcal{E}$*, is an étale* φ*-module, if there is a* φ*-module* (D', φ') *on* $\mathcal{O}_{\mathcal{E}}$ *such that* $(D, \varphi) \cong (D', \varphi') \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{E}$.

Proof. Everything is almost a formal consequence of characteristic p coefficient theory. □

Remark 2.2. That we need to introduce the notion of étale φ-modules is very much expected. Indeed, if φ acts on a φ-module over $\mathcal{E}$ with an eigenvalue, say, p, then obviously it does not come from φ-module over $\mathcal{O}_{\mathcal{E}}$. In some sense being étale φ-module is to assert that there is an "integral model." On the other hand, for any rational Galois representation, there is a Galois stable lattice by usual compactness argument.

3. (φ, Γ)-**modules.**

Characteristic p theory is very satisfying. But what about mixed characteristic local fields? Let $K/\mathbb{Q}_p$ be a finite extension, then how do we describe continuous G_K-reprsentations on finitely generated $\mathbb{F}_p$-vector spaces/$\mathbb{Z}_p$-modules/$\mathbb{Q}_p$-representaitons?

The idea is to find a (deeply ramified, infinite) Galois extension K_∞/K such that

- one can transfer the situation to equicharacteristic local fields, i.e. there is an equicharacteristic local field F such that $G_{K_\infty} \cong G_F$,
- the transferred continuous action of G_{K_∞} on $\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}$ extends to a continuous G_K-action on $\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}$ commuting with φ,
- and hopefully $\Gamma = \mathrm{Gal}(K_\infty/K)$ is as simple as possible.

If this is the case, then we get a continuous Γ-action on $(\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})^{G_{K_\infty}} = \mathcal{O}_{\mathcal{E}}$ commuting with φ.

Definition 3.1. *If the above situation holds, a* (φ, Γ)-***module*** *over* $\mathcal{O}_{\mathcal{E}}$ *($\mathcal{E}$, resp.) is a* φ*-module over* $\mathcal{O}_{\mathcal{E}}$ *($\mathcal{E}$, resp.) with a semilinear* Γ*-action commuting with* φ.

A (φ, Γ)*-module over* $\mathcal{E}$ *is* ***étale*** *if the underlying* φ*-module is étale.*

Remark 3.1. The definition of étale (φ, Γ)-module is sensible, as Γ is compact.

Theorem 3.1. *(1) There is an equivalence of categories*

$$\left\{\begin{array}{c}\text{continuous}\\ G_K\text{-representations on finitely}\\ \text{generated } \mathbb{Z}_p\text{-modules}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}(\varphi, \Gamma)\text{-modules over}\\ \mathcal{O}_{\mathcal{E}}\end{array}\right\},$$

$$\Lambda \longmapsto (\Lambda \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})^{G_{K_\infty}},$$

with quasi-inverse

$$D \longmapsto (D \otimes_{\mathcal{O}_{\mathcal{E}}} \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})^{\varphi = \mathrm{id}},$$

where G_K acts on the target diagonally (G_K acts via its quotient Γ on D).

(2) There is an equivalence of categories

$$\left\{\begin{array}{c}\text{continuous}\\ G_K\text{-representations on finite}\\ \text{dimensional } \mathbb{Q}_p\text{-modules}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{étale } (\varphi, \Gamma)\text{-modules}\\ \text{over } \mathcal{E}\end{array}\right\},$$

$$V \longmapsto (V \otimes_{\mathbb{Q}_p} \widehat{\mathcal{E}}^{\mathrm{un}})^{G_{K_\infty}},$$

with quasi-inverse

$$D \longmapsto (D \otimes_{\mathcal{E}} \widehat{\mathcal{E}}^{\mathrm{un}})^{\varphi = \mathrm{id}}.$$

Proof. Everything is a very formal consequence. ∎

4. **Tilting equivalence.**

Characteristic 0 situation can be salvaged as a formal consequence of having K_∞, but is there such K_∞? Classically, Fontaine and Wintenberger used "norm fields" to find K_∞. Nowadays we have a more conceptual perspective due to Scholze using tilting equivalence of perfectoid fields.

Definition 4.1. *A **perfectoid field** K is a complete nonarchimedean field of residue charateristic p, complete with respect to a valuation v_k (resp. norm $|-|$) such that*

- *v_k is non-discrete, i.e. the maximal ideal $\mathfrak{m} \subset \mathcal{O}_K$ satisfies $\mathfrak{m}^2 = \mathfrak{m}$,*
- Frob $: \mathcal{O}_K/p \longrightarrow \mathcal{O}_K/p$, $x \mapsto x^p$, *is surjective.*

Example 4.1. The following are examples of perfectoid fields.

1. $\mathbb{F}_q((X^{1/p^\infty})) := \left(\cup_{n\geq 0}\mathbb{F}_q((X^{1/p^n}))\right)^\wedge$, the X-adic completion.
2. $\overline{\mathbb{F}_q((X))}^\wedge$.
3. $\overline{\mathbb{Q}}_p^\wedge$, the p-adic completion.
4. Given a finite extension $F/\mathbb{Q}_p$, $F(\pi^{1/p^\infty})^\wedge$, for a uniformizer $\pi \in F$.
5. Given a finite extension $F/\mathbb{Q}_p$, and a compatible system (ε_n) of p-power primitive roots of 1, $F(\varepsilon_n, n \geq 1)^\wedge$.

Remark 4.1. If K is a complete nonarchimedean field of characteristic p with non-discrete valuation, then K is perfectoid if and only if K is perfect.

Definition 4.2. *Let K be a perfectoid field. Fix a pseudo-uniformizer $\varpi \in \mathcal{O}_K$, namely $|p| \leq |\varpi| < 1$. Define*

$$\mathcal{O}_{K^\flat} := \varprojlim_{\text{Frob}} \mathcal{O}_K/\varpi.$$

*Choose a compatible system of p-power roots of ϖ, $\varpi^\flat = (0, \varpi_1^\flat, \cdots) \in \mathcal{O}_{K^\flat}$, such that $\varpi_1^\flat \neq 0$. Then, we define the **tilt** of K to be*

$$K^\flat = \mathcal{O}_{K^\flat}[1/\varpi^\flat].$$

Lemma 4.1. *Fix a valuation v_K on a perfectoid field K.*

(1) $\mathcal{O}_{K^\flat}$ has a valuation defined by

$$(x_0, x_1, \cdots) \mapsto \lim_{n\to\infty} v_K(\tilde{x}_n^{p^n}),$$

where $\tilde{x}_n \in \mathcal{O}_K$ is a lift of x_n. Namely, the limit always exists, and does not depend on a choice of lifts.

(2) $\mathcal{O}_{K^\flat}$ is complete with respect to this valuation, and $\mathcal{O}_{K^\flat}$ does not depend on choices of ϖ. The topology defined by this valuation is independent of choice of v_K.

(3) $K^\flat$ is a perfectoid field of characteristic p, and does not depend on choice of $\varpi^\flat$.

Remark 4.2. If K is a perfectoid field of characteristic p, then $K^\flat \cong K$. More generally, if K is characteristic 0,

$$\mathcal{O}_{K^\flat} = \varprojlim_{\text{Frob}} \mathcal{O}_K/\varpi \xleftarrow{\sim} \varprojlim_{x\mapsto x^p} \mathcal{O}_K,$$

because the rightmost object is a priori just a multiplicative monoid but $x \mapsto x^p$ commutes with addition in characteristic p. In general, if $(x^{(n)}), (y^{(n)})$ is in $\varprojlim_{x\mapsto x^p} \mathcal{O}_K$, then we can give the additive structure by

$$(x^{(n)}) + (y^{(n)}) = \left(\lim_{m\to\infty}\left(x^{(n+m)} + y^{(n+m)}\right)^{p^m}\right).$$

Theorem 4.1 (Scholze)**.** *Let K be a perefctoid field.*

(1) If L/K is a finite extension, then L is perfectoid.
*(2) **(Tilting equivalence)** There is a degree-preserving equivalence of categories*

$$\{\text{finite extensions of } K\} \longrightarrow \left\{\text{finite extension of } K^\flat\right\},$$

$$L \longmapsto L^\flat.$$

In particular, we have a canonical isomorphism $G_K \xrightarrow{\sim} G_{K^\flat}$.

Remark 4.3. (1) One can give an explicit quasi-inverse. Namely, given K there is a **canonical ring homomotphism**

$$\theta_K : W(\mathcal{O}_{K^\flat}) \twoheadrightarrow \mathcal{O}_K,$$

which is defined on Teichmuller representatives as

$$[(x_0, x_1, \cdots)] \longmapsto \lim_{n\to\infty} \tilde{x}_n^{p^n},$$

where $\tilde{x}_n \in \mathcal{O}_K$ is a lift of x_n. Then, given a finite extension $E/k^\flat$ with the ring of integers $\mathcal{O}_E$,

$$L = W(\mathcal{O}_E) \otimes_{W(\mathcal{O}_{K^\flat}),\theta_K} K,$$

is an untilt of E.

(2) In fact, we have a more general tilting equivalence,

$$\{\text{perfectoid } K\text{-algebras}\} \xrightarrow{\sim} \left\{\text{perfectoid } K^\flat\text{-algebras}\right\},$$

$$R \longmapsto R^\flat,$$

where a **perfectoid K-algebra** is a Banach K-algebra R such that
- $R^\circ \subset R$ is open and bounded, where R° is the set of power-bounded elements,
- and $\mathrm{Frob} : R^\circ/\varpi \longrightarrow R^\circ/\varpi$ is surjective.

More generally, there is the **almost purity theorem**: given a perfectoid K-algebra R, tilting induces an equivalence

$$\{\text{finite étale } R\text{-algebras}\} \leftrightarrow \left\{\text{finite étale } R^\flat\text{-algebras}\right\}.$$

(3) Without fixing a base field K, tilting is not an equivalence: there are many different fields giving the same tilt. For example, we can take $K_1 = \mathbb{Q}_p(\varepsilon_n, n \geq 1)^\wedge$, where ε_n is the compatible system of p-power roots of 1, and $K_2 = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$. Then, $(K_1)^\flat \cong (K_2)^\flat \cong (K_3)^\flat = K_3$, where $K_3 = \mathbb{F}_p((X^{1/p^\infty}))$. For example, $\mathbb{F}_p((X)) \xrightarrow{X \mapsto \varpi^\flat} K_2^\flat$, for $\varpi^\flat = (p, p^{1/p}, \cdots)$, induces an isomorphism $K_3 \xrightarrow{\sim} K_2^\flat$.

Idea of proof of tilting equivalence. We fix $K, \mathcal{O}_K, \varpi, K^\flat, \mathcal{O}_{K^\flat}, \varpi^\flat$. Then, we have an isomorphism $\mathcal{O}_K/\varpi \cong \mathcal{O}_{K^\flat}/\varpi^\flat$.

We prove that $L \mapsto L^\flat$ is an equivalence as follows. We eventually prove that $\mathcal{O}_L \mapsto \mathcal{O}_{L^\flat}$ is an equivalence. Given $\mathcal{O}_L$, we show that $\mathcal{O}_L/\varpi^n$ is the unique flat lift of $\mathcal{O}_L/\varpi$ over $\mathcal{O}_K/\varpi^n$. This is shown by the naturality of cotangent complexes and perfectoidness assumption, the cotangent complexes have to vanish. Thus from the isomorphism $\mathcal{O}_K/\varpi \cong \mathcal{O}_{K^\flat}/\varpi^\flat$, the equivalence follows. ∎

5. **Back to (φ, Γ)-modules.**

Now using the tilitng equivlance, there is a hope to find "K_∞". Let $K/\mathbb{Q}_p$ be a finite extension, and $F/\mathbb{F}_p((X))$ be also a finite extension. Let $C_1 = \mathbb{C}_p = \widehat{\overline{K}}$, and $C_2 = \widehat{F^s}$. Then, the G_K-action on $\overline{K}$ extends to a continuous action of G_K on C_1, and the G_F-action on $\overline{F}$ extends to a continuous action of G_F on C_2.

Lemma 5.1. *(1) (Krasner's lemma) C_1 and C_2 are algebraically closed. In particular, $C_2 = \widehat{\overline{F}}$.*

(2) C_1 and C_2 are perfectoid, and $C_2 \cong C_1^\flat$. On the other hand, the isomorphism depends on the choice of $\varpi^\flat \in C_1^\flat$, which is highly non-unique.

(3) (Ax-Sen lemma)Let $H \subset G_K$ (resp. $H \subset G_F$) be a closed subgroup, then $C_1^H = (\overline{K}^H)^\wedge$ (resp. $C_2^H = (((F^s)^H)^{\mathrm{perf}})^\wedge$; need to deal with inseparable extensions).

Now fix $\varepsilon_n \in \overline{K}$ a compatible sequence of p-power roots of unity. Our sought-after K_∞ can now be taken as follows.

Definition 5.1. *Let K_∞ be defined as*

$$K_\infty = K(\varepsilon_n, n \geq 1).$$

We see that K_∞ satisfies the following properties.

- K_∞/K is a Galois extension.
- The Galois group $\Gamma = \mathrm{Gal}(K_\infty/K)$ can be realized as an open subgroup of $\mathbb{Z}_p^\times$ via the **cyclotomic character**

$$\chi : \Gamma \longrightarrow \mathbb{Z}_p^\times,$$
$$g \cdot \varepsilon_n = \varepsilon_n^{\chi(g)} \text{ for all } n.$$

  For example, if $K = \mathbb{Q}_p$, then $\Gamma = \mathbb{Z}_p^\times$.
- $\widehat{K_\infty}$ is a perfectoid field.
- Completion does not harm Galois group, namely we have an equivalence of categories

$$\{\text{finite separable extensions of } K_\infty\} \xrightarrow{\sim} \left\{\text{finite separable extensions of } \widehat{K_\infty}\right\},$$
$$L \mapsto \hat{L}.$$

  Thus, via the tilting equivalence, we have an equivalence of categories

$$\{\text{finite separable extensions of } K_\infty\} \xrightarrow{\sim} \left\{\text{finite separable extensions of } \widehat{K_\infty}^\flat\right\},$$
$$L \mapsto \hat{L}^\flat.$$

- Let $\mathbb{F}_q$ be the residue field of K_∞.
  - If $K/\mathbb{Q}_p$ is unramified, then there is a natural map

$$\mathbb{F}_q((X)) \longrightarrow K_\infty^\flat,$$
$$X \mapsto (\varepsilon_0, \varepsilon_1, \cdots) - 1.$$

    This induces the isomorphism $\mathbb{F}_q((X^{1/p^\infty})) \xrightarrow{\sim} \widehat{K}_\infty^\flat$; you can check this by hand.
  - Indeed, it is enough to show that $\bigcup_{n\geq 0} \mathbb{F}_q[[X^{1/p^n}]] \subset \mathcal{O}_{\widehat{K}_\infty^\flat}$ is dense. As $\mathcal{O}_{\widehat{K}_\infty^\flat} = \varprojlim_\varphi \mathcal{O}_{K_\infty}/p$, you need to show that $\overline{\pi}_n \in \mathrm{pr}_m(\bigcup_j \mathbb{F}_q[[X^{1/p^j}]])$, where $\mathrm{pr}_m : \varprojlim_\varphi \mathcal{O}_{K_\infty}/p \longrightarrow \mathcal{O}_{K_\infty}/p = \bigcup_{n\geq 0} W(\mathbb{F}_q)[\pi_n]/p$ is the m-th projection map and $\overline{\pi}_n$ is the class of π_n. But, $\mathrm{pr}_m(X^{p^{m-n}}) = \overline{\pi}_n$ for all n.

  - More generally, if $K/\mathbb{Q}_p$ is a general finite extension, then using the above result for L, the maximal unramified subextension of $K/\mathbb{Q}_p$, we get the similar result.

  Thus, by the same reason as above an equivalence of categories

$$\left\{\text{finite separable extensions of } \mathbb{F}_q((X))^{\mathrm{perf}}\right\} \xrightarrow{\sim} \left\{\text{finite separable extensions of } \widehat{K_\infty}^\flat\right\},$$

$$L \mapsto \hat{L}.$$

- Finally, as the étale topology does not see inseparable extensions, we have an equivalence of categories

$$\left\{\text{finite separable extensions of } \mathbb{F}_q((X))\right\} \xrightarrow{\sim} \left\{\text{finite separable extensions of } \mathbb{F}_q((X))^{\mathrm{perf}}\right\},$$

$$L \mapsto L^{\mathrm{perf}}.$$

This gives the half of the desired properties for K_∞:

Proposition 5.1. *For $F = \mathbb{F}_q((X))$, $G_{K_\infty} \cong G_F$.*

What about the other half?

- We note that G_K acts on $\mathbb{C}_p$, $\mathbb{C}_p^\flat \supset \widehat{K}_\infty^\flat \supset F$, $\mathbb{C}_p^\flat \supset F^s$.
- Thus, G_K acts on $W(\mathbb{C}_p^\flat)$ and $W(\mathcal{O}_{\mathbb{C}_p^\flat})$.
- The G_K action on $\mathbb{C}_p^\flat$ preserves F^s, and the restriction of G_K-action to F^s induces the canonical $G_F \cong G_{K_\infty} \subset G_K$ on F^s.
- Let $\mathcal{O}_\mathcal{E}$ be our Cohen ring, $\mathcal{O}_\mathcal{E} = W(\mathbb{F}_q)((X))^\wedge$. Indeed, this embeds into $W(\mathbb{C}_p^\flat)$, with Γ and φ-stable image.
  - If $K/\mathbb{Q}_p$ is unramified, the map can be given via

$$X \mapsto [(\varepsilon_0, \varepsilon_1, \cdots)] - 1.$$

    The reason why we take Teichmuller lift is because we cannot take any lift, as we want X to be topologically nilpotent on the LHS.
  - The reason why the image is Γ and φ-stable is because you explicitly know how Γ and Frobenius act on this thing, namely $\gamma.X = (1+X)^{\chi(\gamma)} - 1 \in W(k)((X))^\wedge$ and $\varphi(X) = (1+X)^p - 1 \in W(k)((X))^\wedge$.
  - For $K/\mathbb{Q}_p$ a general finite extension, one uses the maximal unramified subextension and draw the same conclusion (but with less explicit map).
- $\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}$, the p-adic completion of the maximal unramified extension of $\mathcal{O}_\mathcal{E}$ in $W(\mathbb{C}_p^\flat)$, is stable under G_K and Frobenius, and the restriction of G_K-action on $\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}$ to $G_F \cong G_{K_\infty} \subset G_K$ is the canonical action.

Thus we have the desired other half. Namely, we can use this specific K_∞ for the abstract theory of (φ, Γ)-modules.

Theorem 5.1. *There are bijections*

$$\left\{\begin{array}{c}\text{continuous}\\ \text{representations of}\\ G_K \text{ on finitely}\\ \text{generated}\\ \mathbb{Z}_p\text{-modules}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}(\varphi,\Gamma)\text{-modules}\\ \text{over } \mathcal{O}_\mathcal{E}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}(\varphi, G_K)\text{-}\\ \text{modules over}\\ \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}\end{array}\right\} = \left\{\begin{array}{c}\varphi\text{-module over } \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}\\ \text{with continuous}\\ \text{semilinear}\\ G_K\text{-action}\\ \text{commuting with } \varphi\end{array}\right\}.$$

6. **φ-modules and the Fargues-Fontaine curve.**
Let F be a perfectoid field of characteristic p.

Definition 6.1. *Let $A_{\mathrm{inf}} = A_{\mathrm{inf}}(F) := W(\mathcal{O}_F)$, equipped with Frobenius.*

Choose a pseudouniformizer $\varpi \in \mathcal{O}_F$.

Definition 6.2. *Let*
$$Y_F = \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})\backslash V(p[\varpi]),$$
an adic space, independent of choice of ϖ, equipped with an action φ such that $\varphi^{\mathbb{Z}}$ acts totally discontinuously and freely.

We will try to justify this construction for a while.

- As a set, $\mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$ is the set of continuous multiplicative valuations $v : A_{\mathrm{inf}} \to \Gamma_v \cup \{0\}$ such that $v(f) \le 1$ for all $f \in A_{\mathrm{inf}}$, where Γ_v is a totally ordered abelian group with $0 < \gamma$ for all $\gamma \in \Gamma_v$. Here, continuity means, for all $\gamma \in \Gamma_v$, $\{f \in A_{\mathrm{inf}} \mid v(f) \le \gamma\} \subset A_{\mathrm{inf}}$ is open (for the $(p, [\varpi])$-adic topology on A_{inf}).
- The set $V(p[\varpi])$ is $\{v \in \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) \mid v \text{ factors through } A_{\mathrm{inf}}/p[\varpi]\}$.
- We want to make Y_F into a locally ringed space as follows. Let
$$B^b = A_{\mathrm{inf}}[\frac{1}{p[\varpi]}] = \left\{ \sum_{n \gg -\infty} [x_n]p^n \in W(F)[1/p] \mid x_n \text{ is bounded} \right\}.$$
This should be thought of as (bounded) functions on Y_F. For $0 \le \rho \le 1$, define a norm $|-|_\rho$ on B^b as
$$\left| \sum_{n \gg -\infty} [x_n]p^n \right|_\rho := \max_n |x_n|\rho^n \in \mathbb{R}_{\ge 0}.$$
For a finite union of closed intervals $I \subset [0,1]$, define $B_I = B_{F,I}$ to be the completion of B^b with rrespect to the family of norms $\{|-|_\rho\}_{\rho \in I}$.

  Theorem 6.1 (Fargues-Fontaine)**.** *If $1 \notin I$, and if the endpoints of I are in $|F^\times|$, then B_I is a PID.*

  Definition 6.3. *We define*
  $$Y_{F,I} = \mathrm{Spa}(B_I, B_I^\circ) = \{v \in Y_F \text{ that extends to a continuous valuation on } B_I\}.$$

  Then, $Y_{F,I_1} \cap Y_{F,I_2} = Y_{F,I_1 \cap I_2}$ and $Y_F = \bigcup_{I \subset (0,1)} Y_{F,I}$. The topology on Y_F can be defined as the topology generated by regarding $Y_{F,I}$ as open subsets of Y_F. Furthermore, one can equip Y_F with a structure sheaf $\mathcal{O}_{Y_F}$ such that $\Gamma(Y_{F,I}, \mathcal{O}_{Y_F}) = B_I$, which makes Y_F into a locally ringed space.
- For $f \in B^b$, $|\varphi(f)|_{\rho^p} = |f|_\rho^p$, so φ extends to $B_{[a,b]} \xrightarrow{\sim} B_{[a^p,b^p]}$, so $\varphi : Y_{F,[a^p,b^p]} \xrightarrow{\sim} Y_{F,[a,b]}$, and $\varphi : Y_F \xrightarrow{\sim} Y_F$, giving a totally disconnected and free action of $\mathbb{Z}$.

Now we have a locally ringed space with totally disconnected free action by $\mathbb{Z}$, the following makes sense.

Definition 6.4. *The (adic)* ***Fargues-Fontaine curve*** *is the locally ringed space $X_F = Y_F/\varphi^{\mathbb{Z}}$.*

Remark 6.1. If $F = \overline{F}$, then there is a bijection
$$\{\text{“classical points” of } Y_F\} = \{(p - [a]) \subset A_{\mathrm{inf}}, a \in \mathcal{O}_F, 0 < |a| < 1\},$$

where a classical point means a point $v \in \mathrm{Spa}(B_I, B_I^\circ)$ for some nice I which factors through a maximal ideal of B_I (this comes from the PID-ness of B_I's). Then, there is a bijection

$$\{(p - [a]) \subset A_{\mathrm{inf}}, a \in \mathcal{O}_F, 0 < |a| < 1\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{perfectoid fields } E \\ \text{of characteristic } 0, \\ E^\flat \cong F \end{array} \right\},$$
$$(p - [a]) \mapsto W(\mathcal{O}_F)/(p - [a]).$$

In a certain senes, Y_F is a "punctured open unit disc," with the coordinate function being p.

Let $B = \Gamma(Y_F, \mathcal{O}_{Y_F}) = \varprojlim_I B_I$.

Theorem 6.2 (Fargues-Fontaine)**.** *We have* $\Gamma(X_F, \mathcal{O}_{X_F}) = B^{\varphi=\mathrm{id}} = \mathbb{Q}_p$.

Now we want to make sense of the notion of "vector bundles over X_F". Note that, at least formally, a vector bundle over X_F must be a vector bundle V on Y_F with $\varphi^* V \xrightarrow{\sim} V$.

Theorem 6.3. *The embedding*

$$\left\{ \begin{array}{c} \varphi\text{-module over } B, \\ \text{finite projective over} \\ B \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} \text{vector bundle } V \text{ on} \\ Y_F \text{ with } \varphi^* V \xrightarrow{\sim} V \end{array} \right\},$$

is a bijection. That is, a vector bundle V on Y_F is finite projective over B.

Remark 6.2. This is not an equivalence of categories, in particular this is not full (the Hom space on the vector bundle side is huge).

An advantage of this approach is that we can use geometric intuition.

Proposition 6.1. *Any closed line bundle $\mathscr{L}$ on X_F is associated to a divisor $D = \sum_{i<\infty} n_i x_i$, $n_i \in \mathbb{Z}$, and $x_i \in X_F$ classical points. For a classical point x, $\mathcal{O}(-x)$ is the ideal sheaf of functions vanishing at x.*

Definition 6.5. *If $\mathscr{L} \cong (\sum n_i x_i)$ is a line bundle, let* $\deg \mathscr{L} = \sum n_i$. *If V is a vector bundle on X_F, define* $\deg V = \deg(\Lambda^{\mathrm{rk}\, V} V)$, *and* $\mu(V) = \frac{\deg(V)}{\mathrm{rk}(V)}$, *the* ***slope*** *of V.*

Definition 6.6. *A vector bundle V is called* ***semistable of slope*** *μ if $\mu(V) = \mu$, and for any subbundle $V' \subset V$, $\mu(V') \leq \mu(V)$.*

Theorem 6.4 (Fargues-Fontaine)**.**
(1) Every vector bundle on X_F decomposes as a direct sum of semistable vector bundles.
(2) For every $\mu \in \mathbb{Q}$, there is a unique indecomposable semistable vector bundle V_μ of slope μ.

We review the construction of V_μ.

Definition 6.7. *Let $\mu = \frac{r}{s}$ where $r, s \in \mathbb{Z}$, $s \geq 1$, $(r, s) = 1$. Let*

$$D_\mu = (W(\overline{\mathbb{F}}_p)[1/p])^s.$$

This can be made into a φ-module over $W(\overline{\mathbb{F}}_p)[1/p]$ by giving

$$\varphi = \begin{pmatrix} 0 & \cdots & \cdots & 0 & p^r \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1 & 0 \end{pmatrix},$$

with respect to the canonical basis. Let $\mathcal{D}_\mu$ be the pullback (as a φ-module) of D_μ into $A_{\inf}[\frac{1}{p[\varpi]}]$ along

$$\begin{tikzcd} A_{\inf} \arrow[d, two heads] & W(\overline{\mathbb{F}}_p) \arrow[l] \arrow[d, two heads] \\ \mathcal{O}_F & \overline{\mathbb{F}}_p \arrow[l] \end{tikzcd}$$

This defines a vector bundle $\mathcal{D}_\mu$ on Y_F with $\varphi^*\mathcal{D}_\mu \xrightarrow{\sim} \mathcal{D}_\mu$, and this descends to a vector bundle V_μ on X_F.

Definition 6.8 (Rational subdomains)**.** *If $f_1, \cdots, f_n, g \in B_I$, $(f_1, \cdots, f_n, g) = (1)$, we deinfe*

$$U\left(\frac{f_1, \cdots, f_n}{g}\right) := \{x \in Y_{F,I} \mid |f_i(x)| \le |g(x)|\},$$

where $x = v \in Y_{F,I}$ is a valuation, $|f(x)| = v(f)$.

One could easily check that if $I' \subset I$ is a subinterval with endpoints in $|F^\times|$, $Y_{F,I'} \subset Y_{F,I}$ is of this form.

Example 6.1. For an algebraically closed field F, a classical point $x \in Y_F$ is of form $(p - [a])$ for $a \in \mathcal{O}_F$ with $0 < |a| < 1$. Then, $\mathcal{O}_E := A_{\inf}/(p - [a])$ defines a perfectoid field $E = \mathcal{O}_E[1/p]$ where $E^\flat \cong F$ via $\mathcal{O}_E/p \cong \mathcal{O}_F/a$. For $f \in A_{\inf}$, let $\overline{f}$ be the image of f in $\mathcal{O}_E/p = \mathcal{O}_F/a$. If $\overline{f} \neq 0$, then for a lift $\overline{f}' \in \mathcal{O}_F$, we have $|f(x)| = |\overline{f}'|$, which should be independent of the lift.

The classification of vector bundles on X_F, for F algebraically closed, implies

Corollary 6.1. *There is a bijection*

$$\left\{\begin{matrix} \textit{semistable vector} \\ \textit{bundles on } X_F \textit{ of} \\ \textit{slope 0} \end{matrix}\right\} \xrightarrow{\sim} \left\{\textit{finite dimensional } \mathbb{Q}_p\textit{-vector spaces}\right\},$$

$$V \mapsto \Gamma(X_F, V),$$
$$V \otimes_{\mathbb{Q}_p} \mathcal{O}_{X_F} \leftarrow\!\shortmid V.$$

Remark 6.3. In general you should not expect $\Gamma(X_F, V)$ to be finite-dimensional.

Remark 6.4. For an arbitrary F, the curve X_F has an algebraic variant

$$X_F^{\mathrm{alg}} = \operatorname{Proj} \bigoplus_{d \ge 0} B^{\varphi = p^d}.$$

This enjoys a strong finiteness property.

Theorem 6.5 (Fargues-Fontaine)**.** *(1) X_F^{alg} is a regular 1-dimensional noetherian scheme.*

(2) There is a morphism of locally ringed spaces $X_F \longrightarrow X_F^{\mathrm{alg}}$ which induces a bijection

$$\{\textit{classical points of } X_F\} \xrightarrow{\sim} \left\{\textit{closed points of } X_F^{\mathrm{alg}}\right\},$$

which further induces a bijection of completed local rings at classical points.

Theorem 6.6 (GAGA; Kedlaya-Liu)**.** *For F algebraically closed, pullback along $X_F \longrightarrow X_F^{\mathrm{alg}}$ gives an equivalence*

$$\left\{\textit{vector bundles on } X_F^{\mathrm{alg}}\right\} \xrightarrow{\sim} \{\textit{vector bundles on } X_F\}.$$

Remark 6.5. For F not algebraically closed, this may not hold.

The reason we might be interested in schematic Fargues-Fontaine curve is because we can give a vector bundle on it by giving a vector bundle on an open subset, the complement of a point, and a modification datum at that point. This does not work in the world of adic spaces, because functions can have essential singularity around a point.

Definition 6.9. *Let $\infty \in X_F^{\mathrm{alg}}$ be any closed point. Define*

$$B_e = \Gamma(X_F^{\mathrm{alg}} \backslash \{\infty\}, \mathcal{O}_{X_F^{\mathrm{alg}}}) = B[1/t]^{\varphi=\mathrm{id}},$$

for some $t \in B$ such that $V(t) = \mathrm{pr}^{-1}(\infty) \subset Y_F$.

Thus, we have an equivalence

$$\left\{\text{vector bundles on } X_F^{\mathrm{alg}}\right\} \xrightarrow{\sim} \left\{\begin{matrix} \text{finite projective} \\ B_e\text{-module } M \text{ with} \\ \widehat{\mathcal{O}}_{X_F^{\mathrm{alg}},\infty}\text{-lattice} \\ \Lambda \subset M \otimes_{B_e} \widehat{\mathcal{O}}_{X_F^{\mathrm{alg}},\infty}[1/t] \end{matrix}\right\},$$

where $t \in \widehat{\mathcal{O}}_{X_F^{\mathrm{alg}},\infty}$ is a pseudouniformizer.

7. Equivariant vector bundles.

Let $K/\mathbb{Q}_p$ be a finite extension, and $\mathbb{C}_p = \widehat{\overline{K}}$ which has an action by G_K. Then, G_K acts on $\mathbb{C}_p^\flat$ and on $W(\mathcal{O}_{\mathbb{C}_p^\flat}) = A_{\mathrm{inf}}(\mathbb{C}_p^\flat)$, thus on $Y_{\mathbb{C}_p^\flat}$ and $X_{\mathbb{C}_p^\flat}$.

Corollary 7.1. *There is a bijection,*

$$\left\{\begin{matrix} \textit{continuous} \\ \textit{representations of} \\ G_K \textit{ on finite} \\ \textit{dimensional} \\ \mathbb{Q}_p\textit{-vector spaces} \end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix} G_K\textit{-equivariant} \\ \textit{vector bundle on} \\ X_{\mathbb{C}_p^\flat}\textit{, semistable of} \\ \textit{slope 0} \end{matrix}\right\},$$

$$V \mapsto V \otimes_{\mathbb{C}_p} \mathcal{O}_{X_{\mathbb{C}_p^\flat}},$$

where G_K acts diagonally on the RHS.

Remark 7.1. If V is an equivariant vector bundle, $U \subset X_{\mathbb{C}_p^\flat}$ open and $H \subset G_K$ is the stabilizer of U, then H is asked to act continuously on $\Gamma(U, V)$.

Corollary 7.2. *Let $\theta : W(\mathcal{O}_{\mathbb{C}_p^\flat}) \to \mathcal{O}_{\mathbb{C}_p}$ be the continuous surjective map which defines a classical point $x_0 \in Y_{\mathbb{C}_p^\flat}$. Let $\infty \in X_{\mathbb{C}_p^\flat}$ be its image. Then, ∞ is stabilized by G_K, and this endows a G_K-action on $B_e = \Gamma(X_{\mathbb{C}_p^\flat}^{\mathrm{alg}} \backslash \{\infty\}, \mathcal{O}_{X_{\mathbb{C}_p^\flat}^{\mathrm{alg}}})$. Then,*

$$\left\{\begin{matrix} G_K\textit{-equivariant} \\ \textit{vector bundles on} \\ X_{\mathbb{C}_p^\flat} \end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix} (M, \Lambda) \textit{ where } M \textit{ is finite projective} \\ B_e\textit{-module with semilinear } G_K\textit{-action,} \\ \Lambda \subset M \otimes_{B_e} \widehat{\mathcal{O}}_{X_{\mathbb{C}_p^\flat},\infty}[1/t] \textit{ is a } G_K\textit{-stable} \\ \widehat{\mathcal{O}}_{X_{\mathbb{C}_p^\flat},\infty}\textit{-lattice} \end{matrix}\right\},$$

is a bijection.

Definition 7.1. *We write*
$$B_{\mathrm{dR}}^{+} = \widehat{\mathcal{O}}_{X_{\mathbb{C}_p^\flat},\infty} = \widehat{\mathcal{O}}_{Y_{\mathbb{C}_p^\flat},x_0},$$
which is the completion of $W(\mathcal{O}_{\mathbb{C}_p^\flat})[1/p]$ *with respect to* $\ker\theta$.

Example 7.1. Let us denote $\underline{\varepsilon} = (\varepsilon_n, n \geq 0) \in \mathcal{O}_{\mathbb{C}_p^\flat}$, and let $t = \log([\underline{\varepsilon}])$, which means
$$t = \sum_{n\geq 1}(-1)^{n+1}\frac{([\underline{\varepsilon}]-1)^n}{n},$$
which does not converge in A_{inf} or B^b. However, it converges in all B_I with $I \subset (0,1)$, so it converges in B. In particular, t is a uniformizer in $\widehat{\mathcal{O}}_{Y_{\mathbb{C}_p^\flat},x_0}$, $\varphi(t) = pt$, and for $g \in G_K$, $gt = \chi(g)t$. Then, $V(t) = \mathrm{pr}^{-1}(\infty) \subset Y_F$.

Consider the following table, which summarizes corresponding objects in different settings.

Equivariant v.b. on $X_{\mathbb{C}_p^\flat}$	φ-module on B	φ-module on $W(\overline{\mathbb{F}}_p)[1/p] = \breve{\mathbb{Q}}_p$
$\mathcal{O}_{X_{\mathbb{C}_p^\flat}}$	$(B, \varphi(1) = 1)$	$D_0 = (\breve{\mathbb{Q}}_p, \varphi(1) = 1)$
$\mathcal{O}_{X_{\mathbb{C}_p^\flat}}(\infty)$	$t^{-1}B$ $(\cong (B, \varphi(1) = p^{-1}))$	$D_{-1} = (\breve{\mathbb{Q}}_p, \varphi(1) = p^{-1})$

In particular, as $\mathcal{O}_{X_{\mathbb{C}_p^\flat}}$ is semistable of slpoe zero, we get a trivial G_K-representaiton on $\mathbb{Q}_p$, whereas as $\mathcal{O}_{X_{\mathbb{C}_p^\flat}}(\infty)$ is semistable of slope 1, $\Gamma(X_{\mathbb{C}_p^\flat}, \mathcal{O}_{X_{\mathbb{C}_p^\flat}}(\infty)) = (t^{-1}B)^{\varphi=\mathrm{id}} = B^{\varphi=p}$ is infinite-dimensional. In particular, $\mathrm{Hom}(D_0, D_{-1}) = 0$, but $\mathrm{Hom}(\mathcal{O}_{X_{\mathbb{C}_p^\flat}}, \mathcal{O}_{X_{\mathbb{C}_p^\flat}}(\infty)) = B^{\varphi=p}$ is huge.

On the other hand, if we consider the line bundle $\mathscr{L}$ on $X_{\mathbb{C}_p^\flat}$ which corresponds to a φ-module over B, $(t^{-1}B, \phi = p\varphi)$, then $\mathscr{L}$ is semistable of slope zero and the corresponding Galois representation is $t^{-1}\mathbb{Q}_p = (t^{-1}B)^{\phi=\mathrm{id}}$, with G_K acting via χ^{-1}.

8. Galois descent, decompletion and deperfection.

Let F be any perfectoid field of characteristic p. Let $C = \widehat{\overline{F}} = \widehat{F^s}$ which has an action of G_F. Then, $F \longrightarrow C$ induces $X_C \longrightarrow X_F$, $X_C^{\mathrm{alg}} \longrightarrow X_F^{\mathrm{alg}}$. These morphisms are equivariant for G_F-action.

Theorem 8.1 ((Pro-)Galois descent)**.** *The natural map*
$$\left\{\text{vector bundles on } X_F^{\mathrm{alg}}\right\} \longrightarrow \left\{\begin{array}{c} G_F\text{-equivariant} \\ \text{vector bundle on} \\ X_C^{\mathrm{alg}} \end{array}\right\},$$
is an equivalence of categories.

Now let $K/\mathbb{Q}_p$ be a finite extension, K_∞/K be a Galois extension, $\Gamma = \mathrm{Gal}(K_\infty/K)$ such that $\widehat{K}_\infty$ is perfectoid.

Corollary 8.1. *There is a natural equivalence of categories*
$$\left\{\begin{array}{c} \Gamma\text{-equivariant} \\ \text{vector bundles on} \\ X_{\widehat{K}_\infty^\flat}^{\mathrm{alg}} \end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c} G_K\text{-equivariant} \\ \text{vector bundles on} \\ X_{\mathbb{C}_p^\flat}^{\mathrm{alg}} \end{array}\right\}.$$

Remark 8.1. We have an equivalence
$$\left\{\begin{array}{c} \text{vector bundles on} \\ X_{\widehat{K}_\infty^\flat} \end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c} \varphi\text{-modules over} \\ B = \Gamma(Y_{\widehat{K}_\infty^\flat}, \mathcal{O}_{Y_{\widehat{K}_\infty^\flat}}) \end{array}\right\}.$$

By restriction, one sees that this is equivalent to

$$\xrightarrow{\sim}\left\{\begin{array}{c}\varphi\text{-modules over}\\ B^{(0,r]}\end{array}\right\}=\left\{\begin{array}{c}B^{(0,r]}\text{-module } M,\\ \text{with } \varphi^*M\xrightarrow{\sim}\\ M\otimes_{B^{(0,r]}}B^{(0,r^p]}\end{array}\right\}\xrightarrow{\sim}\left\{\begin{array}{c}\varphi\text{-module over}\\ \lim_{r\to 0}B^{(0,r]}=:\\ \widetilde{B}^{\dagger}_{\mathrm{rig},K}\end{array}\right\}.$$

A similar statement holds for (φ,Γ)-modules.

Let Δ^* be the punctured open unit disc over $W(\mathbb{F}_q)[1/p]$, where $\mathbb{F}_q$ is residue field of K_∞. Then $\Gamma(\Delta^*,\mathcal{O}_{\Delta^*})\hookrightarrow B$; if $K/\mathbb{Q}_p$ is unramified, then one can explicitly describe this as $X\mapsto[\underline{\varepsilon}]-1$ where X is the coordinate function on Δ^*. Again, each finite level annulus is not stable under φ and Γ, but the limit of these towards boundary is:

$$\mathcal{R}=\lim_{\rho\to 1^-}\Gamma(\{\rho\le|x|<1\},\mathcal{O}_{\Delta^*})\subset\widetilde{B}^{\dagger}_{\mathrm{rig},K},$$

is stable under φ,Γ (although φ is not injective). This is usually referred as the **Robba ring** (or $B^{\dagger}_{\mathrm{rig},K}$ in Berger's terminology). This p-adic limit process gives a compactification by adding characteristic p points (which is why adic space is useful).

Now let K_∞ be the cyclotomic extension. Then, much more can be said, which is specific to cyclotomic extension.

Theorem 8.2 (Decompletion and deperfection)**.**

$$\{(\varphi,\Gamma)\text{-}\textit{modules over }\mathcal{R}\}\xrightarrow{\sim}\left\{(\varphi,\Gamma)\text{-}\textit{modules over }\widetilde{B}^{\dagger}_{\mathrm{rig},K}\right\},$$

is a bijection.

This is really only specific to the cyclotomic extension.

9. Crystalline representations and Fontaine's period rings.

How do we produce G_K-equivariant vector bundles on $X_{\mathbb{C}_p^\flat}$, semistable of slope zero? We saw that a φ-module over $\breve{\mathbb{Q}}_p$ produces a vector bundle on $X_{\mathbb{C}_p^\flat}$. Thus, a G_K-equivariant φ-module over $\breve{\mathbb{Q}}_p$ will produce a G_K-equivariant vector bundle on X. As $\breve{\mathbb{Q}}_p^{G_K}=W(k)[1/p]=K_0$, a φ-module over K_0 will produce a G_K-equivariant vector bundle on $X_{\mathbb{C}_p^\flat}$. Let's call this functor

$$(D,\varphi)\mapsto V(D,\varphi).$$

Of course, there is no guarantee that $V(D,\phi)$ is semistable of slope zero. We can instead try to modify the vector bundle at ∞ and hope to get something semistable of slope zero. Certainly, for a $B^+_{\mathrm{dR}}=\widehat{\mathcal{O}}_{X_{\mathbb{C}_p^\flat},\infty}$-lattice Λ in $V(D,\varphi)\otimes\widehat{\mathcal{O}}_{X_{\mathbb{C}_p^\flat},\infty}[1/t]$, we can modify $V(D,\varphi)$ by Λ, and get a vector bundle $V(D,\varphi,\Lambda)$. As G_K acts on $V(D,\varphi)\otimes\widehat{\mathcal{O}}_{X_{\mathbb{C}_p^\flat},\infty}[1/t]$, if Λ is a G_K-stable lattice, then $V(D,\varphi,\Lambda)$ is a G_K-equivariant vector bundle. If $V(D,\varphi,\Lambda)$ is semistable of slope zero, then $\Gamma(X,V(D,\varphi,\Lambda))$ is a continuous G_K-representation on a finite dimensional $\mathbb{Q}_p$-vector space, with dimension $\operatorname{rk}V(D,\varphi)=\dim_{K_0}D$.

Crystalline representations are precisely the G_K-representations which can arise in this way.

Definition 9.1. *Let A_{cris} be the p-adic completion of the divided power envelope of $A_{\mathrm{inf}}=A_{\mathrm{inf}}(\mathcal{O}_{\mathbb{C}_p^\flat})$ with respect to* $\ker(A_{\mathrm{inf}}\xrightarrow{\theta}\mathcal{O}_{\mathbb{C}_p})$*. Namely,*

$$A_{\mathrm{cris}}=\left(A_{\mathrm{inf}}\left[\frac{\xi^n}{n!},n\ge 1\right]\right)^\wedge,$$

where ξ is a generator of the principal ideal $\ker\theta$.

Let $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p]$, and $B_{\mathrm{cris}} = B^+_{\mathrm{cris}}[1/t] = A_{\mathrm{cris}}[1/t]$, where $t = \log[\underline{\varepsilon}]$.

By construction, one can easily check the following

Proposition 9.1.

- *The G_K-action on A_{inf} extends to A_{cris}, B^+_{cris} and B_{cris}.*
- *The Frobenius $\varphi : A_{\mathrm{inf}} \xrightarrow{\sim} A_{\mathrm{inf}}$ extends to $\varphi : A_{\mathrm{cris}} \hookrightarrow A_{\mathrm{cris}}$ and $\varphi : B^{(+)}_{\mathrm{cris}} \longrightarrow B^{(+)}_{\mathrm{cris}}$, commuting with G_K-action.*
- *There is a natural G_K-equivariant injective ring homomorphism $B^+_{\mathrm{cris}} \hookrightarrow B^+_{\mathrm{dR}}$.*

Remark 9.1. The injection $B^+_{\mathrm{cris}} \hookrightarrow B^+_{\mathrm{dR}}$ is justified by showing that an expansion in B^+_{cris} converges in B^+_{dR}. This is a subtle issue, as the topology of B^+_{cris} is p-adic, whereas the topology of B^+_{dR} is t-adic (or valuation topology). Thus, the injective ring homomorphism is not really compatible with topology.

Definition 9.2. *The $\mathbb{Z}$-graded filtration $\mathrm{Fil}^i B_{\mathrm{cris}}$ is defined by $\mathrm{Fil}^i B_{\mathrm{cris}} = B_{\mathrm{cris}} \cap t^i B^+_{\mathrm{dR}}$.*

Remark 9.2. Even though t makes sense in B^+_{cris}, $\mathrm{Fil}^i B_{\mathrm{cris}}$ is not the same as $t^i B^+_{\mathrm{cris}}$.

Proposition 9.2. *The rings B_{cris} and B_{dR} are G_K-regular. Also,*

- $B_{\mathrm{cris}}^{G_K} = K_0$,
- $B_{\mathrm{dR}}^{G_K} = K$,
- $(B^+_{\mathrm{cris}})^{\varphi=\mathrm{id}} = bQ_p$.

Definition 9.3. *A continuous G_K-representation V over a finite dimensional $\mathbb{Q}_p$-vector space is called* ***de Rham*** *if V is B_{dR}-admissible, and* ***crystalline*** *if V is B_{cris}-admissible. We define functors*

$$D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K},$$

which is a finite-dimensional K-vector space, and

$$D_{\mathrm{cris}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_K},$$

which is a finite-dimensional K_0-vector space.

Remark 9.3. As $B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$ Galois-equivariantly, crystalline representations are automatically de Rham. In this case,

$$D_{\mathrm{cris}}(V) \otimes_{K_0} K = (D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{dR}})^{G_K} = (D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} \otimes_{B_{\mathrm{cris}}} B_{\mathrm{dR}})^{G_K} = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \otimes_{B_{\mathrm{cris}}} B_{\mathrm{dR}})^{G_K} = D_{\mathrm{dR}}(V).$$

But $D_{\mathrm{cris}}(V)$ and $D_{\mathrm{dR}}(V)$ contain more structures. As $\varphi : B_{\mathrm{cris}} \longrightarrow B_{\mathrm{cris}}$ and $t^i B^+_{\mathrm{dR}} \subset B_{\mathrm{dR}}$ are compatible with G_K-action, we have

$$\phi : D_{\mathrm{cris}}(V) \longrightarrow D_{\mathrm{cris}}(V),$$

which is φ-linear bijection, and a $\mathbb{Z}$-filtration of K-vector spaces

$$\mathrm{Fil}^i D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} t^i B^+_{\mathrm{dR}})^{G_K},$$

which is exhaustive and separated. Based on this, we have the following definition.

Definition 9.4. *A* ***filtered φ-module*** *for K is a finite dimensional K_0-vector space D with a φ-linear isomorphism $\phi : D \longrightarrow D$ and an exhaustive separated filtartion $\mathrm{Fil}^i D_K \subset D_K = D \otimes_{K_0} K$ of sub-K-vector spaces.*

We have seen that D_{cris} gives a functor

$$D_{\mathrm{cris}} : \{\text{crystalline representations}\} \longrightarrow \{\text{filtered } \varphi\text{-modules}\}.$$

Theorem 9.1 (Fontaine, Colmez-Fontaine, Berger, Kedlaya, Kisin). *(1) D_{cris} is fully faithful.*

*(2) The essential image of D_{cris} is given by the **weakly admissible** objects, and a quasi-inverse to D_{cris} is given by*

$$V_{\mathrm{cris}}(D,\phi,\mathrm{Fil}^{\bullet}) := \mathrm{Fil}^0((D\otimes_{K_0}B_{\mathrm{cris}})^{\varphi=\mathrm{id}}) = (D\otimes_{K_0}B_{\mathrm{cris}})^{\varphi=\mathrm{id}}\cap \mathrm{Fil}^0(D_K\otimes_K B_{\mathrm{dR}})\subset D\otimes_{K_0}B_{\mathrm{dR}} = D_K\otimes_K B_{\mathrm{dR}}.$$

Definition 9.5. *A filtered φ-module $(D,\phi,\mathrm{Fil}^{\bullet})$ is **weakly admissible** if it is semistable of slope zero for the slope theory defined by*

$$\mu(D,\phi,\mathrm{Fil}^{\bullet}) = v_p(\det\phi) - \sum_{i\in\mathbb{Z}} i\dim_K \mathrm{gr}_i D_K.$$

Remark 9.4. Let X/K be a proper smooth algebraic variety. Then, $V = H^i_{\text{ét}}(X_{\overline{K}},\mathbb{Q}_p)$ is a finite-dimensional continuous G_K-representation which is always de Rham. There is a canonical G_K-equivariant isomorphism

$$H^i_{\text{ét}}(X_{\overline{K}},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p}B_{\mathrm{dR}}\xrightarrow{\sim} H^i_{\mathrm{dR}}(X/K)\otimes_K B_{\mathrm{dR}}.$$

This gives a canonical identification $D_{\mathrm{dR}}(V) = H^i_{\mathrm{dR}}(X/K)$. This in fact identifies the filtration on $D_{\mathrm{dR}}(V)$ and the Hodge filtration on $H^i_{\mathrm{dR}}(X/K)$.

If X furthermore has good reduction so that there is a smooth proper model $\mathcal{X}/\mathcal{O}_K$, then $H^i_{\text{ét}}(X_{\overline{K}},\mathbb{Q}_p)$ is crystalline, and there is a canonical G_K and φ-equivariant isomorphism

$$H^i_{\text{ét}}(X_{\overline{K}},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p}B_{\mathrm{cris}}\cong H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))\otimes_{W(k)}\otimes B_{\mathrm{cris}},$$

so that $D_{\mathrm{cris}}(V) = H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p]$ as φ-modules over K_0.

We want to relate the classical theory to the geometric theory of Fargues-Fontaine curve alluded earlier.

Proposition 9.3. *One has identifications*

$$(B^+_{\mathrm{cris}}[1/t])^{\varphi=\mathrm{id}} = B^{\varphi=\mathrm{id}}_{\mathrm{cris}} = B_e = (B[1/t])^{\varphi=\mathrm{id}},$$

$$(B^+_{\mathrm{cris}})^{\varphi=p^d} = B^{\varphi=p^d}.$$

In particular,

$$X^{\mathrm{alg}}_{\mathbb{C}_p^\flat} = \mathrm{Proj}\bigoplus_d B^{\varphi=p^d} = \mathrm{Proj}\bigoplus_d (B^+_{\mathrm{cris}})^{\varphi=p^d}.$$

From this, one sees that there is a functor

$$\{\varphi\text{-module over }K_0\}\longrightarrow\{\varphi\text{-module over }B^+_{\mathrm{cris}}\}\xrightarrow{\mathrm{gr}^{\bullet}}\left\{\begin{matrix}\text{graded}\\ \bigoplus_d (B^+_{\mathrm{cris}})^{\varphi=p^d}\text{-}\\ \text{module}\end{matrix}\right\}\longrightarrow\left\{\text{vector bundle on }X^{\mathrm{alg}}_{\mathbb{C}_p^\flat}\right\},$$

and this turns out to coincide with $(D,\phi)\mapsto V(D,\phi)$ we built earlier.

Lemma 9.1. *There is a bijection*

$$\left\{\begin{matrix}\textit{exhaustive}\\ \textit{separable filtrations}\\ \textit{of } D_K\end{matrix}\right\}\xrightarrow{\sim}\left\{\begin{matrix}G_K\textit{-stable}\\ B^+_{\mathrm{dR}}\textit{-lattice}\\ \Lambda\subset V(D,\varphi)\otimes B_{\mathrm{dR}} = \\ D_K\otimes_K B_{\mathrm{dR}}\end{matrix}\right\},$$

$$(\mathrm{Fil}^i D_K)_i\mapsto \sum \mathrm{Fil}^i D_K\otimes t^{-i}B^+_{\mathrm{dR}}.$$

We now compute what $V(D,\phi,\Lambda)$ is, for Λ coming from the filtration of a filtered φ-module $(D,\phi,\mathrm{Fil}^\bullet)$:

$$\begin{aligned}\Gamma(X^{\mathrm{alg}}_{\mathbb{C}_p^\flat}, V(D,\phi,\Lambda)) &= \Gamma(X^{\mathrm{alg}}\backslash\{\infty\}, V(D,\phi))\cap\Lambda\\ &= (D\otimes_{K_0} B_{\mathrm{cris}})^{\varphi=\mathrm{id}}\cap\Lambda\\ &= V_{\mathrm{cris}}(D,\phi,\mathrm{Fil}^\bullet).\end{aligned}$$

We see that the slope theory of filtered φ-modules coincides with the slope theory of vector bundles over algebraic Fargues-Fontaine curve, and thus we get the following

Proposition 9.4. *The vector bundle $V(D,\phi,\Lambda)$ is semistable of slope zero if and only if $(D,\phi,\mathrm{Fil}^\bullet)$ is weakly admissible.*

Remark 9.5. If one fills out the details of the proof of this, this will actually give the full proof of Theorem 9.1.

References:

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1. **Objective.**

This lecture will be about illustrating why Taylor-Wiles-Kisin patching might be useful. Recall the so-called "mod p local Langlands correspondence" we have for $\mathrm{GL}_2(\mathbb{Q}_p)$:

$$\left\{ \begin{array}{c} \text{continuous} \\ \overline{\rho} : G_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{smooth admissible} \\ \mathrm{GL}_2(\mathbb{Q}_p)\text{-representation} \\ \text{over } \overline{\mathbb{F}}_p \end{array} \right\},$$

which is supposed to

- be compatible with characteristic zero p-adic local Langlands,
- and be compatible with cohomology of modular curves.

Our hope is that this correspondence generalizes to other groups, but so far this has been extremely mysterious, even for $\mathrm{GL}_2(F)$ for F the unramified quadratic extension of $\mathbb{Q}_p$.

The compatibility with modular curves means the following.

Theorem 1.1 (Emerton). *Let*

$$\varprojlim_t H^1_{\text{ét}}(Y(Np^t)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p) =: \widetilde{H}^1(N),$$

be the completed cohomology, where $N \geq 5$, $p \geq 5$ and $(N,p) = 1$. This has an action of $G_{\mathbb{Q}} \times \mathbb{T}(N) \times \mathrm{GL}_2(\mathbb{Q}_p)$, and in particular the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ is smooth. Then, the completed cohomology realizes the p-adic local Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$; namely, given a residual Galois representation $\overline{r} : G_{\mathbb{Q}} : \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, which is irreducible and odd (+ some technical assumptions), then

$$\mathrm{Hom}_{\mathbb{T}(N)[G_{\mathbb{Q}}]}(\overline{r}, H^1(N)_{\mathfrak{m}_{\overline{r}}}) \cong \pi_p(\overline{r}|_{G_{\mathbb{Q}_p}}),$$

where π_p is the mod p local Langlands correspondence.

The objective of the course is to produce the following **patching functor**.

$$M_\infty : \mathrm{Rep}^{\mathrm{cts}}_{\overline{\mathbb{Z}}_p}(\mathrm{GL}_2(\mathbb{Z}_p)) \longrightarrow \mathrm{Mod}(R^{\square}_{\overline{\rho}}[[x_1, \cdots, x_g]]),$$

which is

(1) exact,

(2) given an inertial type $\tau : I_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$,

$$\mathrm{Supp}(M_\infty(\sigma(\tau))) \subset R_{\overline{\rho}}^{(1,0),\tau}[[x_1, \cdots, x_g]],$$

which is a union of irreducible components, and furthermore $M_\infty(\sigma(\tau))$ is Cohen-Macaulay on each irreducible components, where σ is the inertial local Langlands correspondence; in particular, if $R_{\overline{\rho}}^{(1,0),\tau}[1/p]$ is connected, then we have a modularity lifting theorem,

(3) if V is a Serre weight for $\mathrm{GL}_2(\mathbb{F}_p)$, then

$$\left(\frac{M_\infty(V)}{(\mathfrak{a}, p)}\right)^\vee = \mathrm{Hom}(V, \widetilde{H}^1(N)_{\mathfrak{m}_{\overline{r}}}),$$

where $\mathfrak{a}$ is certain augmentation ideal; this gives Serre weight conjecture and Breuil-Mezard conjecture.

2. **Automorphic forms.**

We pick a CM field $F/F^+/\mathbb{Q}$, and for simplicity we assume that F/F^+ is unramified at all finite places. Let G be the unitary group over $\mathcal{O}_{F^+}$ defined as

$$G(R) = \{g \in \mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathcal{O}_{F^+}} R) \mid (g^c)^T = g^{-1}\},$$

where c means the complex conjugation $c \in \mathrm{Gal}(F/F^+)$. This splits over F, namely

$$G(\mathcal{O}_F) = \{(g_1, g_2) \in \mathrm{GL}_2(\mathcal{O}_F \times \mathcal{O}_F) \mid (g_2^T, g_1^T) = (g_1^{-1}, g_2^{-1})\} \xrightarrow{\iota, \sim} \mathrm{GL}_2(\mathcal{O}_F),$$

and there is a commutatitve diagram

$$\begin{tikzcd}
G \times_{\mathcal{O}_{F^+}} \mathcal{O}_F \arrow[r, "\iota"] \arrow[d, "\mathrm{id} \otimes c"] & \mathrm{GL}_{2,\mathcal{O}_F} \arrow[d, "g \mapsto (g^{-1})^T"] \\
G \times_{\mathcal{O}_{F^+}} \mathcal{O}_F \arrow[r, "\iota"'] & \mathrm{GL}_{2,\mathcal{O}_F}
\end{tikzcd}$$

At finite places, the group looks like the following. If $v = \tilde{v}\tilde{v}^c$ is a finite place of F^+ that splits in F, then

$$\begin{tikzcd}
G(\mathcal{O}_{F_v^+}) \arrow[r, equal] \arrow[dr] & G(\mathcal{O}_{F_{\tilde{v}}}) \arrow[r, "\iota_{\tilde{v}}"] \arrow[d, "c"] & \mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}}) \arrow[d, "g \mapsto (g^{-1})^T"] \\
& G(\mathcal{O}_{F_{\tilde{v}^c}}) \arrow[r] & \mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}^c}})
\end{tikzcd}$$

On the other hand, if v is inert, then via conjugation by $\begin{pmatrix} 1 & b \\ b & -1 \end{pmatrix}$ for a choice of $b \in \mathcal{O}_{F_v}^\times \backslash \mathcal{O}_{F_v^+}^\times$, $bb^c = 1$,

$$G(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \mathrm{U}_2(\mathcal{O}_{F_v^+}) = \left\{ g \in \mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}}) \mid (g^T)^{\mathrm{Frob}} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} g = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right\}.$$

Finally, it is compact at infinity, namely $G(F_v^+) \cong \mathrm{U}_2(\mathbb{C}/\mathbb{R})$.

Let $E/\mathbb{Q}_p$ be a big enough (yet finite) p-adic field, which will be used as a coefficient field. Let $\mathcal{O}$ be its ring of integers, and $\mathbb{F}$ be its residue field. From now on, we further assume that for all $v \mid p$ in F^+, v splits in F as $v = \tilde{v}\tilde{v}^c$. Let S_p^+ be the set of places of F^+ above p, and let $\widetilde{S}_p$ ($\widetilde{S}_p^c$, resp.) be the set of places of F of form $\tilde{v}$ ($\tilde{v}^c$).

We review the theory of algebraic automorphic forms on G. Given a compact open subgroup $U \subset G(\mathbb{A}_{F^+}^\infty)$ such that $U = \prod_{v \text{ finite}} U_v$, $U_v = G(\mathcal{O}_{F_v^+})$ for almost every place v of F^+. By the general theory (cf. Knapp), $G(F^+)$ is a discrete subgroup of $G(\mathbb{A}_{F^+}^\infty)$, and as G is compact at infinity, $G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)$ is compact (or $G(F^+)$ is "cocompact" in $G(\mathbb{A}_{F^+}^\infty)$). In particular, the double quotient $G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U$ is a finite set $\{g_1, \cdots, g_r\}$. Also, $g_i^{-1}G(F^+)g_i \cap U$ is finite.

Definition 2.1. *We say U is **sufficiently small** if $g_i^{-1}G(F^+)g_i \cap U$ does not contain any element of order p.*

Example 2.1. For example, U such that U_v is hyperspecial at every place, except at one place v_1 of F^+ such that it does not divide p, $N\tilde{v}_1 \not\equiv 1 (\mathrm{mod}\, p)$ and splits as $\tilde{v}_1\tilde{v}_1^c = w_1$ in F at which U_{v_1} looks like

$$U_{v_1} = \iota_{\tilde{v}_1}^{-1}\left(\left\{g \in \mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}_1}}) \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\mathrm{mod}\, \varpi_{\tilde{v}_1})\right\}\right),$$

satisfies the property of being sufficiently small.

Definition 2.2. *Let W be a $G(\mathcal{O}_{F^+,p})$-representation over $\mathcal{O}$. Then, the space of **algebraic automorphic forms** of weight W is defined as*

$$S(U, W) := \{f : G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty) \longrightarrow W \mid f(gu) = u_p^{-1}f(g) \text{ for } u = (u_v) \in U\}.$$

Lemma 2.1. *There is an $\mathcal{O}$-linear isomorphism*

$$S(U, W) \xrightarrow{\sim} \bigoplus_{i=1}^{r} W^{(g_i^{-1}G(F^+)g_i)\cap U}.$$

Proof. The map is given by $f \mapsto f(g_i)$. The inverse is given by $\underline{x} = (x_i) \mapsto f_{\underline{x}}$ defined by $f_{\underline{x}}(g_i) = x_i$; this well-defines the function uniquely, because if $u \in U$ lies in $u = g_i^{-1}tg_i$ for $t \in G(F^+)$, then $f_{\underline{x}}(g_iu) = f_{\underline{x}}(tg_i) = f_{\underline{x}}(g_i) = x_i$, and this covers everything. □

Corollary 2.1. *If U is sufficiently small, then the following are true.*

1. *The functor $W \mapsto S(U, W)$ is exact, and $S(U, W)$ is a finitely generated module. If W is free over $\mathcal{O}$, then $S(U, W)$ is a projective module over $\mathcal{O}$.*
2. *For any $\mathcal{O}$-algebra B, $S(U, W) \otimes_{\mathcal{O}} B \cong S(U, W \otimes B)$. This holds without sufficient smallness assumption if B is a flat $\mathcal{O}$-algebra.*

Proof. Everything follows from the fact that, if H is a finite group with order invertible in $\mathcal{O}$, then $\mathrm{Hom}_H(\mathbb{1}, -)$ is an exact functor from the category of H-representations over $\mathcal{O}$, and that $W^H = \cap_{h\in H} \ker(W \xrightarrow{x \mapsto hx - x} W)$. □

Remark 2.1. If $V \subset U$, then $S(U, W) \hookrightarrow S(V, W)$, and if $V \trianglelefteq U$, then there is an action U/V on $S(V, W)$.

Lemma 2.2. *If $g_i^{-1}G(F^+)g_i \cap U$ is tirival, then U/V acts freely.*

Proof. By the hypothesis, we know that there is a further decomposition

$$G(\mathbb{A}_{F^+}^\infty) = \coprod_{1\le i\le r, u_j \in U/V} G(F^+)g_iu_jV.$$

This is really a partition. Therefore,

$$S(V, W) \xrightarrow{\sim} \bigoplus_i W \otimes \mathcal{O}[U/V],$$

$$f \mapsto \sum_{i,j} f(g_iu_j) \otimes u_j^{-1}.$$

The RHS can be rewritten as $\bigoplus_{i,j} W$, where U acts via

$$(u \cdot (\underline{x}))_{i,j} = v_p^{-1} \cdot x_{i[u_{j'}]},$$

where $u_j u = u_{j'} v$ for $v \in V$. In particular, $S(V, W)_{U/V} \xrightarrow{\sim, \mathrm{Tr}_{U/V}} S(U, W) \cong S(V, W)^{U/V}$, where the action of U on $S(V, W)$ is $f \mapsto u_p^{-1} f(-u)$. □

This is one decsription of automorphic forms, and we would like to relate $S(U, W)$ to classical automrophic forms, namely $L^2(G(\mathbb{A}_{F^+}))$. The upshot is that one can exchange information between p and ∞.

Definition 2.3. *For a compact subgroup $J \subset G(\mathbb{A}_{F^+}^\infty)$, one defines*

$$S(J, W) = \lim_{U \supset J \text{ compact open}} S(U, W).$$

For $g \in G(\mathbb{A}_{F^+}^\infty)$ such that $J_p \subset G(\mathcal{O}_{F^+,p})$, there is a well-defined map

$$S(J, W) \longrightarrow S(g^{-1} J g, W),$$
$$f \longmapsto g_p^{-1} f(-g).$$

Thus, $S(1, W)$, a $G(\mathbb{A}_{F^+}^\infty)$-representation, contains everything, in the sense that $S(1, W)^J = S(J, W)$.

Now the weights in consideration are as follows. Let $\mathbb{Z}_+^2 = \{(\lambda_1, \lambda_2) \mid \lambda_1 \geq \lambda_2\}$.

Definition 2.4. *For $\lambda_{\tilde{v}} \in (\mathbb{Z}_+^2)^{\mathrm{Hom}(F_{\tilde{v}}, E)}$, we define*

$$W_{\lambda_{\tilde{v}}} = \bigotimes_{\tau : F_{\tilde{v}} \hookrightarrow E} \left(\mathrm{Sym}^{\lambda_{\tau,1} - \lambda_{\tau,2}} \mathcal{O}_{F_{\tilde{v}}}^2 \otimes \det{}^{\lambda_{\tau,2}} \right) \otimes_{\mathcal{O}_{F_{\tilde{v}}}, \tau} \mathcal{O}.$$

This, when transferred to infinity, will exactly correspond to classical automorphic forms of corresponding infinitesimal character.

Remark 2.2. The definition used an identification of $G(\mathcal{O}_{F^+,p})$ with GL_2. We would like to convince ourself that this definition is a sensible good definition.

- The dual of $W_{\lambda_{\tilde{v}}}$ is $W_{-\begin{pmatrix} \lambda_{\tilde{v},2} \\ \lambda_{\tilde{v},1} \end{pmatrix}}$, and $W_{\lambda_{\tilde{v}^c}} \circ \iota_{\tilde{v}}$ is $W_{\lambda_{\tilde{v}^c}}$ with action inverse-transposed.
- If $(\mathbb{Z}_+^2)_0^{\mathrm{Hom}(F,E)} := \{\lambda_\tau \mid \lambda_{\tau,i} = -\lambda_{\tau \circ c, 3-i} \text{ for } i = 1, 2\}$, which is naturally identified with $\bigoplus_{\tilde{v} \in \tilde{S}_p} (\mathbb{Z}_+^2)^{\mathrm{Hom}(F_{\tilde{v}^c}, E)}$, then

$$W_\lambda := \bigotimes_{\tilde{v} \in \tilde{S}_p} W_{\lambda_{\tilde{v}}},$$

  for $\lambda \in (\mathbb{Z}_+^2)_0^{\mathrm{Hom}(F,E)}$, is a well-defined representation of $G(\mathcal{O}_{F^+,p})$.

Definition 2.5. *Let the space of* ***classical automorphic forms*** *be defined as*

$$\mathscr{A} = \{f : G(F^+) \backslash G(\mathbb{A}_{F^+}) \longrightarrow \mathbb{C} \mid f|_{G(\mathbb{A}_F^\infty)} \text{ locally constant}, \dim_{\mathbb{C}} \langle G(F_\infty^+) f \rangle < \infty\}.$$

The following is a very standard fact.

Theorem 2.1. *The space of classical automorphism forms decomposes, as a $G(\mathbb{A}_{F^+})$-representation, as*

$$\mathscr{A} \cong \bigoplus_{\pi \text{ irr. adm. of } G(\mathbb{A}_{F^+})} m(\pi) \pi.$$

Remark 2.3. In our setting of rank 2 unitary groups, it is known that $m(\pi) \leq 1$ (Rogawski, *Automorphic Representation of Unitary Groups in Three Variables*, Thm 13.3).

Now we would like to consider two versions of automorphic forms, namely compare $\mathscr{A}$ and $S(1, W)$. After fixing $\iota : \overline{E} \xrightarrow{\sim} \mathbb{C}$, we can compare two versions of algebraic representations:

$$\iota_* : \bigoplus_{\tilde{v} \in \tilde{S}_p} (\mathbb{Z}^2_+)^{\mathrm{Hom}(F_{\tilde{v}}, E)} \longrightarrow (\mathbb{Z}^2_+)^{\mathrm{Hom}(F^+, \mathbb{R})},$$

$$(\lambda_{\tau_{\tilde{v}}})_{\tilde{v}} \mapsto (\lambda_{\iota \circ \tau_{\tilde{v}}}),$$

$$\theta : W_\lambda \otimes_\iota \mathbb{C} \xrightarrow{\sim} W_{\iota_* \lambda},$$

which is $G(F^+)$-equivariant, where $W_{\iota_* \lambda}$ is the irreducible algebraic representation of $G(F^+_\infty)$ of highest weight $\iota_* \lambda$.

Theorem 2.2. *Let σ be a finite-dimensional smooth $G(\mathcal{O}_{F^+,p})$-representation. Then,*

$$S(1, W_\lambda \otimes \sigma) \otimes_\iota \mathbb{C} \xrightarrow{\sim} \mathrm{Hom}_{G(F^+_\infty)}((W^\vee_{\iota_* \lambda} \otimes \sigma^\vee \otimes \mathbb{C}), \mathscr{A}),$$

which is $G(\mathbb{A}^{\infty,p}_{F^+})$-equivariant.

Proof. The map is given by

$$f \mapsto \left[\phi \mapsto \left[g \mapsto \phi\left(g_\infty \theta\left(g_p f\left(g^\infty\right)\right)\right)\right]\right].$$

The inverse map is given by

$$\phi \mapsto f_\phi,$$

where f_ϕ is a function characterized by

$$\langle v^*, f_\phi(g) \rangle = \phi\left((\theta^\vee)^{-1}\left(g_p * v^*\right)\right)(g, 1_\infty),$$

for $v^* \in (W_\lambda \otimes \sigma)^\vee$. □

Remark 2.4. For J a compact subgroup in $G(\mathbb{A}^\infty_{F^+})$,

$$S(J, W_\lambda \otimes \sigma) \otimes \mathbb{C} \xrightarrow{\sim} S(1, W_\lambda \otimes \sigma)^J \xrightarrow{\sim} \mathrm{Hom}_{G(F^+)}(W^\vee_\lambda \otimes \sigma^\vee, \mathscr{A})^J \xrightarrow{\sim} \mathrm{Hom}_{G(F^+_\infty) \times J}(W^\vee_\lambda \otimes \sigma^\vee, \mathscr{A})$$

$$\xrightarrow{\sim} \bigoplus_{\pi_\infty \cong W^\vee_{\iota_* \lambda}} m(\pi)(\pi^{\infty,p})^{J^p} \otimes \mathrm{Hom}_{J_p}(\sigma^\vee, \pi_p).$$

3. **Galois representations attached to automorphic forms.**

Now we transfer this to a situation where we know how to attach Galois representations, namely we use the group

$$G^* = \left\{ g \in \mathrm{GL}_2(F) \mid g \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} g^{-T} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \right\}.$$

We then have

$$G^*(F^+_v) = G(F^+_v),$$

if v is finite, and

$$G^*(F^+_v) \cong \mathrm{U}(1,1)(\mathbb{C}/\mathbb{R}),$$

if v is infinite (as opposed to $G(F^+_v) \cong \mathrm{U}(2)(\mathbb{C}/\mathbb{R})$).

Definition 3.1. *For $v < \infty$, an L-packet for $G^*(F^+_v)$ is a* $\mathrm{PGL}_2(F^+_v)$*-orbit of irreducible admissible representaiton of $G^*(F^+_v)$.*

Example 3.1. The L-packets are of the following forms,

$$\left\{ \mathrm{nInd}_{B(F_v^+)}^{G^*(F_v^+)} \lambda \right\} \text{ for } \lambda \neq |-|^{\pm 1}, \omega_{F_{\tilde{v}}/F},$$

or

$$\left\{ \mathrm{cInd}_{G^*(\mathcal{O}_{F_v^+})}^{G^*(F_v^+)} \sigma, \mathrm{cInd}_{\begin{pmatrix} 1 & \\ & p \end{pmatrix} G^*(\mathcal{O}_{F_v^+}) \begin{pmatrix} 1 & \\ & p \end{pmatrix}}^{G^*(F_v^+)} \sigma \right\} \text{ for } \sigma \text{ cuspidal rep of } G^*(\mathbb{F}_{\tilde{v}}/\mathbb{F}_v),$$

which depends on a character of $\mathrm{U}(1) \times \mathrm{U}(1)$.

Now we do the Jacquet-Langlands transfer.

Definition 3.2. *The space of automorphic forms for G^* is*

$$L^2(G^*(F^+)\backslash G^*(\mathbb{A}_{F^+}))$$

$$:= \left\{ f : G^*(F^+)\backslash G^*(\mathbb{A}_{F^+}) \longrightarrow \mathbb{C} \;\middle|\; \int_{Z(\mathbb{A}_{F^+})G^*(F^+)\backslash G^*(\mathbb{A}_{F^+})} \|f(g)\|^2 d\mu(g) < \infty + \textit{growth condition} \right\}.$$

The space of cusp forms $L_0(G^*) \subset L^2(G^*(F^+)\backslash G^*(\mathbb{A}_{F^+}))$ is consisted of those f such that

$$\int_{N^*(F^+)\backslash N^*(\mathbb{A}_{F^+})} f(ng)d\mu(n) = 0,$$

for almost every g, for $B^* = N^*T^*$ Borels of G^*.

Theorem 3.1 (Jacquet-Langlands correspondence)**.** *There is a map*

$$\mathrm{JL} : \{L\textit{-packets of } G(\mathbb{A}_{F^+})\} \longrightarrow \{\textit{cuspidal } L\textit{-packets of } G^*(\mathbb{A}_{F^+})\},$$

where $\mathrm{JL}([\pi])_v = [\pi_v]$ *for v finite, and* $\mathrm{JL}([\pi])_v = (\iota_*\lambda_v)$ *for $v \mid \infty$, where $\pi_v = W_{\iota_*\lambda}$.*

Theorem 3.2 (Base change)**.** *There is a base change transfer*

$$\{L\textit{-packets of } G^*(F_v^+)\} \longrightarrow \left\{ \textit{conjugate stable representations of } \mathrm{Res}_{F_v/F_v^+} \mathrm{GL}_2 \right\},$$

where the conjugation is given by $g \mapsto g^{-T}$.

- *If $v = \tilde{v}\tilde{v}^c$, then*

$$\pi \longmapsto (\pi \circ \iota_{\tilde{v}}^{-1}) \otimes (\pi \circ \iota_{\tilde{v}^c}^{-1}).$$

- *If v is inert, then*

$$\mathrm{nInd}_{B(F_v^+)}^{G^*(F_v^+)} \lambda \longmapsto \mathrm{nInd}_{B(F_v)}^{\mathrm{GL}_2(F_v)}(\lambda \otimes \lambda^{-\mathrm{Frob}}),$$

$$\mathrm{ind}_{G^*(\mathcal{O}_{F_v^+})}^{G(F_v^+)} \sigma\theta \longmapsto \mathrm{nInd}_{B(F_v)}^{\mathrm{GL}_2(F_v)} \theta \circ (x \longmapsto x\underline{x}^{-\mathrm{Frob}}),$$

where θ is a character of $\mathrm{U}(1) \times \mathrm{U}(1)$.

The upshot is that we have a transfer to the situation where we precisely know what happens at each local place. We now try to attach Galois representation to transferred automorphic forms.

Theorem 3.3. *Given a potentially semistable $\rho_{\tilde{v}} : G_{F_{\tilde{v}}} \longrightarrow \mathrm{GL}_2(E)$, which becomes semistable after K, then Fontaine associated the filtered (φ, N)-module*

$$(D(\rho_{\tilde{v}}), \varphi, N),$$

where $D(\rho_{\tilde{v}})$ is an $F_{\tilde{v}}^{\mathrm{un}} \otimes_{\mathbb{Q}_p} E$-module, where $F_{\tilde{v}}^{\mathrm{un}}$ is the maximal unramified extension in K and $D \otimes K$ has a filtration, φ is a semilinear action on $D(\rho_{\tilde{v}})$ nd N is a nilpotent operator such that $N\varphi = \mathbf{N}(\tilde{v})\varphi N$. Then, we define the corresponding ***Weil-Deligne representation*** $\mathrm{WD}(\rho_{\tilde{v}})$ *is an action on one factor of $D(\rho_{\tilde{v}})$ such that $g \in \mathrm{WD}_{F_{\tilde{v}}}$ acts via $(\overline{g}\varphi^{-\mathrm{val}(g)})$, where $\overline{g} \in \mathrm{Gal}(K/F_{\overline{v}})$ is the natural image via the quotient map.*

Now recall that there is an association

$$\mathscr{L}_{\overline{\mathbb{Q}}_p} : \mathrm{Rep}_{\overline{\mathbb{Q}}_p}^{\text{irr. adm.}}(\mathrm{GL}_2(F_{\tilde{v}})) \longrightarrow \mathrm{WDRep}_{\overline{\mathbb{Q}}_p}(W_{F_{\tilde{v}}})^{\mathrm{Fss}},$$

$$(\pi \otimes |\det|^{1/2}) \longmapsto \iota^{-1}(\mathscr{L}_{\mathbb{C}}(\iota(\pi))).$$

This is closely related to the so-called **inertial local Langlands correspondence**.

Theorem 3.4 (Henniart, Paskunas, Caraiani-Emerton-Gee-Geraghty-Paskunas-Shin)**.** *Given $\pi \in \mathrm{Rep}^{\mathrm{irr}}(\mathrm{GL}_2(F_{\tilde{v}}))$ and smooth $\tau_{\tilde{v}} : I_{F_{\tilde{v}}} \longrightarrow \mathrm{GL}_2(E)$, there is a unique irreducible $\mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}})$-representation $\sigma(\tau_{\tilde{v}})$ such that $\mathscr{L}_{\overline{\mathbb{Q}}_p}(\pi)|_{I_{F_{\tilde{v}}}} \cong \tau_{\tilde{v}}$ if and only if $\pi|_{\mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}})} \supset \sigma(\tau_{\tilde{v}})$, in which case $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}})}(\sigma(\tau_{\tilde{v}}), \pi)$ is 1-dimensional.*

Now we can finally state the association of Galois representation.

Theorem 3.5. *Let $\lambda \in (\mathbb{Z}^2_+)^{\mathrm{Hom}(F,E)}$, $\tau = \otimes_{\tilde{v}\in\tilde{S}_p}\tau_{\tilde{v}}$ be an inertial type, and $\sigma^\circ \subset \otimes_{\tilde{v}\in\tilde{S}_p}\sigma(\tau_{\tilde{v}})$ is an $\mathcal{O}$-lattice. For $\pi \subset S(G(\mathcal{O}_{F^+,p}), W_\lambda \otimes \sigma^\circ) \otimes E$, there is a Galois representation $r_\pi : G_F \longrightarrow \mathrm{GL}_2(E)$ such that*

- $r_\pi^c \cong r_\pi^\vee \chi_{\mathrm{cyc}}$,
- *if $v = \tilde{v}\tilde{v}^c$, then*

$$\mathrm{WD}(r_{\pi|_{G_{F_{\tilde{v}}}}})^{\mathrm{Fss}} = \mathscr{L}_{\overline{\mathbb{Q}}_p}((\pi_v \circ \iota_{\tilde{v}}^{-1}) \otimes |\det|^{-1/2}),$$

- *if v is inert,*

$$\mathrm{WD}(r_{\pi|_{G_{F_v}}})^{\mathrm{Fss}} = \mathscr{L}_{\overline{\mathbb{Q}}_p}(\mathrm{BC}_{F_v/F_v^+}([\pi_v] \otimes |\det|^{-1/2})),$$

- *if $\tilde{v} \in \tilde{S}_p$, $r_{\pi|_{G_{F_{\tilde{v}}}}}$ is potentially crystalline, $\mathrm{HT}_{\tilde{v}}(r_{\pi|_{G_{F_{\tilde{v}}}}}) = \lambda_{\tilde{v}} + (1,0)$, and $\mathrm{WD}(r_{\pi|_{G_{F_{\tilde{v}}}}})|_{I_{F_{\tilde{v}}}} = \tau_{\tilde{v}}$.*

These properties characterize r_π uniquely.

We can summarize the general strategy of attaching Galois representations to automorphic forms on definite unitary groups of rank 2 into the following.

$$G\text{-automorphic form} \xrightarrow{\mathrm{JL}} G^*\text{-automorphic form} \xrightarrow{\text{base change}}$$

$$\text{self-dual cuspidal } \mathrm{GL}_{2,F}\text{-automorphic form} \longrightarrow \text{Galois representation.}$$

4. **Galois representations valued in Hecke algebras.**

Let $\prod_v U_v = U \subset G(\mathbb{A}_{F^+}^\infty)$ be a compact open subgroup.

Definition 4.1. *Let $v = \tilde{v}\tilde{v}^c$ in F, and $G(\mathcal{O}_{F_v^+}) = U_v$. Then, for $f \in S(U, W_\lambda \otimes \sigma^\circ)$, $T_{\tilde{v}} \cdot f := \sum_i h_{\tilde{v},i} \cdot f$, where $h_{\tilde{v},i}$ runs over representatives of*

$$U_v \iota_{\tilde{v}}^{-1} \begin{pmatrix} \varpi_{\tilde{v}} & \\ & 1 \end{pmatrix} U_v U^v / U_v U^v.$$

Explicitly, these can be given as

$$\left\{ \iota_{\tilde{v}}^{-1} \begin{pmatrix} \varpi_{\tilde{v}} & [\lambda] \\ & 1 \end{pmatrix} U_v U^v \middle| \lambda \in \mathbb{F}_{\tilde{v}} \right\} \cup \left\{ \iota_{\tilde{v}}^{-1} \begin{pmatrix} & 1 \\ \varpi_{\tilde{v}} & \end{pmatrix} U_v U^v \right\}.$$

We similarly define $S_{\tilde{v}}$ for $\begin{pmatrix} \varpi_{\tilde{v}} & \\ & \varpi_{\tilde{v}} \end{pmatrix}$.

Remark 4.1.
- These define an $\mathcal{O}$-linear endomorphism of $S(U, W_\lambda \otimes \sigma^\circ)$.
- $T_{\tilde{v}^c} = T_{\tilde{v}} S_{\tilde{v}}^{-1}$.

Definition 4.2. *We fix T^+, a set of places of F^+ containing S_p^+, the places of F^+ over p, and such that for all $v \notin T^+$, $v = \tilde{v}\tilde{v}^c$ in F, and a compact open $U \subset G(\mathbb{A}_{F^+}^\infty)$ which is hyperspecial at all $v \notin T^+$. Then, we let the Hecke algebra $\mathbb{T}_{\lambda,\sigma}^{T^+}(U) \subset \mathrm{End}_{\mathcal{O}}(S(U, W_\lambda \otimes \sigma^\circ))$ be the $\mathcal{O}$-algebra generaed by $T_{\tilde{v}}$'s and $S_{\tilde{v}}$'s for $\tilde{v}|_{F^+} \notin T^+$.*

It is important to understand this Hecke algebra because this is the Hecke algebra that will be compared with Galois deformation rings.

Recall that we had an isomorphism

$$S(U, W_\lambda \otimes \sigma^\circ) \otimes \mathbb{C} \xrightarrow{\sim} \bigoplus_{\pi \subset L_{\mathrm{cusp}}(G(\mathbb{A}_{F^+})), (\pi^{\infty,p})^{U^p} \neq 0} \mathrm{Hom}_{G(F_\infty^+)}\left((W_{\iota_* \lambda}^\vee \otimes \sigma) \otimes \mathbb{C}, \pi_\infty \otimes \pi_p\right) \otimes (\pi^{\infty,p})^{U^p}.$$

As this is a Hecke eigenspace decomposition, we can observe that this yields an isomorphism

$$\begin{aligned} \mathbb{T}_{\lambda,\sigma}^{T^+}(U) \otimes \overline{E} \xrightarrow{\sim} \prod_{\pi \subset L_{\mathrm{cusp}}(G(\mathbb{A}_{F^+})), (\pi^{\infty,p})^{U^p} \neq 0} & \overline{E}, \\ T_{\tilde{v}} \longmapsto & \, t_{\tilde{v}}, \\ S_{\tilde{v}} \longmapsto & \, s_{\tilde{v}}, \end{aligned}$$

where t and s are Satake parameters of $\pi_v^{U_v}$.

Corollary 4.1.
(1) $\mathbb{T}_{\lambda,\sigma}^{T^+}(U)$ is reduced.
(2) We have a bijection

$$\left\{ \begin{array}{c} \pi \subset L_{\mathrm{cusp}}(G) \text{ such} \\ \text{that } (\pi^{\infty,p})^{U^p} \neq 0 \end{array} \right\} \leftrightarrow \mathrm{Hom}(\mathbb{T}_{\lambda,\sigma}^{T^+}(U) \otimes \overline{E}, \overline{E}).$$

(3) $\mathbb{T}_{\lambda,\sigma}^{T^+}(U)$ is a semilocal ring.

Proof. To see (1), note that it is p-torsion-free and is reduced after base-changing to $\overline{E}$. ☐

Lemma 4.1. *The map $\mathbb{T}_{\lambda,\sigma}^{T^+}(U) \hookrightarrow \mathbb{T}_{\lambda,\sigma}^{T^+}(U) \otimes \overline{E}$ induces a bijection*

$$\left\{ \begin{array}{c} \text{minimal primes of} \\ \mathbb{T}_{\lambda,\sigma}^{T^+}(U) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{maximal ideals of} \\ \mathbb{T}_{\lambda,\sigma}^{T^+}(U) \otimes \overline{E} \end{array} \right\}.$$

Proof. It is enough to show that $\mathfrak{p} \subset \mathbb{T}^{T^+}_{\lambda,\sigma}(U)$ is a minimal prime if and only if $\mathfrak{p} \cap \mathcal{O} = 0$.

If $\mathfrak{p}$ is a minimal prime, then as $\mathcal{O} \longrightarrow \mathbb{T}^{T^+}_{\lambda,\sigma}(U)$ is finite flat, we use going down theorem and get the result.

If $\mathfrak{p} \cap \mathcal{O} = 0$, then if there is $\mathfrak{p}_1 \cap \mathfrak{p}$, as $\mathcal{O} \longrightarrow \mathbb{T}^{T^+}_{\lambda,\sigma}(U)$ is integral, the quotient map $\mathbb{T}^{T^+}_{\lambda,\sigma}(U)/\mathfrak{p}_1 \twoheadrightarrow \mathbb{T}^{T^+}_{\lambda,\sigma}(U)/\mathfrak{p}$ is a surjective map between domains of the same dimension. □

Thus, minimal primes of the Hecke algebra are precisely Hecke eigenclasses with appropriate level, namely

$$\left\{\text{minimal primes of } \mathbb{T}^{T^+}_{\lambda,\sigma}(U)\right\} = \left\{\begin{array}{c}\ker(\lambda_\pi : \mathbb{T}^{T^+}_{\lambda,\sigma}(U) \longrightarrow \overline{E}),\ T_{\tilde{v}} \mapsto t_{\tilde{v}},\ S_{\tilde{v}} \mapsto s_{\tilde{v}}, \text{ such} \\ \text{that } t_{\tilde{v}} \text{ and } s_{\tilde{v}} \text{ are Satake parameters at } \tilde{v} \text{ for } \pi \text{ an} \\ \text{algebraic automorphic form with } (\pi^{\infty,p})^{U^p} \neq 0\end{array}\right\}.$$

We now fix a residual Galois representation

$$\overline{r} : G_F \longrightarrow \mathrm{GL}_2(\mathbb{F}),$$

which is absolutely irreducible, continuous and unramified outside T^+. Let $\mathfrak{m} = \mathfrak{m}_{\overline{r}}$ be a maximal ideal of $\mathbb{T}^{T^+}_{\lambda,\sigma}(U)$ corresponding to $\overline{r}$, namely

$$\mathfrak{m}_{\overline{r}} = \langle \varpi, T_{\overline{v}} - \mathrm{tr}(\overline{r}(\mathrm{Frob}_{\tilde{v}})), S_{\tilde{v}} - (N\overline{v})\det(\overline{r}(\mathrm{Frob}_{\tilde{v}}))\rangle.$$

We now group all algebraic automorphic forms that reduce to $\overline{r}$, to get a massive Galois representation

$$r^0_{\mathfrak{m}} : G_F \longrightarrow \prod_{\pi \text{ algebraic automorphic form}, (\pi^{\infty,p})^{U^p}\neq 0, \mathfrak{p}_\pi \subset \mathfrak{m}} \mathrm{GL}_2(\mathbb{T}^{T^+}_{\lambda,\sigma}(U)/\mathfrak{p}_\pi \otimes \overline{E}) = \mathrm{GL}_2(\mathbb{T}^{T^+}_{\lambda,\sigma}(U)_{\mathfrak{m}} \otimes \overline{E}).$$

We would like to conjugate $r^0_{\mathfrak{m}}$ so that the image of $r^0_{\mathfrak{m}}$ lands into $\mathrm{GL}_2(\mathbb{T}^{T^+}_{\lambda,\sigma}(U))$. This is possible by a theorem of Carayol.

Theorem 4.1 (Carayol)**.** *Let $S \hookrightarrow \prod_{i=1}^r R_i$, where $r < \infty$ and all rings involved are complete local noetherian $\mathcal{O}$-algebras of finite residue field. Suppose that we have a collection of Galois representations*

$$r_i : G_F \longrightarrow \mathrm{GL}_2(R_i),$$

such that $\overline{r}_i$'s are all irreducible and isomorphic to each other, and $\mathrm{tr}(r_i(\mathrm{Frob}_w)) \in S$ *for almost every w. Then, one can always conjugate $\prod_{i=1}^r r_i : G_F \longrightarrow \prod_{i=1}^r \mathrm{GL}_2(R_i) = \mathrm{GL}_2(\prod_{i=1}^r R_i)$ into $G_F \longrightarrow \mathrm{GL}_2(S)$.*

We can apply this theorem to our situation as $r^0_{\mathfrak{m}}$ is actually valued in some finite extension of $\mathbb{T}^{T^+}_{\lambda,\sigma}(U)_{\mathfrak{m}}$ by compactness of G_F. Thus, we can conjugate into get

Proposition 4.1. *We have a Galois representation*

$$r_{\mathfrak{m}} : G_F \longrightarrow \mathrm{GL}_2(\mathbb{T}^{T^+}_{\lambda,\sigma}(U)_{\mathfrak{m}}),$$

such that

(1) $r_{\mathfrak{m}} \equiv \overline{r} (\mathrm{mod}\, \mathfrak{m})$,
(2) $r_{\mathfrak{m}}|_{G_{F_{\tilde{v}}}}$ *is unramified for all $\tilde{v}$ such that $\tilde{v}|_{F^+} \notin T^+$,*
(3) the characteristic polynomial of $r_{\mathfrak{m}}(\mathrm{Frob}_{\tilde{v}})$ *is $X^2 - T_{\tilde{v}}X + N\tilde{v}S_{\tilde{v}}$,*
(4) if $x : \mathbb{T}^{T^+}_{\lambda,\sigma}(U)_{\mathfrak{m}} \longrightarrow \overline{E}$ is an algebra homomorphism, then for $\tilde{v} \in \tilde{S}_p$, $x \circ r_{\mathfrak{m}}|_{G_{F_{\tilde{v}}}}$ is potentially crystalline with Hodge-Tate weights $\lambda_{\tilde{v}} + (1,0)$ and

$$\mathrm{WD}(x \circ r_{\mathfrak{m}}|_{G_{F_{\tilde{v}}}})|_{I_{F_{\tilde{v}}}} \cong \mathscr{T}_{\tilde{v}},$$

the inertial type associated to $\sigma_{\tilde{v}}$.

5. **Comparing different levels.**

We fix an auxiliary finite set Q of split places of F^+ not in T^+ such that the following conditions are satisfied.

(1) The Hecke polynomial $X^2 - T_{\tilde{v}}X + N\tilde{v}S_{\tilde{v}}$ has distinct roots mod $\mathfrak{m}$, and
(2) $Nv \equiv 1 (\mathrm{mod}\, p)$ for all $v \in Q$.

If $U_0(Q), U_1(Q) \subset U$ are compact open subgroups such that $(U_i(Q))_v = U_v$ for all $v \notin Q$, we consider adding ramification via $U_i(Q)$:

$$\mathbb{T}^{T^+\cup Q}_{\lambda,\sigma}(U_i(Q)) \twoheadrightarrow \mathbb{T}^{T^+\cup Q}_{\lambda,\sigma}(U) \hookrightarrow \mathbb{T}^{T^+}_{\lambda,\sigma}(U),$$

$$\mathfrak{m}_Q \mapsto \mathfrak{m}.$$

By the above, we similarly have a Galois representation

$$r_{\mathfrak{m}_Q} : G_F \longrightarrow \mathrm{GL}_2(\mathbb{T}^{T^+\cup Q}_{\lambda,\sigma}(U_i(Q))_{\mathfrak{m}_Q}),$$

where

$$\mathrm{char.\,poly}(r_{\mathfrak{m}_Q}(\mathrm{Frob}_{\tilde{v}})) = (X - A_{\tilde{v}})(X - B_{\tilde{v}}),$$

for $\tilde{v} \in Q$ (which is possible by Hensel's lemma). For $\tilde{v}$ with $\tilde{v}|_{F^+} \in Q$, we consider

$$V_{\varpi_{\tilde{v}}} := (U_i(Q)_v)\iota_{\tilde{v}}^{-1}\begin{pmatrix}1 & \\ & \varpi_{\tilde{v}}\end{pmatrix}U_i(Q)_vU_i(Q)^v \in \mathrm{End}_{\mathcal{O}}(S(U_i(Q), W_\lambda \otimes \sigma_j^\circ)).$$

We now set levels at Q: for $v \in Q$,

$$U_0(Q)_v = \iota_{\tilde{v}}^{-1}\left(\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}}),\ \text{congruent to upper triangular matrix modulo } \varpi_{\tilde{v}}\right\}\right),$$

or the Iwahori level, and

$$U_1(Q)_v = \iota_{\tilde{v}}\left(\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in U_0(Q)_v, ad^{-1} \in \ker(\mathcal{O}_{F_{\tilde{v}}}^\times \twoheadrightarrow \mathbb{F}_{\tilde{v}}^\times \twoheadrightarrow \mathbb{F}_{\tilde{v}}^\times(p) =: \Delta_{\tilde{v}})\right\}\right),$$

a normal subgroup of $U_0(Q)_v$, where $\Delta_{\tilde{v}}$ is the p-part of $\mathbb{F}_{\tilde{v}}^\times$. Now we modify our Hecke algebra slightly:

$$\mathbb{T}^{T^+}_{\lambda,\sigma}(U) \hookleftarrow \mathbb{T}^{T^+\cup Q}_{\lambda,\sigma}(U) \twoheadleftarrow \mathbb{T}^{T^+\cup Q}_{\lambda,\sigma}(U_i(Q)) \hookrightarrow \widetilde{\mathbb{T}}^{Q\cup T^+}_{\lambda,\sigma}(U_i(Q)),$$

where $\widetilde{\mathbb{T}}$ is the Hecke algebra generated rather by $T_{\tilde{v}}, S_{\tilde{v}}$'s for $\tilde{v}$ with $\tilde{v}|_{F^+} \notin T^+ \cup Q$, and $V_{\varpi_{\tilde{v}}}$'s for $\tilde{v}$ with $\tilde{v}|_{F^+} \in Q$. Let $\widetilde{\mathfrak{m}}_Q \subset \widetilde{\mathbb{T}}^{Q\cup T^+}_{\lambda,\sigma}(U_i(Q))$ be a maximal ideal generated by $\mathfrak{m}_Q$ and $V_{\varpi_{\tilde{v}}} - A_{\tilde{v}}$'s.

Proposition 5.1. *The natural map*

$$\prod_{\tilde{v},\tilde{v}|_{F^+}\in Q}(V_{\varpi_{\tilde{v}}} - B_{\tilde{v}}) : S(U, W_\lambda \otimes \sigma^\circ) \longrightarrow S(U_0(Q), W_\lambda \otimes \sigma^\circ),$$

induces an isomorphism

$$S(U, W_\lambda \otimes \sigma^\circ)_{\mathfrak{m}} \xrightarrow{\sim} S(U_0(Q), W_\lambda \otimes \sigma^\circ)_{\widetilde{\mathfrak{m}}_Q}.$$

Remark 5.1. If one does not enhance the Hecke algebra on the RHS side with $V_{\varpi_{\tilde{v}}}$'s, one then gets an injective map after localization.

6. **Deformation of Galois representations and Hecke algebras.**

Suppose that we are given an absolutely irreducible Galois representations $\overline{r} : G_F \longrightarrow \mathrm{GL}_2(\mathbb{F})$ such that $\overline{r}^c \cong \overline{r}^\vee \overline{\chi}_{\mathrm{cyc}}^{-1}$. Then, up to increasing $\mathbb{F}$, there exists a unique extension

$$\overline{r} : G_{F^+} \longrightarrow \mathcal{G}_2(\mathbb{F}),$$

where $\mathcal{G}_2(\mathbb{F}) = (\mathrm{GL}_2(\mathbb{F}) \times \mathrm{GL}_1(\mathbb{F})) \rtimes (\mathbb{Z}/2\mathbb{Z})$ is the group scheme appeared in Clozel-Harris-Taylor, namely, if we denote $\mathbb{Z}/2\mathbb{Z} = \{1, j\}$, $j(g,\mu)j^{-1} = (\mu(g^{-1})^T, \mu)$, such that

- $\overline{r}|_{G_F} = (\overline{r}, \overline{\chi}_{\mathrm{cyc}}^{-1})$,
- $\nu \circ \overline{r} = \overline{\chi}_{\mathrm{cyc}}^{-1}$, where $\mu : \mathcal{G}_2(\mathbb{F}) \longrightarrow \mathrm{GL}_1(\mathbb{F})$ is the natural projection to the second factor.

The deformation functor in our concern is

$$\mathrm{Def}_{\overline{r}}^{\square_T}(R) = \left\{ \begin{array}{c} r : G_{F^+} \longrightarrow \mathcal{G}_2(R), \text{with } r \equiv \overline{r} (\mathrm{mod}\, \mathfrak{m}),\ \nu \circ r = \chi_{\mathrm{cyc}}^{-1}, \\ \text{equipped with } (\alpha_v)_{v \in T} \text{ where} \\ \alpha_v \in \ker(\mathcal{G}_2(R) \longrightarrow \mathcal{G}_2(\mathbb{F})) \end{array} \right\} / \sim_T,$$

for R a complete noetherian local ring over $\mathcal{O}$ with finite residue field equal to $\mathbb{F}$, where $r_1 \sim_T r_2$ if

1. there exists $\sigma \in \ker(\mathcal{G}_2(R) \longrightarrow \mathcal{G}_2(\mathbb{F}))$ such that $r_2(g) = \sigma r_1(g) \sigma^{-1}$ for all $g \in G_{F^+}$,
2. and $\alpha_{1,v} = \sigma \alpha_{2,v}$ for all $v \in T$.

Remark 6.1. Given the global datum $(r, \{\alpha_v\})$, $\alpha_v^{-1} r \alpha_v|_{G_{F_v^+}}$ gives a local datum. Thus, if the deformation functors are representable, then we get a natural map $R_T^{\mathrm{loc}} \longrightarrow R_{\mathscr{S},T}^{\square_T}$, where $\mathscr{S}$ is some global deformation problem, and $R_T^{\mathrm{loc}} = \widehat{\otimes}_{v \in T} R^{\square}_{\overline{r}|_{G_{F_v^+}}}$ for some appropriate local deformation rings with conditions.

We make the above remark more precise, by defining local deformation functors.

Definition 6.1. *Let $\overline{\rho}_v = \overline{r}|_{G_{F_v^+}}$, and $\mathrm{Def}_{\overline{\rho}_v}^{\square}$ be the universal framed deformation functor of $\overline{\rho}_v$. Let $D_v \subset \mathrm{Def}_{\overline{\rho}_v}^{\square}$ be a subfunctor such that*

1. *$\overline{\rho}_v \in D_v$,*
2. *D_v is closed under fiber product and inverse limit,*
3. *D_v is closed under $\ker(\mathcal{G}_2(R) \longrightarrow \mathcal{G}_2(\mathbb{F}))$,*
4. *if there is a map $f : R \longrightarrow S$ in the category of complete noetherian local rings over $\mathcal{O}$, then*
    - (a) *$\rho_v \in D_v(R)$ implies $f \circ \rho_v \in D_v(S)$,*
    - (b) *and the converse holds if f is injective.*

Then, we call D_v a ***local deformation problem****.*

Lemma 6.1. *There is a bijection*

$$\left\{ \begin{array}{c} \textit{local deformation} \\ \textit{problems at } v \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{radical } I \subset R_v^{\square} \textit{ with } I \neq \mathfrak{m} \\ \textit{which is stable by} \\ \ker(\mathcal{G}_2(R_v^{\square}) \longrightarrow \mathcal{G}_2(\mathbb{F})) \end{array} \right\},$$

where $R_v^{\square}$ is the ring representing $\mathrm{Def}_{\overline{\rho}_v}^{\square}$ (which is well-known).

Proof. The map is given by the minimal element of the set $\{J \subset R_v^{\square} \mid \rho_v^{\mathrm{univ}}/J \in D_v(R_v^{\square}/J)\}$. $\square$

This is useful because of the following

Theorem 6.1 (Kisin). *There is a unique $\mathcal{O}$-flat reduced quotient $R^{\lambda_v,\tau_v,\square}_{\overline{\rho}_v}$ of $R^{\square}_{\overline{\rho}_V}$ such that $x \in R^{\square}_{\overline{\rho}_v}(\overline{E})$ gives rise to a potentially crystalline, Hodge-Tate weights at v $\lambda_v + (1,0)$ and inertial type τ_v (namely the associated Weil-Deligne representation restricted to I_v is equal to τ_v) if and only if x factors through $R^{\square}_{\overline{\rho}_v} \twoheadrightarrow R^{\lambda_v,\tau_v,\square}_{\overline{\rho}_v}$.*

Now we specify the global deformation conditions,

$$\mathscr{S}_S = (F/F^+, S, \overline{r}, \chi_{\mathrm{cyc}}^{-1}, \{D_v\}_{v\in S}),$$

where S is a finite set of split primes containing T.

Theorem 6.2. *If $\overline{r}$ satisfies $\mathrm{End}(\overline{r}) = \mathbb{F}$, then $R^{\square_T}_{\mathscr{S}_S,T}$ and $R_{\mathscr{S}_S,T}$ (unframed deformation ring) are representable by complete noetherian local rings over $\mathcal{O}$ with finite residue field $\mathbb{F}$.*

Definition 6.2. *We define $H^i_{\mathscr{S}_S,T}(G_{F^+,S}, \mathrm{ad}^0(\overline{r}))$ to be the i-th cohomology of the cone of the map of complexes*

$$C^\bullet(G_{F^+,S}, \mathrm{ad}^0(\overline{r})) \xrightarrow{\oplus_{v\in S}\mathrm{res}} \bigoplus_{v\in S}(C^\bullet(G_{F_v^+}, \mathrm{ad}^0(\overline{r}))/M_v^\bullet),$$

where $M_v^0 = C^0(G_{F_v^+}, \mathrm{ad}^0(\overline{r}))$ and $M_v^1 = I_v/I_v \cap (\mathfrak{m}^2_{R_v^\square}, \varpi_E)$ if $v \in S\backslash T$, and $M_v^\bullet = 0$ for every other v.

Remark 6.2. Do not forget that our r goes into $\mathcal{G}_2$, so taking ad^0 is rather forgetting GL_1 factor; thus $\mathrm{ad}^0(\overline{r}) \cong \mathfrak{gl}_2$.

The tangent space calculation can be done using the following observation,

$$\left(\mathfrak{m}_{R_v^\square}/(\mathfrak{m}^2_{R_v^\square}, \varpi)\right)^\vee \xrightarrow{\sim} \mathrm{Def}_v^\square(\mathbb{F}[\varepsilon]/\varepsilon^2) \xleftarrow{\sim} Z^1(G_{F_v^+}, \mathrm{ad}^0(\overline{r})).$$

Theorem 6.3. *$R^{\square_T}_{\mathscr{S}_S,T}$ is topologically generated over R^{loc}_T by $\dim_{\mathbb{F}}(H^1_{\mathscr{S}_S,T}(G_{F^+,S}, \mathrm{ad}^0(\overline{r})))$ many elements.*

We can compute the value of $\dim_{\mathbb{F}}(H^1_{\mathscr{S}_S,T}(G_{F^+,S}, \mathrm{ad}^0(\overline{r})))$ when $S\backslash T$ is chosen carefully.

Definition 6.3. *Define*

$$H^1_{\mathscr{S}_S^\perp,T}(G_{F^+,S}, \mathrm{ad}^0\,\overline{r}(1)) = \ker\left(H^1(G_{F^+,S}, \mathrm{ad}^0(\overline{r})(1)) \longrightarrow \bigoplus_{v\in S\backslash T} H^1(G_{F_v^+}, \mathrm{ad}^0(\overline{r})(1))/(\overline{M}_v^1)^\perp\right),$$

where $\perp$ means the annihilator using the cup product pairing

$$H^1(G_{F_v^+}, \mathrm{ad}^0(\overline{r})) \times H^1(G_{F_v^+}, \mathrm{ad}^0(\overline{r})(1)) \xrightarrow{\mathrm{tr}} \mathbb{F}(1).$$

Lemma 6.2. *The value $\dim_{\mathbb{F}} H^1_{\mathscr{S},T}(G_{F^+,S}, \mathrm{ad}^0(\overline{r}))$ is the same as*

$$\dim_{\mathbb{F}} H^1_{\mathscr{S}_S^\perp,T}(G_{F^+,S}, \mathrm{ad}^0(\overline{r})(1)) - \dim_{\mathbb{F}} H^0(G_{F^+,S}, \mathrm{ad}^0(\overline{r})) - [F^+ : \mathbb{Q}] - \sum_{v\in S\backslash T}\left(\dim(H^0(G_{F_v^+}, \mathrm{ad}^0(\overline{r}))) - \dim \overline{M}_v^1\right).$$

Proof. This is just a very convoluted application of Poitou-Tate duality. First, by the definition of $H^i_{\mathscr{S}_S,T}$, there is a long exact sequence

$$H^1_{\mathscr{S}_S,T}(G_{F^+,S}, \mathrm{ad}^0(\overline{r})) \longrightarrow H^1(G_{F^+,S}, \mathrm{ad}^0(\overline{r})) \longrightarrow \bigoplus_{v\in T} H^1(G_{F_v^+}, \mathrm{ad}^0(\overline{r})) \oplus \bigoplus_{v\in S\backslash T} H^1(G_{F_v^+}, \mathrm{ad}^0(\overline{r}))/\overline{M}_v^1 \longrightarrow$$

$$H^2_{\mathscr{S}_S,T}(G_{F^+,S}, \mathrm{ad}^0(\overline{r})) \longrightarrow H^2(G_{F^+,S}, \mathrm{ad}^0(\overline{r})) \longrightarrow \bigoplus_{v\in S} H^2(G_{F_v^+}, \mathrm{ad}^0(\overline{r})) \longrightarrow H^3_{\mathscr{S}_S,T}(G_{F^+,S}, \mathrm{ad}^0(\overline{r})).$$

From Poitou-Tate duality (with slight modification), we get

$$H^1(G_{F^+,S}, \mathrm{ad}^0(\overline{r})) \longrightarrow \bigoplus_{v\in T} H^1(G_{F_v^+}, \mathrm{ad}^0(\overline{r})) \oplus \bigoplus_{v\in S\setminus T} H^1(G_{F_v^+}, \mathrm{ad}^0(\overline{r}))/\overline{M}_v^1 \longrightarrow \left(H^1_{\mathscr{S}_T^\perp,T}(G_{F^+,S}, \mathrm{ad}^0(\overline{r})(1))\right)^\vee \longrightarrow$$

$$H^2(G_{F^+,S}, \mathrm{ad}^0(\overline{r})) \longrightarrow \bigoplus_{v\in S} H^2(G_{F_v^+}, \mathrm{ad}^0(\overline{r})) \longrightarrow H^0(G_{F^+,S}, \mathrm{ad}^0(\overline{r})(1))^\vee \longrightarrow 0.$$

Comparing two exact sequences and using Euler characteristic formula, one gets

$$\dim H^3_{\mathscr{S}_S,T} = \dim H^0(\mathrm{ad}^0(\overline{r})(1)),$$

$$\dim H^2_{\mathscr{S}_S,T} = \dim H^1_{\mathscr{S}_S^\perp,T},$$

and

$$\chi_{\mathscr{S}_S,T}(G_{F^+,S}, \mathrm{ad}^0(\overline{r})) = \chi(G_{F^+,S}, \mathrm{ad}^0(\overline{r})) - \sum_{v\in S} \chi(G_{F_v^+}, \mathrm{ad}^0(\overline{r})) + \sum_{v\in S\setminus T} \left(\dim H^0(G_{F_v^+}, \mathrm{ad}^0(\overline{r})) - \dim \overline{M}_v^1\right).$$

Now we know $\sum_{v\in S} \chi(G_{F_v^+}, \mathrm{ad}^0(\overline{r})) = 4[F^+ : \mathbb{Q}]$ and

$$\chi(G_{F^+,S}, \mathrm{ad}^0(\overline{r})) = \sum_{v|\infty} H^0(G_{F_v^+}, \mathrm{ad}^0(\overline{r})) + 4[F^+ : \mathbb{Q}] = \sum_{v|\infty} 1 + 4[F^+ : \mathbb{Q}].$$

$\square$

Corollary 6.1. *If* $\dim M_v^1 - \dim H^0(G_{F_v^+}, \mathrm{ad}^0(\overline{r})) = 0$ *for all* $v \in S\setminus T$*, then* $R^{\square_T}_{\mathscr{S}_S,T}$ *is topologically generated over* R_T^{loc} *by* $\left(\dim H^1_{\mathscr{S}_S^\perp,T} + \#(S\setminus T) - [F^+ : \mathbb{Q}] - \dim H^0(G_{F^+,S}, \mathrm{ad}^0(\overline{r})(1))\right)$*-many elements. In particular, if* $\overline{r}$ *is absolutely irreducible, then the number is* $\dim H^1_{\mathscr{S}_S^\perp,T} + \#(S\setminus T) - [F^+ : \mathbb{Q}]$.

Now we choose **auxiliary primes**:

Definition 6.4. *A set of* ***auxiliary primes*** Q *is a finite set of split finite primes such that*

(1) $Q \cap T = \varnothing$
(2) $Nv \equiv 1 (\mathrm{mod}\, p)$ *for all* $v \in Q$,
(3) $\overline{r}|_{G_{F_v^+}} = \overline{\psi_v} \oplus \overline{\psi_v'}$ *where* $\overline{\psi_v}$ *and* $\overline{\psi_v'}$ *are distinct unramified characters.*

Proposition 6.1. *Assume that* $\overline{r}|_{G_{F^+(\zeta_p)}}$ *is absolutely irreducible. Let*

$$q = \max(\dim H^1_{\mathscr{S}_T^\perp,T}(G_{F^+,T}, \mathrm{ad}^0(\overline{r})(1)), [F^+ : \mathbb{Q}]).$$

Then, (for p *big enough) for all* $N \geq 1$*, there exists a set of auxiliary primes* Q_N *such that*

(1) $\#Q_N = q$,
(2) $Nv \equiv 1 (\mathrm{mod}\, p^N)$,
(3) $R^{\square_T}_{\mathscr{S}_{Q_N \amalg T},T}$ *is topologically generated over* R_T^{loc} *by* $(q - [F^+ : \mathbb{Q}])$*-many elements.*

Lemma 6.3. *For a complete local noetherian* $\mathcal{O}$*-algebra* R *with finite residue field* $\mathbb{F}$*, if* $r : G_{F^+,S} \longrightarrow \mathrm{GL}_2(R)$ *reduces to* $\overline{r} : G_{F^+,T} \longrightarrow \mathrm{GL}_2(\mathbb{F})$*, where for all* $v \in S\setminus T$*,* $Nv \equiv 1 (\mathrm{mod}\, p)$ *and* $\overline{r}|_{G_{F_v^+}}$ *is unramified with distinct Frobenius eigenvalues, then*

$$r|_{G_{F_v^+}} \cong \begin{pmatrix} \psi_v & 0 \\ 0 & \psi_v' \end{pmatrix},$$

for distinct unramified characters ψ_v, ψ_v'.

Now the action of $\widetilde{\mathbb{T}}^{Q_N \coprod T}(U_1(Q_N))_{\widetilde{\mathfrak{m}}_{Q_N}}$ on $S_{\lambda,\sigma}(U_1(Q_N), \mathcal{O})_{\widetilde{\mathfrak{m}}_{Q_N}}$ gives the action of $R^{\text{univ}}_{\mathcal{S}_{Q_N \coprod T}}$ on it. Also, the character $\psi_v : G_{F_v^+} \longrightarrow \left(R^{\text{univ}}_{\mathcal{S}_{Q_N \coprod T}}\right)^\times$ coming from the above lemma, when restricted to $I_{F_v^+}$, factors through

$$I_{F_v^+} \twoheadrightarrow I^{\text{ab}}_{F_v^+} \twoheadrightarrow \mathcal{O}_{F_v^+} \twoheadrightarrow \mathbb{F}_v \twoheadrightarrow \Delta_v,$$

as $1 + \mathfrak{m}_{R^{\text{univ}}_{\mathcal{S}_{Q_N \coprod T}}}$ is pro-p. Thus, there is an action of

$$\mathcal{O}[\prod_{v \in Q_N} \Delta_v] \cong \frac{\mathcal{O}[y_1, \cdots, y_q]}{((y_1+1)^{p^N} - 1, \cdots, (y_q+1)^{p^N} - 1)},$$

on $R^{\text{univ}}_{\mathcal{S}_{Q_N \coprod T}}$, and if we denote the augmentation ideal as $\mathfrak{a}_{Q_N}$, then quotienting out by the augmentation ideal gives back the original deformation ring:

Lemma 6.4. *The map $R^{\square_T}_{\mathcal{S}_{Q_N \coprod T}} \longrightarrow R^{\square_T}_{\mathcal{S}_T}$ is the quotient map $R^{\square_T}_{\mathcal{S}_{Q_N \coprod T}} \twoheadrightarrow R^{\square_T}_{\mathcal{S}_{Q_N \coprod T}}/\mathfrak{a}_{Q_N}$.*

Proof. One shows that the quotient satisfies the universal property of the deformation ring for the deformation problem $\mathcal{S}_T$. $\square$

Recall that our deformation condition $\mathcal{S}_{Q_N \coprod T}$ is given by

$$\mathcal{S}_{Q_N \coprod T} = (F/F^+, T \coprod Q_N, \overline{r}, \chi^{-1}_{\text{cyc}}, \{R^{\square}_v\}_{v \in T} \cup \{R^{\psi_v}_v\}_{v \in Q_N}),$$

where $R^{\psi_v}_v$ is the quotient of $R^{\square}_v$ parametrizing lifts of $\overline{r}$ such that $r|_{G_{F_v}^+} \cong \psi_v \oplus \psi'_v$, with ψ'_v unramified. We denote $S_{\lambda,\sigma}(U_1(Q_N), \mathcal{O})_{\widetilde{\mathfrak{m}}_{Q_N}}$ as pr_{1,Q_N} and $\widetilde{\mathbb{T}}^{T \coprod Q_N}(U_1(Q_N))_{\widetilde{\mathfrak{m}}_{Q_N}}$ as $\mathbb{T}_{1,Q_N}$.

Proposition 6.2. *For all $v \in Q_N$, there exists $\psi_v : F^\times_{\overline{v}} \longrightarrow \mathbb{T}^\times_{1,Q_N}$ such that*

(1) $V_{\varpi_{\tilde{v}}} = \psi_v(\varpi_{\tilde{v}})$ on pr_{1,Q_N} and
(2) $(r_{\mathfrak{m}_{Q_N \coprod T}} \otimes \mathbb{T}_{1,Q_N})|_{W_{F_v^+}} \cong \psi_v \oplus \psi'_v$ for an unramified character ψ'_v.

This implies that $R^{\text{univ}}_{\mathcal{S}_{Q_N \coprod T}} \twoheadrightarrow \mathbb{T}_{1,Q_N}$, and the two actions

$$\mathcal{O}[\Delta_{Q_N}] \xrightarrow{\prod_{v \in Q_N} \psi_v} R^{\text{univ}}_{\mathcal{S}_{Q_N \coprod T}} \longrightarrow \text{End}(\text{pr}_{1,Q_N}),$$

coming from $R \twoheadrightarrow \mathbb{T}$, and

$$\mathcal{O}[\Delta_{Q_N}] \xrightarrow{V_{\varpi_{\tilde{v}}} \mapsto \left(\begin{smallmatrix} 1 & \\ & \varpi_{\tilde{v}} \end{smallmatrix}\right)} \text{End}(\text{pr}_{1,Q_N}),$$

coming from diamond action $U_0(Q_N)/U_1(Q_N)$, are the same.

7. **Patching.**

Now we can patch. Before patching, we fix several things.

(1) Fix $v_1 \notin S_p^+$, such that
- $v_1 = \widetilde{v}_1 \widetilde{v}_1^{\,c}$,
- $N v_1 \not\equiv 1 (\text{mod}\, p)$,
- $\overline{r}|_{G_{F^+_{v_1}}}$ has Frobenius eigenvalues whose ratio is not 1 or $(N v_1)^{\pm 1}$.

This means that,
- by picking $U_{v_1} = \iota^{-1}_{\tilde{v}_1}(I_{\omega_{\tilde{v}_1}})$, U is sufficiently small, and
- $R_{\overline{r}|_{G_{F^+_{v_1}}}} \cong \mathcal{O}[[x_{i,j} \mid 1 \le i, j \le 2]]$.

(2) Fix $\mathfrak{p} \in S_p^+$; then we send $\{\tau_v\}_{v\in S_p^+\setminus\{\mathfrak{p}\}} \mapsto \bigotimes_{v\in S_p^+\setminus\{\mathfrak{p}\}} \sigma(\tau_v)^{K_0} \otimes W_{\lambda_v}$ and $\{\lambda_v\}_{v\in S_p^+\setminus\{\mathfrak{p}\}} \longrightarrow (\mathbb{Z}_+^2)^{\mathrm{Hom}(F^+,E)}$, where τ_v's are inertial types and σ is inertial local Langlands.

This gives rise to

$$S(U_m, \sigma^{\circ,\mathfrak{p}} \otimes \mathcal{O}/\varpi^m)_{\mathfrak{m}} = S_{\lambda,\sigma,\mathfrak{p}}(U_m, \mathcal{O}/\varpi^m)_{\mathfrak{m}},$$

for

$$(U_m)_{\mathfrak{p}} = \iota_{\tilde{\mathfrak{p}}}^{-1}\left(\ker\left(\mathrm{GL}_2(\mathcal{O}_{F_{\tilde{\mathfrak{p}}}}) \twoheadrightarrow \mathrm{GL}_2(\mathcal{O}_{F_{\tilde{\mathfrak{p}}}}/\varpi_{\tilde{\mathfrak{p}}}^m)\right)\right).$$

We form modules to be patched,

$$M_{1,Q_N}^{\square} := S_{\lambda,\sigma,\mathfrak{p}}((U_1(Q_N))_{2N}, \mathcal{O}/\varpi^N)^{\vee}_{\tilde{\mathfrak{m}}_{Q_N}},$$

which has an action by $R^{\square_T}_{\mathscr{S}_{Q_N}\coprod T}$.

Lemma 7.1. *The maps*

$$S_{\lambda,\sigma,\mathfrak{p}}(U_{2N}, \mathcal{O}/\varpi^N) \hookrightarrow S_{\lambda,\sigma,\mathfrak{p}}(U_0(Q_N), \mathcal{O}/\varpi^N) \hookrightarrow S_{\lambda,\sigma,\mathfrak{p}}(U_1(Q_N)_{2N}, \mathcal{O}/\varpi^N),$$

induce

$$M_{1,Q_N}^{\square}/\mathfrak{a}_{Q_N} \xrightarrow{\sim} S_{\lambda,\sigma,\mathfrak{p}}(U_{2N}, \mathcal{O}/\varpi^N)^{\vee}_{\mathfrak{m}}.$$

Proof. Note that $M_{1,Q_N}^{\square}/\mathfrak{a}_{Q_N} = S_{\lambda,\sigma,\mathfrak{p}}(U_1(Q_N)_{2N}, \mathcal{O}/\varpi^N)^{\vee}_{\tilde{\mathfrak{m}}_{Q_N}}/\mathfrak{a}_{Q_N}$, and taking quotient by $\mathfrak{a}_{Q_N}$ in the dual is the same as taking invariants under $\mathfrak{a}_{Q_N}$ (as we've taken Pontryagin dual!), and this recovers U_0-level. Namely,

$$S_{\lambda,\sigma,\mathfrak{p}}(U_1(Q_N)_{2N}, \mathcal{O}/\varpi^N)^{\vee}_{\tilde{\mathfrak{m}}_{Q_N}}/\mathfrak{a}_{Q_N} = (S(U_1(Q_N)_{2N}, \mathcal{O}/\varpi^N)^{\Delta_Q})^{\vee}_{\tilde{\mathfrak{m}}_{Q_N}} = S(U_0(Q_N)_{2N}, \mathcal{O}/\varpi^N)^{\vee}_{\tilde{\mathfrak{m}}_{Q_N}} \cong S(U_{2N}, \mathcal{O}/\varpi^N)_{\mathfrak{m}}.$$

$\square$

Now we denote $K = G(\mathcal{O}_{F^+_{\mathfrak{p}}})$ and

$$G_N = \coprod_{i\leq N} K \iota_{\tilde{\mathfrak{p}}}^{-1}\begin{pmatrix} \varpi_{\mathfrak{p}}^i & \\ & 1 \end{pmatrix} K.$$

Let

$$R_\infty := \left(\left(\widehat{\bigotimes}_{v\in S_p\setminus\{\mathfrak{p}\}} R^{\lambda_v+(1,0),\tau_v,\square}_{\overline{\rho}_v}\right) \otimes R^{\square}_{\overline{\rho}_{\mathfrak{p}}} \otimes R^{\square}_{\overline{\rho}_v}\right)[[x_1, \cdots, x_{q-[F^+:\mathbb{Q}]}]],$$

and

$$S_\infty = \mathcal{O}[[y_1, \cdots, y_q, z_1, \cdots, z_{\#T}]],$$

where $T = S_p \coprod \{v_1\}$.

Proposition 7.1. *There is a commutative diagram*

$$\begin{tikzcd} M_{1,Q_N}^{\square} \arrow[r, "\alpha_N"] \arrow[d] & \mathrm{ind}_{KZ}^{G_N Z}(M_{1,Q_N}^{\square})_{K_N} \arrow[d] \\ S_{\lambda,\sigma,\mathfrak{p}}(U_{2N}, \mathcal{O}/\varpi^N)^{\vee}_{\mathfrak{m}} \arrow[r] & \mathrm{ind}_{KZ}^{G_N Z}(S_{\lambda,\sigma,\mathfrak{p}}(U_N, \mathcal{O}/\varpi^N)^{\vee}_{\mathfrak{m}}) \end{tikzcd}$$

whose maps are KZ-equivariant. Here, $\mathrm{ind}_{KZ}^{G_N Z}$ *means functions on $G_N Z$ which respect KZ-action on the original module.*

Let $\mathfrak{b}_N = (\varpi^N, (1+y_i)^{\#\Delta_i}-1, z_j^N) \subset S_\infty$, and

$$\delta_N = \mathfrak{m}^N_{R^{\mathrm{univ}}_{\mathscr{S}_T}} \cap \mathrm{Ann}_{R^{\mathrm{univ}}_{\mathscr{S}_T}}(M_\varnothing),$$

where

$$M_\varnothing := S_{\lambda,\sigma,\mathfrak{p}}(U_{2N}, \mathcal{O}/\varpi^N)^\vee_{\mathfrak{m}},$$

and consider, for $N' \geq N$,

$$M^\square_{N',N} = (M^\square_{1,Q_{N'}}/\mathfrak{b}_{N'}) \otimes_{S_\infty/\mathfrak{b}_{N'}} S_\infty/\mathfrak{b}_N.$$

Now consider

$$\begin{tikzcd}
\cdots & \mathrm{ind}_{KZ}^{G_{N-1}Z}(M^\square_{N,N-1})_{K_{N-1}} \arrow[l] & \mathrm{ind}_{KZ}^{G_N Z}(M^\square_{N,N})_{K_N} \arrow[l] \\
\cdots & M^\square_{N,N-1} \arrow[l] \arrow[u, "\alpha_{N-1}"] \arrow[d] & M^\square_{N,N} \arrow[l, "\mathrm{pr}_{N,N-1};\ \mathrm{mod}\ \mathfrak{b}_{N-1}"'] \arrow[u, "\alpha_N"] \arrow[d] \\
\cdots & M_\varnothing/\delta_{N-1} \arrow[l] & M_\varnothing/\delta_N \arrow[l]
\end{tikzcd}$$

Here, the first row has S_∞-action, the second row has R_∞-action, and the third row has $R^{\mathrm{univ}}_{\mathscr{S}_T}/\delta_m$-action, which are all compatible via $S_\infty \longrightarrow R_\infty \twoheadrightarrow R^{\square_T}_{\mathscr{S}_{Q_N}} \longrightarrow R^{\mathrm{univ}}_{\mathscr{S}_T}/\delta_m$.

Upshot. $S_\infty/\mathfrak{b}_N$, $R^{\mathrm{univ}}_{\mathscr{S}_T}/\delta_N$ are all finite sets, and there are finitely many isomorphism classes of $M^\square_{N',N} \longrightarrow \mathrm{ind}_{KZ}^{G_N Z} M^\square_{N',N}$, for a fixed N.

Therefore, by pigeonhole principle, there is a projective system $(M^\square_i) = (M^\square_{N(i),i})$ such that

$$\begin{tikzcd}
\cdots & \mathrm{ind}_{KZ}^{G_i Z}(M^\square_i) \arrow[l] & \mathrm{ind}_{KZ}^{G_{i+1} Z}(M^\square_{i+1}) \arrow[l] & \cdots \arrow[l] \\
\cdots & M^\square_i \arrow[l] \arrow[u] \arrow[d] & M^\square_{i+1} \arrow[l] \arrow[u] \arrow[d] & \cdots \arrow[l] \\
\cdots & M_\varnothing/\delta_i \arrow[l] & M_\varnothing/\delta_{i+1} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Thus we can make sense of the patched module.

Definition 7.1. *The* ***patched module*** *is defined by* $M_\infty = \varprojlim_i M^\square_i$.

The patched module M_∞ has the following properties, due to the way we chose the projective system.

(1) M_∞ has a continuous action by

$$\varprojlim_{\mathfrak{b}_N} S_\infty/\mathfrak{b}_N[\mathrm{GL}_2(\mathcal{O}_{F_{\tilde{\mathfrak{p}}}}/\varpi^N_{\tilde{\mathfrak{p}}})] = S_\infty[[K]],$$

(2) M_∞ has compatible actions by S_∞ and R_∞, as the S_∞-action is continuous, so

$$\mathrm{im}(S_\infty \longrightarrow \mathrm{End}_{S_\infty}(M_\infty)),$$

is closed.

(3) $M_\infty/\mathfrak{b}_i \xrightarrow{\sim} (M^\square_{1,Q_{N(i)}}/\mathfrak{b}_i)_{K_{2i}}$,

(4) $M_\infty/(\mathfrak{b}_1+\mathfrak{a}_\infty) \longrightarrow S_{\lambda,\sigma,\mathfrak{p}}(U_1(Q_{N(1)})_2,\mathbb{F})^\vee_{\tilde{\mathfrak{m}}_{Q_{N(1)}}}$, which recovers $S_{\lambda,\sigma,\mathfrak{p}}(U_2,\mathbb{F})^\vee_{\mathfrak{m}}$ after reduction mod $\mathfrak{a}_{Q_{N(1)}}$.

(5) There is a KZ-equivariant commutative diagram

$$\begin{tikzcd} M_\infty \arrow[rr, "\alpha_\infty"] \arrow[d, "\mathrm{pr}_i"'] & & \mathrm{Ind}_{KZ}^G M_\infty \arrow[d] \\ (M^\square_{1,Q_{N(1)}}/\mathfrak{b}_i)_{K_{2i}} \arrow[rr, "\alpha_i"] & & \mathrm{ind}_{KZ}^{G_iZ}(M_{1,Q_{N(1)}}/\mathfrak{b}_i)_{K_i} \end{tikzcd}$$

In other words, if $m = \lim_{i\to\infty} m_i$, $\alpha_\infty(m) = \lim_{i\to\infty}\alpha_i(m_i)$.

By hand, one can also check the following.

Proposition 7.2. *(1) M_∞ is fintiely generated and projective over $S_\infty[[K]]$.*
(2) $\alpha_\infty : M_\infty \longrightarrow \mathrm{Ind}_{KZ}^G M_\infty$ is injective.
(3) $\alpha_\infty(M_\infty)$ is $\mathrm{GL}_2(F_{\tilde{\mathfrak{p}}})$-stable.

We note its compatibility with completed cohomology. Recall that the **completed cohomology** can be defined as

$$\tilde{S}_{\lambda,\sigma,\mathfrak{p}}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}} = \varprojlim_N \varinjlim_i S_{\lambda,\sigma,\mathfrak{p}}(U_{2i},\mathcal{O}/\varpi^N)_{\mathfrak{m}},$$

which has an action of $\mathbb{T}^T(U^{\mathfrak{p}})_{\mathfrak{m}} = \varprojlim_i \mathbb{T}^T(U_{2i}) \twoheadleftarrow R^{\mathrm{univ}}_{\mathscr{S}_T}$.

Theorem 7.1 (Caraiani-Emerton-Gee-Geraghty-Paskunas-Shin). *There is a $\mathrm{GL}_2(F_{\tilde{\mathfrak{p}}})$-isomorphism*

$$M_\infty/\mathfrak{a}_\infty \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}}(\tilde{S}_{\lambda,\sigma,\mathfrak{p}}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}},\mathcal{O}),$$

which respects the actions of $R^\infty/\mathfrak{a}_\infty$ and $R^{\mathrm{univ}}_{\mathscr{S}_T}$ via the natural quotient map $R^\infty/\mathfrak{a}_\infty \twoheadrightarrow R^{\mathrm{univ}}_{\mathscr{S}_T}$.

Part 2. **Seminars**

Pascal Boyer, *About Ihara's lemma in higher dimension*

1. Introduction.

Suppose you have a smooth model $\mathbb{X}$ over $\mathbb{Z}[1/N]$ of smooth projective $\mathbb{Q}$-variety X. One defines a **Hasse-Weil zeta function** of X, $Z_X^*(s)$, as a formal product

$$Z_X^*(s) = \prod_{p\nmid N} Z_{X,p}(p^{-s}),$$

where the local Euler factor $Z_{X,p}(T)$ is defined formally as

$$Z_{X,p}(T) = \exp\left(\sum_{n\geq 1} \frac{|\mathbb{X}_p(\mathbb{F}_{p^n})|}{n} T^n\right),$$

where $\mathbb{X}_p = \mathbb{X} \times_{\mathbb{Z}[1/N]} \mathbb{F}_p$. More generally, for many cases there is a way to define the local Euler factor at "bad primes", i.e. $p \mid N$, and obtain the complete Hasse-Weil zeta function $Z_X(s)$. It is clear from the notation that bad Euler factors should be defined such that $Z_X(s)$ does not depend on a choice of $\mathbb{X}$ or N. It is a consequence of Weil's conjecture (proved by Deligne) that this formal product is holomorphic for $\operatorname{Re} s \gg 0$.

Example 1.1. If $X = \operatorname{Spec}\mathbb{Q}$, then $Z_X(s)$ is the Riemann zeta function.

Conjecture 1.1 (Hasse-Weil conjecture)**.** *The zeta function $Z_X(s)$ can be meromorphically continued to the whole complex plane. Furthermore, it should satisfy some appropriate functional equation.*

Remark 1.1. Even though $Z_X(s)$ needs Euler factors at bad primes, the subjects of Hasse-Weil conjecture, namely meromorphic continuation and functional equation, can tolerate any finite number of Euler factors missing. Thus, one might as well just not worry about Euler factors at bad primes and work with $Z_X^*(s)$ using a specific smooth integral model.

Even though the conjecture is formulated purely in terms of algebraic geometry, essentially the only way it can be proved is by relating it to **automorphic L-functions**, where one knows from general theory of Langlands that meromorphic continuation and functional equation indeed are true. One can recover $Z_X(s)$ from the Galois representation $G_{\mathbb{Q}} \longrightarrow \mathrm{GL}(H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell))$, so it is more convenient to compare Galois representations and automorphic forms. This is a very general picture of Langlands program.

There are typically two directions in the Langlands program, namely automorphic to Galois direction and Galois to automorphic direction. A celebrated result of Wiles says that, given an irreducible Galois representation $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, if $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ is modular and if ρ is nice, then ρ is modular. In proving this (for GL_2), one crucial ingredient is **Ihara's lemma**. Roughly, this is a statement about level raising; if ρ_1, ρ_2 are two 2-dimensional Galois representations of level N_1, N_2, respectively, $N_1 < N_2$, and if $\overline{\rho}_1 \equiv \overline{\rho}_2 (\mathrm{mod}\, \ell)$, then the modularity of ρ_1 implies the modularity of ρ_2.

2. Ihara's lemma of Clozel-Harris-Taylor.

Clozel-Harris-Taylor formulated and conjectured a generalized version of Ihara's lemma, which we would like to formulate precisely.

We work with a definite unitary group as follows. Let $F = F^+E$ be a CM field with F^+ totally real and E imaginary quadratic. Let $\overline{B}$ be a division algebra over F of dimension d^2, and let $*$ be an involution on it. Given $\beta \in \overline{B}$ with $\beta^* = -\beta$, we define a $\mathbb{Q}$-group $\overline{G}$ as

$$\overline{G}(R) = \left\{ \begin{array}{c} (\lambda, x) \in R^\times \times (\overline{B}^{\mathrm{op}})^\times \text{ such that} \\ xx^{\#_\beta} = \lambda \end{array} \right\},$$

for any $\mathbb{Q}$-algebra R, where $\overline{B}^{\mathrm{op}} = \overline{B} \otimes_{F,c} F$ (c is the complex conjugation) and $x^{\#_\beta} = \beta x^* \beta^{-1}$. We want this to be definite, i.e. the associated unitary group is compact at infinity.

For a prime p which splits into $\mu\mu^c$ in E,

$$\overline{G}(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times \prod_{w|\mu} \overline{B}_w^{\mathrm{op},\times}.$$

We choose rational primes p, q such that both split completely in E, and there are primes $w \mid p$ and $v \mid q$ such that $\overline{B}_w^{\mathrm{op},\times}$ is the division algebra of invariant $1/d$ and $\overline{B}_v^{\mathrm{op},\times}$ is $\mathrm{GL}_d(F_v)$.

Pick $\ell \neq p, q$, and for S a finite set of rational places (including bad primes and ∞), let $\mathbb{T}_S$ be the $\mathbb{Z}_\ell$-algebra generated by Hecke operators away from S inside the $\mathbb{Z}_\ell$-endomorphism algebra on the space of algebraic automorphic forms. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_S$, such that the associated Galois representation $\overline{\rho}_\mathfrak{m} : G_F \longrightarrow \mathrm{GL}_d(\overline{\mathbb{F}}_\ell)$ is absolutely irreducible.

Conjecture 2.1 (Ihara's lemma; Clozel-Harris-Taylor)**.** *Let $\overline{U}$ be an open compact subgroup of $\overline{G}(\mathbb{A})$, and $\overline{\Pi}$ be an irreducible subrepresentation of $C^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/U^v, \overline{\mathbb{F}}_\ell)_\mathfrak{m}$. Then, $\overline{\Pi}_v$ is* ***generic****.*

3. **Mirabolic subgroups and genericity.**

We discuss the meaning of "generic" in Conjecture 2.1.

Definition 3.1. *The* ***mirabolic subgroup*** *$M_d \subset \mathrm{GL}_d(F_v)$ is the subgroup of the form*

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Let V_d be the unipotent radical of M_d. Fix a nontrivial character χ of $\mathbb{G}_a$, and let θ be the character of V_d defined by $\theta(M) = \chi(M_{d-1,d})$, where the subscript means the entry corresponding to the subscript.

Remark 3.1. If τ is a smooth irreducible M_d-representation over $\overline{\mathbb{Q}}_\ell$, then $\overline{\tau}$ is a smooth irreducible M_d-representation over $\overline{\mathbb{F}}_\ell$.

Definition 3.2. *Let Φ^- (Ψ^-, respectively) be the functor of taking (V_d, id)-coinvariants ((V_d, θ)-coinvariants, respectively) from the category of smooth representations of M_d to the category of smooth representations of GL_{d-1} (M_{d-1}, respectively). Given a smooth M_d-representation τ, and $1 \leq h \leq d$, let $\tau^{(h)}$, a smooth GL_{d-h}-representation, be the h-**th derivative**, i.e. $\Phi^- \circ \Psi^- \circ \cdots \circ \Psi^-(\tau)$, where Ψ^- is composed $h-1$ times.*

Proposition 3.1. *If τ is an irreducible smooth M_d-representations, then there is unique $1 \leq h \leq d$ such that $\tau^{(h)} \neq 0$, and τ can be recovered from $\tau^{(h)}$ by applying certain functors ("$\tau = (\Psi^+)^{h-1}\Phi^+(\tau^{(h)})$").*

In particular, this implies that there is **unique** smooth irreducible M_d-representation with nonvanishing d-th derivative. We call this the **generic** (or **nondegenerate**) representation τ_{nd}.

Definition 3.3. *Given a smooth M_d-representation τ, let $\lambda(\tau)$ be the largest integer such that $\tau^{(\lambda(\tau))} \neq 0$.*

Theorem 3.1. *For an irreducible smooth* $\mathrm{GL}_d(F_v)$*-representation* Π_v*, and for any irreducible subrepresentation* τ *of* $\Pi_v|_{M_d}$*,* $\lambda(\tau) = \lambda(\Pi_v|_{M_d})$*.*

Definition 3.4. *A smooth irreducible* GL_d*-representaiton* Π *is* ***generic*** *if* $\Pi|_{M_d}$ *has* τ_{nd} *as its subquotient.*

Example 3.1.
- For GL_2, every representation not of finite dimensional is generic.
- For GL_3, given a character χ of $F_v^\times$, $\mathrm{Ind}_B^{\mathrm{GL}_3}(\chi(-1)\otimes\chi\otimes\chi)$ has four irreducible subquotients, and only one is generic, which is the **Steinberg representation** $\mathrm{St}_3(\chi)$. Even after mod ℓ there is only one generic irreducible subquotient.
- Similarly, for GL_d, $\mathrm{Ind}_B^{\mathrm{GL}_d}(\chi(-\frac{d-1}{2})\otimes\cdots\otimes\chi(\frac{d-1}{2}))$ has 2^{d-1} irreducible subquotients, and only one is generic, which is the Steinberg representation $\mathrm{St}_d(\chi)$. Even after mod ℓ there is only one generic irreducible subquotient.

In $\overline{\mathbb{Q}}_\ell$-coefficients, for any automorphic Π for $\overline{G}$, τ_{nd} is the only irreducible subspace of $\Pi|_{M_d}$, because everything is cuspidal (no cusps for 0-dimensional locally symmetric space). This does not apply to $\overline{\mathbb{F}}_\ell$-coefficients, but nevertheless says that $\overline{\tau}_{nd}$ is an irreducible subspace of $\overline{\Pi}|_{M_d}$. Thus, the genericity statement in Conjecture 2.1 is rather saying that "there is a unique irreducible subspace."

4. A strategy for Conjecture 2.1.

A Shimura variety associated to $\overline{G}$ is 0-dimensional, so there is no geometry. Instead, we consider a similar unitary group G such that
- $G(\mathbb{A}^{\infty,w}) = \overline{G}(\mathbb{A}^{\infty,w})$,
- $G(\mathbb{R})$ is of signature $(1, d-1), (0, d), \cdots, (0, d)$,
- and $G(F_w) = \mathrm{GL}_d(F_w)$.

Let Sh_G be a Shimura variety associated to G. This can be referred as a **Kottwitz-Harris-Taylor Shimura variety** (or KHT Shimura variety in short). Then, given $\overline{\Pi}$ an irreducible smooth representation of $\overline{G}$, the w-component $\overline{\Pi}_w$ gives rise to a $\overline{\mathbb{Z}}_\ell$-local system $\mathrm{HT}(\Pi_w)$ on $\mathrm{Sh}_w^{=d}$, the supersingular points of the special fiber of Sh_G at w, which coincides with the Shimura variety associated to $\overline{G}$. To be a little more precise, we send $\overline{\Pi}_w$ via mod ℓ Jacquet-Langlands transfer (Dat, $\cdots$) to a mod ℓ representation τ_w of $\mathrm{GL}_g(F_w)$, which is of form $\mathrm{Speh}_s(\rho)$, where ρ is an irreducible supercuspidal representation, $s \mid d$ and Speh_s is the "super-Speh" representation. Then, this gives rise to a local system $\mathrm{HT}(\tau_w, s)$ on $\mathrm{Sh}_w^{=d}$.

The $\overline{\mathbb{Z}}_\ell$-local system $\mathrm{HT}(\Pi_w)$ is a strict subsheaf of $\Psi_{\overline{\mathbb{Z}}_\ell}$, the vanishing cycle (perverse) sheaf, which means that it is a sub with free cokernel. Thus, $H^0(\mathrm{Sh}_w^{=d}, \mathrm{HT}(\Pi_w))_\mathfrak{m} \hookrightarrow H^{d-1}(\mathrm{Sh}_{G,\overline{F}}, \overline{\mathbb{Z}}_\ell)_\mathfrak{m}$ has free cokernel, if the following hypothesis is satisfied:

Hypothesis 3. $\rho, \rho(1), \cdots, \rho(s-1)$ are pairwise different.

Remark 4.1. There is little hope of weakening this hypothesis.

Remark 4.2. Before trying to prove Conjecture 2.1, one might want to prove that any irreducible sub of $H^0(\mathrm{Sh}_v^{=d}, \mathrm{HT}(\Pi_v)_{\overline{\mathbb{F}}_\ell})$ is isomorphic to mod ℓ reduction of τ_{nd}. By a theorem of Berkovich, this is a problem about étale cohomology of Lubin-Tate spaces, with $\overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{Z}}_\ell$-coefficients. This local analogue of Ihara's lemma is known to be true.

Theorem 4.1 (Boyer)**.** *Suppose that* $\mathfrak{m}$ *and its associated mod* ℓ *Galois representation* $\overline{\rho}_\mathfrak{m}$ *satisfy the following hypotheses.*

- ***Hypothesis 1.*** *$\overline{\rho}_{\mathfrak{m}}$ is irreducible, and $\ell \geq d+1$.*
- ***Hypothesis 2.*** *$\overline{\rho}_{\mathfrak{m},v}$ is multiplicity-free.*
- ***Hypothesis 3.*** *Stated as above.*

Then, Conjecture 2.1 is true.

Idea of proof. We want to construct a filtration of $H^{d-1}(\mathrm{Sh}_{G,\overline{F}}, \overline{\mathbb{Z}}_\ell)_{\mathfrak{m}}$ such that graded pieces are all free such that each has nondegeneracy property of irreducible submodules (i.e. each graded piece satisfies Ihara's lemma).

We filter the vanishing cycle sheaf Ψ using geometry of the special fiber. Namely, using the natural stratification

$$\mathrm{Sh}_{\overline{w}} = \mathrm{Sh}^{\geq 1} \supset \mathrm{Sh}^{\geq 2} \supset \cdots \supset \mathrm{Sh}^{\geq d},$$

using $j^{\geq h} : \mathrm{Sh}^{=h} \hookrightarrow \mathrm{Sh}^{\geq h}$, by adjointness we have a natural morphism

$$(j^{\geq h})_!(j^{\geq h})^* \longrightarrow \mathrm{id}\,.$$

We want to say that for example the image of $(j^{\geq 1})_!(j^{\geq 1})^*\Psi \longrightarrow \Psi$ to be $\mathrm{Fil}^1\,\Psi$, but the cokernel might not be free. On the other hand, there is a general process of "saturation" for perverse sheaves, so that if we take the saturation of the image, we get a sub whose cokernel is free. In this way we get a filtration of Ψ whose graded pieces are all free. Using spectral sequences we subsequently get a filtration on $H^{d-1}(\mathrm{Sh}_{G,\overline{F}}, \overline{\mathbb{Z}}_\ell)_{\mathfrak{m}}$. The filtration has free graded pieces because of Hypothesis 2.

Remark 4.3. In general, the graded pieces are parabolically induced modules. In particular, if $p \equiv 1 (\mathrm{mod}\ \ell)$, then some of the parabolic inductions mod ℓ are semisimple, so there is no hope of proving Ihara's lemma in this case (which is unfortunate as this is the setting for most arithmetic applications).

The cohomology $H^*(\mathrm{Sh}_{G,\overline{F}}, \overline{\mathbb{Z}}_\ell)_{\mathfrak{m}}$ is torsion-free.

Remark 4.4. This is no longer true if you don't localize at $\mathfrak{m}$, as for any weight ξ, the automorphic sheaf associated to ξ always has torsion cohomology for sufficiently small level.

If there is a place of F at which the mod ℓ Satake parameters coming from $\mathfrak{m}$ do not have $p^{\pm 1}$ as a ratio of two Satake parameters, then the cohomology is torsion-free. If ℓ is big enough, one can indeed find such a place and get the torsion-freeness.

Remark 4.5. This is almost the same condition as Caraiani-Scholze.

Remark 4.6. In general, the cohomology is expected to be free if $\overline{\rho}_{\mathfrak{m}}$ is irreducible.

□

5. **Applications.**

Let Π_v be an irreducible tempered $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_d(F_v)$. Then Π_v is the Langlands quotient of some parabolic induction of tensor product of unlinked Steinbergs, but after reduction mod ℓ, there might be new linkage appearing.

Proposition 5.1 (Level raising and "fixing")**.** *If $\mathfrak{m}$ satisfies Hypotheses 1 and 2 of Theorem 4.1, then there is a characteristic 0 representation exhibiting all the linkages of mod ℓ reduction of Π_v. Furthermore, "this is the only possible lift."*

Remark 5.1. The level raising part of Proposition 5.1 appears in Clozel-Harris-Taylor for one segment.

Remark 5.2. If there is a torsion class in cohomology, then there is an extra automorphic congruence. Conversely, Hypothesis 1 says that there is no torsion class and thus no extra automorphic congruences.

1. **Bianchi modular forms.**

Let F be an imaginary quadratic field, and $\underline{G} = \mathrm{GL}_{2,F}$. Consider $G_{\mathbb{A}} = \underline{G}(\mathbb{A}_F) = G_f \times G_\infty$ and $G = \underline{G}(F)$. Let Z_∞ be the center of $G_\infty = \mathrm{GL}_2(\mathbb{C})$, and $K_\infty = \mathrm{U}_2 \subset G_\infty$ be a maximal compact subgroup. Then the associated symmetric space $G_\infty/Z_\infty K_\infty \cong \mathfrak{h}_3$, the three-dimensional upper half space, which is identified via

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mapsto (x, y), x \in \mathbb{C}, y \in \mathbb{R}^{+\times},$$

which is possible via Iwasawa decomposition. Then G_∞ naturally acts on $\mathfrak{h}_3$.

In any case, for a small enough compact open level $U \subset G_{\mathbb{A}}$, we define $X_U = G\backslash G_{\mathbb{A}}/UZ_\infty K_\infty$. It is identified with $\coprod_{i=1}^r \Gamma_i\backslash\mathfrak{h}_3$, where $G_{\mathbb{A}} = \coprod_{i=1}^r Gt_iUG_\infty$, and $\Gamma_i = \mathrm{Stab}_G(t_iUG_\infty)$. It has no hope of being an algebraic variety as it is odd-dimensional.

Consider a large enough coefficient field $L \subset \overline{\mathbb{Q}}_p$, and let $\mathcal{O} = \mathcal{O}_L$, $k = \mathcal{O}/(\varpi)$ where ϖ is a uniformizer of L. We are interested in Betti cohomology groups $H^i(X_U, \mathcal{O})$, for $i = 1, 2$. The Poincaré duality induces a pairing $H^1(X_U, \mathcal{O}) \times H_c^2(X_U, \mathcal{O}) \longrightarrow \mathcal{O}$, which is perfect **modulo torsion**. It turns out that H^1 has no torsion, because

$$H^1[\varpi] = \mathrm{coker}(H^0(\mathcal{O}) \longrightarrow H^0(\mathcal{O}/\varpi)) = 0,$$

while there are lot of torsion in H^2.

Remark 1.1. We will localize everything at a non-Eisenstein maximal ideal of Hecke algebra, so we will deliberately not distinguish between H_c^* and H^*.

The Hecke action by $g_f \in G_f$ on $H^i(X_U, \mathcal{O})$ is defined by the double coset operator $[Ug_fU]$, which is the geometric correspondence from the diagram

$$\begin{tikzcd} & X_{U\cap g_fUg_f^{-1}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_U & & X_U \end{tikzcd}$$

where π_1 comes from the natural map and π_2 comes from the natural map after conjugating by g_f, and $[Ug_fU] = (\pi_1)_*\pi_2^*$.

Definition 1.1. *For v at which U is not ramified, the double coset operator corresponding to the element of G_f which is $\begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix}$ at v and 1 everywhere else is denoted as T_v, and the double coset operator corresponding to the element of G_f which is $\begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix}$ at v at 1 everywhere else is denoted as S_v.*

Let $\mathfrak{h}^i(U, \mathcal{O})$ be the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(H^i(X_U, \mathcal{O}))$ generated by such T_v, S_v's.

The above observations then imply that $\mathfrak{h}^1$ is a finite flat $\mathcal{O}$-algebra, while $\mathfrak{h}^2$ is a finite $\mathcal{O}$-algebra (which is not necessarily flat).

Proposition 1.1. *There is a twist of Poincaré duality which gives rise to a perfect-modulo-torsion pairing $H^1 \times H_c^2 \longrightarrow \mathcal{O}$ which is* ***Hecke-equivariant****, i.e. $\langle Tx, y\rangle = \langle x, Ty\rangle$ for $T = T_v$ or S_v.*

This is done by twisting by Atkin-Lehner involution.

Corollary 1.1. *The torsion-free $\mathcal{O}$-algebra $\mathfrak{h}^1$ is the torsion-free quotient of $\mathfrak{h}^2$.*

Thus if one knows $\mathfrak{h}^2$, one knows everything.

Definition 1.2. *Let ω be a character of $Z_{\mathbb{A}}$. The* ***space of Bianchi modular forms*** *of weight 2, level U and nebentypus ω, $S_2(U,\omega)$, is defined by*

$$S_2(U,\omega) = \bigoplus_{\pi_\infty = \rho^0_\infty} H^U_\pi,$$

where

- $\rho^0_\infty = \mathrm{Ind}^{G_\infty}_{B_\infty} \lambda_0$, *where* $\lambda_0 : B_\infty \longrightarrow \mathbb{C}^\times$ *is defined by* $\lambda_0 \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} = \frac{t_1}{t_2}$,
- H_π *for a cuspidal automorphic representation π is the space realizing π,*
- *and π runs over cuspidal automorphic representations of central character ω whose infinity type is ρ^0_∞.*

Equivalently, $S_2(U,\omega)$ is the space of functions $f : G\backslash G_{\mathbb{A}}/U \longrightarrow \mathbb{C}^2$ where

- $\mathbb{C}^2$ *is regarded to be a representation of $K_\infty = \mathrm{U}_2$ via restricting the standard representation structure of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}^2$,*
- $f(gk_\infty) = k_\infty^T \cdot f(g)$ *for* $k_\infty \in K_\infty$,
- $f(gz) = f(g)\omega(z)$ *for* $z \in Z_{\mathbb{A}}$,
- $\int_{Z_{\mathbb{A}} G\backslash G_{\mathbb{A}}} \|f\|^2 dg < \infty$ *for a choice of norm on* $\mathbb{C}^2$,
- $f(g_f g_\infty) \in C^\infty(G_\infty)$ *is "rapidly decreasing" for any choice of* $g_f \in G_f$.

The first definition is "Langlands-style", whereas the second definition is "Harish-Chandra-style." The second definition is more appropriate for algebraic interpretation.

2. Integral structures on the space of Bianchi modular forms.

Now let $\omega = \mathbb{1}$ and $U = U_0(\mathfrak{n})$. We might as well just denote $S_2(U,\mathbb{1}) = S_2(U,\mathbb{C})$. To define transcendental (Deligne) periods, one compares two integral/rational strucures,

- one coming from integral model of Shimura varieties,
- one coming from integral coefficients.

But the problem is that there is no Shimura variety here. Thus, we use a different method. We start by observing the following.

Theorem 2.1 (Harder)**.** *There is an explicit $\mathfrak{h}^i$-equivariant isomorphism*

$$\omega^i : S_2(U,\mathbb{C}) \to H^i_{\mathrm{dR,cusp}}(X_U/\mathbb{C}),$$

where here H^i_{dR} is just in terms of differential forms.

Of course X_U is not like an algebraic variety, so there is no integral model, and therefore there is no integral structre on $H^i_{\mathrm{dR}}(X_U/\mathbb{C})$ coming from geometry (although one can still take the other integral structure, namely the one coming from differential forms with coefficients in $\mathcal{O}$).

On the other hand, all automorphic representations appearing in $S_2(U,\mathbb{C})$ have **Whittaker models**. In particular, there are **integral Whittaker models** $\mathcal{W}_\pi$ for each such automorphic representation π, so that one has an explicit basis of the Whittaker model. This enables us to define an "integral structure" on $S_2(U,\mathbb{C})$, $S_2(U,\mathcal{O})$. By Harder's theorem, this integral structure naturally transports to $H^i_{\mathrm{dR}}(X_U/\mathbb{C})$. This gives us a foundation to define transcendental periods even though there is no algebraic geometry involved.

Let $\mathbb{f}$ be a Bianchi cusp eigen-newform which is of weight 2 and level $U_0(\mathfrak{n})$. Let $\lambda_{\mathbb{f}} : \mathfrak{h}^2(U_0(\mathfrak{n}), \mathcal{O}) \longrightarrow \mathcal{O}$ be the corresponding Hecke eigenvalue. As it is new, $S_2(U_0(\mathfrak{n}), \mathbb{C})[\lambda_{\mathbb{f}}] = \mathbb{C}\mathbb{f}$.

Definition 2.1. *Let $\delta^i(\mathbb{f})$ be an $\mathcal{O}$-integral basis, well-defined up to $\mathcal{O}^\times$, of $(H^i_B(X_U, \mathcal{O})/\mathrm{tors})[\lambda_{\mathbb{f}}]$, where B means Betti cohomology and* tors *means the torsion-free quotient.*

Let $u^i(\mathbb{f}) \in \mathbb{C}^\times/\mathcal{O}^\times$ be such that $\omega^i(\mathbb{f}) = u^i(\mathbb{f})\delta^i(\mathbb{f})$.

That the two vectors are parallel is precisely the consequence of newness.

Remark 2.1. The periods $u^i(\mathbb{f})$ "do not see torsion."

3. Congruence modules.

Now we can try to develop a theory of congruences and ultimately define **congruence modules**. Before we proceed, we emphasize again that our Bianchi modular form $\mathbb{f}$ is **non-Eisenstein**, in the following sense. Namely, the maximal ideal $\mathfrak{m}$ which is defined to be the kernel of the reduction of the Hecke eigenvalues $\overline{\rho}_{\mathbb{f}} : \mathfrak{h}^2(U, \mathcal{O}) \longrightarrow k$ is not an Eisenstein ideal, or more concretely

$$\overline{\lambda}_{\mathbb{f}}(T_v) \neq 1 + Nv,$$

for some v. Therefore there is no harm in identifying compactly supported cohomology/parabolic cohomology with just usual Betti cohomology.

Let $T = \mathfrak{h}^2_{\mathfrak{m}}$, which acts faithfully on $H^2_{\mathfrak{m}}$, and $\widetilde{T} = T/\mathrm{tors} = \mathfrak{h}^1_{\mathfrak{m}}$, which acts faithfully on $H^1_{\mathfrak{m}}$. We have Petersson inner product here as well, so $\mathfrak{h}^i \otimes_{\mathcal{O}} L$ is semisimple. Thus, $T \otimes_{\mathcal{O}} L$ is semisimple, and in particular, $\lambda_{\mathbb{f}} : T_L \twoheadrightarrow L$ splits. We can write it as

$$T_L = L \times T'_L,$$

where the first projection is precisely $\lambda_{\mathbb{f}}$. Morally, T'_L detects eigenvalues of all other Bianchi forms. In particular, if we consider $T' = \mathrm{im}(T \longrightarrow T'_L)$, then this detects congruences between $\mathbb{f}$ and all other Bianchi modular forms.

Definition 3.1. *Let $T^{\mathbb{f}} = \mathbb{1}_{\mathbb{f}}\widetilde{T}$, where $\mathbb{1}_{\mathbb{f}}$ is the idempotent in T_L corresponding to the first factor L of $T_L = L \times T'_L$. Let $T_{\mathbb{f}} = \widetilde{T} \cap \mathbb{1}_{\mathbb{f}} T_L$, which is a submodule of $T^{\mathbb{f}}$. The* ***(first) congruence module*** *$c^0_{\mathbb{f}}$ is defined by $T^{\mathbb{f}}/T_{\mathbb{f}}$.*

More generally, for a finite free $\mathcal{O}$-module M with a T-action, then $M^{\mathbb{f}} = \mathbb{1}_{\mathbb{f}}M \subset M_L$, $M_{\mathbb{f}} = M \cap \mathbb{1}_{\mathbb{f}}M_L$, and $c^0_{\mathbb{f}}(M) = M^{\mathbb{f}}/M_{\mathbb{f}}$.

Remark 3.1. (1) The notion of congruences "does not see torsion."
(2) $c^0_{\mathbb{f}} = T' \otimes_T \mathcal{O} = T'/\mathfrak{c}'$, where $\mathfrak{c}' = T \cap (\{0\} \times T') = \ker \lambda_{\mathbb{f}} \subset T'$. In particular, $c^0_{\mathbb{f}}$ is a ring, and $c^0_{\mathbb{f}}(M)$ is a $c^0_{\mathbb{f}}$-module.
(3) If $M^* = \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$, then $c^0_{\mathbb{f}}(M)$ and $c^0_{\mathbb{f}}(M^*)$ are Pontryagin duals to each other.

Definition 3.2. *The* ***(first) congruence ideal*** *of a finite free $\mathcal{O}$-module with T-action M is $\eta_{\mathbb{f}}(M) = \mathrm{Fitt}(c^0_{\mathbb{f}}(M))$. If $M = \widetilde{T}$ we drop M in the notation.*

In particular, $\eta_{\mathbb{f}}(M) = \eta_{\mathbb{f}}(M^*)$.

Theorem 3.1 (Urban)**.** *If $\mathfrak{m}$ is non-Eisenstein, $p > 3$ and $p \nmid \varphi(N\mathfrak{n})$,*

$$\frac{L(\mathrm{Ad}^0 \mathbb{f}, 1)}{\pi^2 u^1(\mathbb{f}) u^2(\mathbb{f})} \sim \eta_{\mathbb{f}}(H^2_{\mathfrak{m}})(\sim \eta_{\mathbb{f}}(H^1_{\mathfrak{m}})),$$

where $\sim$ means equality up to $\mathcal{O}^\times$.

Remark 3.2. (1) Even up to this point, we "don't see torsion."
(2) If we know $H^2_{\mathfrak{m}}$ is free over T (which in fact will be true), then the adjoint L-value is equal to $\eta_{\mathbb{f}}$. In the usual Taylor-Wiles formalism, $\eta_{\mathbb{f}}$ should be related to the Fitting ideal of the corresponding Selmer group. However, in this Bianchi case, due to torsion, $\eta_{\mathbb{f}}$ will not be directly related to the Fitting ideal of Selmer group. Thus, we have to modify periods to make them "see torsion", to match with Bloch-Kato conjecture/Iwasawa Main Conjecture.

We thus need to use "higher congruence module" (also defined by Hida).

Definition 3.3. *The **second congruence module** $c^1_{\mathbb{f}}$ is*

$$c^1_{\mathbb{f}} = \Omega^1_{T/\mathcal{O}} \otimes_{T,\lambda_{\mathbb{f}}} \mathcal{O} = \mathfrak{c}'/\mathfrak{c}'^2.$$

*The **second congruence ideal** is $\eta^1_{\mathbb{f}} = \mathrm{Fitt}(c^1_{\mathbb{f}})$.*

What is a relation between two congruence modules/ideals?

Proposition 3.1. *(1) $\eta^0_{\mathbb{f}} \mid \eta^1_{\mathbb{f}}$*
(2) (Wiles, Tate, $\cdots$) $\eta^0_{\mathbb{f}} = \eta^1_{\mathbb{f}}$, if T is locally complete intersection.

The failure of T being lci is precisely the source of our failure.

Definition 3.4. *The **Wiles defect** $\delta_{\mathbb{f}}$ is*

$$\delta_{\mathbb{f}} = \eta^1_{\mathbb{f}}/\eta^0_{\mathbb{f}} \subset \mathcal{O}.$$

Theorem 3.2 (Tilouine-Urban)**.** *Let $\tilde{u}^2(\mathbb{f}) = u^2(\mathbb{f})/\delta_{\mathbb{f}}$. Let $\rho_{\mathbb{f}} : G_F \longrightarrow \mathrm{GL}_2(\mathcal{O})$ be the Galois representation associated to $\mathbb{f}$. Suppose*

- *$\mathfrak{n}$ is squarefree,*
- *for all $v \mid \mathfrak{n}$, $\rho_{\mathbb{f}}|_{I_v} \sim \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ is nonsplit,*
- *and $\mathbb{f}$ is $\mathfrak{n}$-minimal, namely, for all $v \mid \mathfrak{n}$, $\overline{\rho}_{\mathbb{f}}|_{I_v} \sim \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ is nonsplit.*

Then,

$$\frac{L(\mathrm{Ad}^0 \mathbb{f}, 1)}{\pi^2 u^1(\mathbb{f})\tilde{u}^2(\mathbb{f})} = \mathrm{Fitt}(\mathrm{Sel}(\mathrm{Ad}^0 \rho_{\mathbb{f}} \otimes L/\mathcal{O})).$$

Remark 3.3. As in the classical case, the idea is simple: the deformation theory almost formally relates the RHS to something like the second congruence module but using instead deformation ring. Then by "$R = T$" (or something similar) the equation follows. The reason why we have some assumptions in Theorem 3.2 is because we need such kind of theorem.

Theorem 3.3 (Scholze, Newton-Thorne, "the 10 authors")**.** *There is a Galois representation $\rho_{\mathfrak{m}} : G_F \longrightarrow \mathrm{GL}_2(T)$ lifting $\overline{\rho}_{\mathbb{f}}$ satisfying the same Hecke=Frobenius compatibility.*

Remark 3.4. The local-global compatibility might be only known up to a nilpotent ideal, but let's just ignore this issue. Namely, assume that for $v \mid p$, $\rho_{\mathfrak{m}}|_{G_{F_v}}$ is Fontaine-Laffaille with Hodge-Tate weights 0,1, and for $v \mid \mathfrak{n}$, $\rho_{\mathfrak{m}}|_{I_v} \sim \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ is nonsplit.

Conside the deformation functor $\mathscr{D}$ sending

$$A \mapsto \left\{ \begin{array}{c} \text{lifts of } \overline{\rho}_{\mathbb{f}} \text{ to } \mathrm{GL}_2(A) \text{ which are} \\ \text{unramified outside } \mathfrak{n}p\text{, minimal and} \\ \text{Fontaine-Laffaille with Hodge-Tate} \\ \text{weights 0,1} \end{array} \right\},$$

for A an Artinian $\mathcal{O}$-algebra with residue field k. Schlessinger's criterion can be applied to this functor, so that this is pro-representable by (R, ρ^u). Thus, the existence of T-valued Galois representation implies that there is a natural map $R \longrightarrow T$.

Best hope. $R \xrightarrow{\sim} T \cong H^2_{\mathfrak{m}}$.

If this holds, then the "Bloch-Kato formula" will tell you that

$$\frac{L(M,0)}{\Omega_M R_M} \sim \operatorname{Fitt} Ш^1_{\mathrm{BK}}(M),$$

where M is the "motive" $\mathrm{Ad}^0\, \rho_{\mathfrak{f}}(1)$ (of course, in this case, there is no motive but just a Galois representation), R_M is the regulator, $Ш^1_{\mathrm{BK}}(M)$ is the Bloch-Kato Tate-Shafarevich group, which is Pontryagin dual to $\mathrm{Sel}(\mathrm{Ad}^0\, \rho_{\mathfrak{f}} \otimes L/\mathcal{O})$, and Ω_M is some period defined in terms of Galois theoretic terms. Thus, to get our formula, we need to show

$$\Omega_M R_M = \pi^2 u^1(\mathfrak{f})\tilde{u}^2(\mathfrak{f}).$$

Thus, we will eventually need to relate topology of Bianchi manifold with Galois cohomology.

4. Calegari-Geraghty method.

We study the map $R \longrightarrow T$ in detail. Let $r = \dim_k H^1_f(F, \mathrm{Ad}^0\, \overline{\rho}_{\mathfrak{f}})$. In our case, this local condition is the same as Fontaine-Laffaille at primes above p and unramified everywhere else.

Proposition 4.1. *For all $n \geq 1$, there exist infinitely many* ***Taylor-Wiles sets****, i.e. sets $Q_n = \{v_1, \cdots, v_r\}$ of distinct primes in F satisfying the following conditions.*

- $Nv_i \equiv 1 (\mathrm{mod}\, p^n)$, $Nv_i \not\equiv (\mathrm{mod}\, p^{n+1})$,
- $\overline{\rho}(\mathrm{Frob}_{v_i})$ *has 2 distinct eigenvalues* $\overline{\alpha}_{v_i}, \overline{\beta}_{v_i} \in k$,
- $H^1_{f, Q_n^\perp}(F, \mathrm{Ad}^0\, \overline{\rho}_{\mathfrak{f}}(1)) = 0$, *where you additionally demand that the cocycle is locally zero at places in Q_n.*

Now we consider the modified deformation functor $\mathscr{D}_{Q_n}$ for a Taylor-Wiles set Q_n, sending

$$A \mapsto \left\{ \begin{array}{c} \text{lifts } \rho \text{ of } \overline{\rho}_{\mathfrak{f}} \text{ to } \mathrm{GL}_2(A) \text{ which are} \\ \text{unramified outside } \mathfrak{n}p \prod_{v_i \in Q_n} v_i, \\ \text{minimal, Fontaine-Laffaille with} \\ \text{Hodge-Tate weights 0,1, and } \rho|_{G_{F_v}} \text{ can} \\ \text{be anything for } v \in Q_n \end{array} \right\}.$$

This is also pro-representable by $(R_{Q_n}, \rho^u_{Q_n})$. By the condition on distinct eigenvalues, it is forced that

$$\rho^u_{Q_n}|_{I_v} \sim \begin{pmatrix} 1 & 0 \\ 0 & \chi^u_v \end{pmatrix},$$

for any $v \in Q_n$. Thus, we get a map

$$\prod_{v \in Q_n} \Delta_{Q_n} \longrightarrow R^\times_{Q_n},$$

where

$$\Delta_{Q_n} = \prod_{v \in Q_n} (p\text{-Sylow subgroup of } k_v^\times),$$

where k_v is the multiplicative group of residue field at v. Thus, we get an action of the group algebra $\mathcal{O}[\Delta_{Q_n}]$ on R_{Q_n},

$$\alpha_{Q_n} : \mathcal{O}[\Delta_{Q_n}] \longrightarrow R_{Q_n},$$

and

$$\mathcal{O}[\Delta_{Q_n}] = S_\infty/((1+s_1)^{p^n} - 1, \cdots, (1+s_r)^{p^n} - 1),$$

where $S_\infty = \mathcal{O}[[s_1, \cdots, s_r]]$ is the p-adically completed formal power series ring.

On the other hand, we know that R_{Q_n} is generated by $r-1$ elements over $\mathcal{O}$, so that R_{Q_n} can be regarded as a quotient of $R_\infty := \mathcal{O}[[x_1, \cdots, x_{r-1}]]$. Here the difference between the numbers r and $r-1$ is precisely coming from what is usually denoted as "ℓ_0", which in the case of $\mathrm{SL}_{2,F}$ is equal to 1.

Theorem 4.1 (Calegari-Geraghty). *There is a suitable choice of Q_n's for each n, and a suitable algebra homomorphism $\alpha : S_\infty \longrightarrow R_\infty$, such that for each n, the following diagram commutes,*

$$\begin{tikzcd}
\mathcal{O}[\Delta_{Q_n}] \arrow[r, "\alpha_{Q_n}"] & R_{Q_n} \\
S_\infty \arrow[u, two heads] \arrow[r, "\alpha"'] & R_\infty \arrow[u, two heads]
\end{tikzcd}$$

Furthermore,

(1) $\mathrm{Tor}_0^{S_\infty}(R_\infty, \mathcal{O}) = R_\infty \otimes_{S_\infty} \mathcal{O} = R = T \cong H^2_{\mathfrak{m}}$,
(2) $\mathrm{Tor}_i^{S_\infty}(R_\infty, \mathcal{O}) = 0$ *for all* $i > 1$,
(3) $\mathrm{Tor}_1^{S_\infty}(R_\infty, \mathcal{O}) \cong H^1_{\mathfrak{m}}$, *as* $\mathrm{Tor}_0^{S_\infty}(R_\infty, \mathcal{O}) = T$*-modules.*

By Poincare duality, $\mathrm{Tor}_1 \cong \mathrm{Hom}(T, \mathcal{O})$ as T-modules, which is not necessarily T as T might not be Gorenstein. The Tor-algebra $\mathrm{Tor}_\bullet^{S_\infty}(R_\infty, \mathcal{O})$ is of simple form $T \oplus \mathrm{Hom}(T, \mathcal{O})$.

The problem is that a priori the choice of Taylor-Wiles primes is not canonical, so R_∞ and consequently Tor_1 are not canonical. This can be remedied by using the theory of Galatius-Venkatesh. In a more "Venkatesh-style" way, what we have now can be more cleanly expressed using homology,

$$H_{\bullet,\mathfrak{m}} \cong \mathrm{Tor}_\bullet \otimes H_{1,\mathfrak{m}},$$

where $H_{k,\mathfrak{m}} = H^{3-k}_{\mathfrak{m}}$; again, we do not need to care about Borel-Moore homology or compactly supported cohomology as we have already localized at a non-Eisenstein maximal ideal. By going through Galatius-Venkatesh theory, we can directly relate cohomology of Bianchi manifolds with Selmer groups.

5. **Simplicial deformation theory.**

We first review some simplicial deformation theory.

Definition 5.1. *Let Δ be the category of finite ordered sets $[n] = \{0, \cdots, n\}$, such that*

$$\mathrm{Hom}_\Delta([n],[m]) = \{\textit{nondecreasing maps } [n] \longrightarrow [m]\}.$$

Given a reasonable category $\mathscr{C}$, a ***simplicial object in*** *$\mathscr{C}$ is a functor $F : \Delta^{\mathrm{op}} \longrightarrow \mathscr{C}$. The category of simplicial objects in $\mathscr{C}$ is denoted as $s\mathscr{C}$, where morphisms are just natural transformations between functors.*

We are mainly interested in two kinds of simplicial objects, simplicial sets $\mathbf{sSets}$ and "simplicial Artin rings over $\mathcal{O}$ with residue k" (which we will define precisely later) $\mathrm{s}\mathbf{Art}_{\mathcal{O},k}$. Given $A. \in \mathrm{s}\mathbf{Art}_{\mathcal{O},k}$, one can form a natural complex $C(A.)$ just by taking alternating sums of face maps, and the homology of the complex $H_*(C(A.))$ is usually referred as the **homotopy group** $\pi_*(A.)$ of $A.$. A priori these are abelian groups, but it turns out that $\pi_n(A.)$ is a ring for all $n \geq 0$, and is an algebra over $\pi_0(A.) = A_0/\operatorname{im}(d_0 - d_1)$. Namely, $\pi_*(A.)$ is a graded $\pi_0(A.)$-algebra.

There is a notion of **Kan fibration** which gives something that is like a projective resolution of a ring. Namely, given a, say, complete noetherian local $\mathcal{O}$-algebra R with residue field k, there is a simplicial ring $A.$ and the augmentation $A_0 \xrightarrow{\varepsilon} R$ such that each A_n is of form $\mathcal{O}[[x_1, \cdots, x_{m_n}]]$ for some finite m_n, $\pi_i(A.) = 0$ for all $i > 0$, and $\pi_0(A.) \cong R$ via ε.

Remark 5.1. There is a canonical resolution which uses uncountably many variables. The fact that there is a resolution with finitely many variables is because R is noetherian. Note that usually in most literatures the rings are given as polynomial rings, not power series rings.

Using the Kan resolution, one can construct **cotangent complex**.

Definition 5.2. *Given a Kan resolution $A. \longrightarrow R$ of $\mathcal{O}$-algebras, the **cotangent complex** is defined as*

$$L_{R/\mathcal{O}} = \Omega_{A./\mathcal{O}} \otimes_{A.} R.$$

Remark 5.2. This is related to Wiles defect as follows.

Proposition 5.1 (Tilouine-Urban). *Let $\ell_0 = 1$. Then $\delta_{\mathfrak{f}} = \operatorname{Fitt}(H^{-1}(L_{T/\mathcal{O}} \otimes_{T,\lambda_{\mathfrak{f}}} \mathcal{O}))$.*

The proof uses that, in $\ell_0 = 1$, there is a locally complete intersection T_0, finite over $\mathcal{O}$, which surjects onto T.

Definition 5.3. *A simplicial $\mathcal{O}$-algebra $A.$ is a **simplicial artinian ring** if $\pi_0 A.$ is artinian and $\pi_* A.$ is a finite $\pi_0 A.$-module.*

Example 5.1.
(1) Given $A \in \mathrm{Art}_{\mathcal{O},k}$ (ordinary artinian ring) and $V. : V_n \longrightarrow \cdots \longrightarrow V_0$ a perfect complex of finite free A-modules, we can construct a simplicial artinian ring $A \oplus V. \in \mathrm{s}\mathbf{Art}_{\mathcal{O},k}$ such that $(A \oplus V.)_i = A \oplus \varepsilon V_i$ (ε means a formal variable with $\varepsilon^2 = 0$) and all face maps $d_j : A \oplus \varepsilon V_i \longrightarrow A \oplus \varepsilon V_{i-1}$ are the projection to A except the last fact map which is the projection to V_{i-1}. Then, quite obviously, $\pi_i(A \oplus V.) = H_i(V.)$. We will particular use $A \oplus \varepsilon M[i] \in \mathrm{s}\mathbf{Art}_{\mathcal{O},k}$, i.e. the construction where in the complex there is only one module concentrated in one degree.
(2) The category $\mathbf{sSets}$ has internal Hom. More precisely, given two simplicial sets $X, Y \in \mathbf{sSets}$, there is a simplicial set $\mathrm{sHom}_{\mathbf{sSets}}(X, Y)_i = \mathrm{Hom}_{\mathbf{sSets}}(X \times \Delta[i], Y)$.

Now we want to define deformation functor. It should be a functor $\mathrm{s}\mathscr{D} : \mathrm{s}\mathbf{Art}_{\mathcal{O},k} \longrightarrow \mathbf{sSets}$ which is something like

$$\mathrm{s}\mathscr{D}(A) = \text{“}\left\{\begin{array}{c}\text{liftings of } \overline{\rho} \text{ to } \mathrm{GL}_2(A) \text{ which is} \\ \text{unramified outside } \mathfrak{n}p\text{, minimal and} \\ \text{Fontaine-Laffaille with Hodge-Tate} \\ \text{weights } 0, 1\end{array}\right\}\text{”},$$

but a Galois representation valued in a simplicial ring is a tricky notion to define. We will not say much about this issue, but roughly speaking it uses the classifying space $B\mathrm{GL}_2(A.)$ which is a simplicial set, and a tower $X_\alpha = \operatorname{Spec}(\mathcal{O}_{F_\alpha}[1/\mathfrak{n}p])$ for F_α/F unramified outside $\mathfrak{n}p$. A Galois representation valued in $\mathrm{GL}_2(A_\cdot)$ is something like "$\{X_{F'} \longrightarrow B\,\mathrm{GL}_2(A_\cdot)\}$." In this way one can also give local conditions.

The analogously defined functor $\mathrm{s}\mathscr{D}$ is pro-representable by Lurie's derived Schlessinger criterion; it says if the functor commutes with homotopy fiber products, and if the tangent complex is concentrated in negative degrees and the tangent space (i.e. the set valued in $k[\epsilon]/(\epsilon^2)$ in degree 0) is finite. There is some issue with projective limits, so in practice it is better to be viewed as representable by a projective system $\mathscr{R} = (R_\alpha)$ of simplicial artinian rings, which is unique up to homotopy. What we mean by representable is that there is a morphism

$$\phi : \varinjlim_{\alpha} \mathrm{sHom}(R_\alpha, -) \longrightarrow \mathrm{s}\mathscr{D},$$

such that $\phi(A)$ is a weak equivalence, i.e. it is an isomorphism on the π_*'s. Note that any classical ring A is simplicial by taking $A_n = A$ and taking identity maps for face maps. Thus, by a very formal reason, $\pi_0\mathscr{R} = R$, the classical deformation ring.

Theorem 5.1 (Galatius-Venkatesh for $T = \mathcal{O}$, Y. Cai for general T)**.** *There is a natural weak equivalence*

$$\mathscr{R} \longrightarrow R_\infty \underline{\otimes}_{S_\infty} \mathcal{O},$$

where $R_\infty \underline{\otimes}_{S_\infty} \mathcal{O}$ is the simplicial ring constructed out of Kan resolution of R_∞ with entries formal power series rings over S_∞.

As $\pi_*(R_\infty \underline{\otimes}_{S_\infty} \mathcal{O}) = \mathrm{Tor}_*^{S_\infty}(R_\infty, \mathcal{O})$, this means that $\pi_*\mathscr{R} = \mathrm{Tor}_*^{S_\infty}(R_\infty, \mathcal{O})$, and $\mathscr{R}$ is the natural object behind the Tor algebra of Calegari-Geraghty. In particular, in our situation of Bianchi modular forms,

$$\pi_*\mathscr{R} = \pi_0 \oplus \pi_1 = T \otimes \mathrm{Hom}(T, \mathcal{O}),$$

as observed above.

6. **Applications.**

Theorem 6.1 (Tilouine-Urban)**.** *There is a T-equivariant exact sequence*

$$0 \longrightarrow \mathrm{Hom}(T, \mathcal{O}) \otimes_{\mathcal{O}} L/\mathcal{O} \xrightarrow{\mathrm{GV}} H^1_f(F, \mathrm{Ad}^0 \rho_{\mathfrak{m}}(1) \otimes L/\mathcal{O}) \longrightarrow Ш^1(\mathrm{Ad}^0 \rho_{\mathfrak{m}}(1) \otimes L/\mathcal{O}) \longrightarrow 0.$$

Remark 6.1.
(1) By the definition of Bloch-Kato Tate-Shafarevich group, it is saying that $\mathrm{Hom}(T, \mathcal{O}) \otimes_{\mathcal{O}} L/\mathcal{O}$ is naturally the p-divisible part of the corank 1 module $H^1_f(F, \mathrm{Ad}^0 \rho_{\mathfrak{m}}(1) \otimes L/\mathcal{O})$. Thus, in some sense the map GV coming out of Galatius-Venkatesh theory is a Bloch-Kato regulator map. If you remember that $\mathrm{Hom}(T, \mathcal{O})$ is $H^1_{\mathfrak{m}}$, this map coming out of simplicial commutative algebra relates Selmer group and topology of Bianchi manifolds.
(2) The map GV is also defined in Galatius-Venkatesh, but not in the context of p-divisible modules, and also under the assumption of $T = \mathcal{O}$ (where they prove that the analogous map is an isomorphism).

We discuss how to define the map GV. As $\mathrm{Hom}(T, \mathcal{O}) \otimes_{\mathcal{O}} L/\mathcal{O}$ is already p-divisible, Theorem 6.1 will follow if we define an injective map GV. It is sufficient to define surjective maps

$$H^2_f(F, \mathrm{Ad}^0 \rho_n \otimes \mathrm{Hom}(T, \mathcal{O}/\varpi^n)) \twoheadrightarrow \mathrm{Hom}_{T\text{-alg}}(\pi_1\mathscr{R}, \mathrm{Hom}(T, \mathcal{O}/\varpi^n)),$$

as the desired injective map will be the dual of inductive limit of these surjections, where $\rho_n = \rho_{\mathfrak{m}} \bmod \varpi^n$. Thus, Theorem 6.1 will follow from the following

Theorem 6.2 (Tilouine-Urban)**.** *Let $A_n = T/\varpi^n$. Let M_n be a finite A_n-module. Then, there is a natural surjection*
$$H^2_f(F, \operatorname{Ad}^0 \rho_n \otimes M_n) \longrightarrow \operatorname{Hom}_{T\text{-alg}}(\pi_1 \mathscr{R}, M_n).$$

Proof. Note that the formalism of simplicial deformation theory tells you that
$$H^{i+1}_f(F, \operatorname{Ad}^0 \rho_n \otimes M_n) \cong \left\{ \begin{array}{c} \text{liftings } \mathscr{R} \longrightarrow A_n \oplus \varepsilon M_n[i] \text{ of} \\ \mathscr{R} \longrightarrow A_n \end{array} \right\},$$
where $\mathscr{R} \longrightarrow A_n$ comes from $\mathscr{R} \longrightarrow T \twoheadrightarrow A_n$. Indeed this makes sense, because if you put $i = 0$ this becomes a more familiar form. On the other hand, $\pi_1\mathscr{R}$ by definition is
$$\pi_1 \mathscr{R} = \left\{ \begin{array}{c} \text{morphisms } \Delta[1] \longrightarrow \mathscr{R} \text{ of simplicial} \\ \text{sets which restrict to } \partial\Delta[1] \longrightarrow 0 \end{array} \right\} /\text{homotopy}.$$
Now the definition is very simple. Namely, given a lifting $\mathscr{R} \longrightarrow A_n \oplus \varepsilon M_n[1]$ and a loop $\Delta[1] \longrightarrow \mathscr{R}$, we compose this to get $\Delta[1] \longrightarrow A_n \oplus \varepsilon M_n[1]$, and we project to $M_n[1]$. This gives a morphism $\Delta[1] \longrightarrow M_n[1]$ of simplicial sets. As $\operatorname{Hom}_{\mathbf{sSets}}(\Delta[1], M_n[1]) = M_n$, we have constructed a natural homotopy invariant map
$$H^2_f(F, \operatorname{Ad}^0 \rho_n \otimes M_n) \longrightarrow \operatorname{Hom}(\pi_1 \mathscr{R}, M_n).$$
The surjectivity of this map will follow if we can prove that, given a homomorphism $\pi_1\mathscr{R} \longrightarrow M_n$, we can lift it to $\mathscr{R} \longrightarrow A_n \oplus \varepsilon M_n[1]$. One can prove that, using an **explicit presentation of** $R_\infty \underline{\otimes}_{S_\infty} \mathcal{O}$, one can lift $\pi_1\mathscr{R} \longrightarrow M_n$ to $R_\infty \underline{\otimes}_{S_\infty} \mathcal{O} \longrightarrow A_n \oplus \varepsilon M_n[1]$. As the direction of the natural weak equivalence is $\mathscr{R} \longrightarrow R_\infty \underline{\otimes}_{S_\infty} \mathcal{O}$, we can compose to get a desired lift. $\square$

Gabriel Dospinescu and Wieslawa Niziol, *Integral p-adic cohomology of Drinfeld half-spaces*

1. **Drinfeld and Lubin-Tate tower.**

The objective of this long-term project is to understand the geometric realization of the p-adic local Langlands and Jacquet-Langlands correspondence via the étale cohomology of the Drinfeld and Lubin-Tate towers. Roughly the big picture is as follows. On the infinite level, we have an isomorphism of the **Drinfeld tower** and the **Lubin-Tate tower** (Fargues, Faltings):

$$\mathrm{LT}_\infty \cong \mathrm{Dr}_\infty,$$

which is a perfectoid space (Scholze, Weinstein).

The Lubin-Tate tower LT_∞ sits over $\coprod_{\mathbb{Z}} \mathbb{D}$, where $\mathbb{D}$ is the open disc of radius 1. The Gross-Hopkins map gives $\coprod_{\mathbb{Z}} \mathbb{D} \longrightarrow \mathbb{P}^1_C$, for $C = \widehat{\overline{\mathbb{Q}}}_p$. The Lubin-Tate tower LT_∞ is then a $G = \mathrm{GL}_2(\mathbb{Q}_p)$-torsor over $\mathbb{P}^1_C$.

For the other direction, the Drinfeld tower Dr_∞ sits over $\coprod_{\mathbb{Z}} \mathbb{H}$, where $\mathbb{H}$ is the **Drinfeld upper-half space**, for which $\mathbb{P}^1_C - \mathbb{P}^1(\mathbb{Q}_p)$ can be used. Then obviously there is the natural map $\coprod_{\mathbb{Z}} \mathbb{H} \longrightarrow \mathbb{H}$. The Drinfeld tower Dr_∞ is then a $D^\times$-torsor over $\mathbb{H}$, where D is the invariant 1/2 division algebra over $\mathbb{Q}_p$ (i.e. the nonsplit quaternion division algebra).

Theorem 1.1 (Scholze-Weinstein). *The Drinfeld tower* Dr_∞ *is the "G-simply connected cover" of* $\mathbb{H}$. *More precisely (although it is still imprecise),*

$$\mathrm{Dr}_\infty = \text{“} \varprojlim_{X \to \mathbb{H}\ G\text{-equivariant fét cover}} X \text{”}.$$

Proof. The Drinfeld tower comes with abundance of explicit intermediate coverings. Then one shows that a G-equivariant finite étale cover is dominated by one of those coverings. □

Thus $\mathrm{LT}_\infty \cong \mathrm{Dr}_\infty$ is an extremely symmetric space where its quotients by G and $D^\times$ are extremely simple spaces. Eventually we would like to compute something like

$$H^1_{\text{ét}}(\mathrm{LT}_\infty, \mathbb{F}_\ell \text{ or } \mathbb{Z}_\ell \text{ or } \mathbb{Q}_\ell),$$

for any prime ℓ. This has an action of $W_{\mathbb{Q}_p} \times G \times D^\times$; the action of Weil group comes from the fact that the tower has an explicit model over the completion of the maximal unramified extension of $\mathbb{Q}_p$. Thus, we would like to understand it as a representation of $W_{\mathbb{Q}_p} \times G \times D^\times$. This is an interesting thing to think about, as the construction works equally well for $\mathrm{GL}_n(F)$ for any finite extension $F/\mathbb{Q}_p$.

For $\ell \neq p$, this is essentially done. This is due to many people, e.g. Drinfeld, Langlands, Deligne, Carayol, Faltings, Fargues, Dat, Harris, Taylor, Mieda, Boyer, ⋯, using a lot of machineries, e.g. vanishing cycles, trace formula, ⋯.

For $\ell = p$, this is very mysterious. Apart from that we have no tools from automorphic side, it has some inherent difficulties.

- We have to use p-adic Hodge theory, and this situation is very nasty, namely it is non-proper, non-algebraic, and even non-quasicompact.
- The relevant representation theory is very hard. This is one reason why we work in $\mathrm{GL}_2(\mathbb{Q}_p)$ case. For example, even for $D^\times$, nobody really knows any good idea on what are p-adic representations of $D^\times$; mod p is OK, but in the course of lifting to characteristic zero, one has to face chain of infinitely many irreducibles.

Let us forget about the tower for the moment, and illustrate why there are serious difficulties in computing p-adic cohomology, even at very elementary level. Namely, what is the first p-adic étale cohomology of $\mathbb{D}$? What about $\mathbb{H}$?

Theorem 1.2 (Colmez-Dospinescu-Niziol). *(1) $H^1_{\text{ét}}(\mathbb{D}, \mathbb{Z}_p(1)) \cong 1 + T\mathcal{O}_C[[T]]$. Or, the p-torsion on* Pic D *is bounded.*

(2) $H^2_{\text{ét}}(\mathbb{D}, \mathbb{Z}_p(1)) = \widehat{\operatorname{Pic}\mathbb{D}}$, which is zero if and only if C is spherically complete.

(3) $H^i_{\text{proét}}(D, \mathbb{Z}_p(1)) = 0$ for $i > 1$, and $H^1_{\text{proét}}(\mathbb{D}, \mathbb{Z}_p(1)) = \mathcal{O}(\mathbb{D})/C$.

(4) (originally due to Drinfeld) $H^1_{\text{ét}}(\mathbb{H}, \mathbb{Z}_p(1)) = (C^\infty(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z}_p)/\text{const. fn.s})^*$, the $\mathbb{Z}_p$-dual. In particular, this is an admissible G-module.

(5) There is an exact sequence

$$0 \longrightarrow \mathcal{O}(\mathbb{H})/C \longrightarrow H^1_{\text{proét}}(\mathbb{H}, \mathbb{Q}_p(1)) \longrightarrow (\mathrm{LC}(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Q}_p)/\text{const.})^* \longrightarrow 0,$$

where LC *means locally constant functions.*

Remark 1.1. (1) This suggests that the cohomology along the Lubin-Tate side is horrible, whereas the cohomology along the Drinfeld side is much nicer. For example, even though one might think $1 + T\mathcal{O}_C[[T]]$ is a nice-looking thing, as a Galois module, it contains every Galois representation having 0 as a Hodge-Tate weight, which is useless. On the other hand, $(C^\infty(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z}_p)/\text{const.})^*$ only has trivial Galois representation inside, which is as expected.

(2) The reason why the pro-étale cohomology of $\mathbb{H}$ has a huge chunk of $\mathcal{O}(\mathbb{H})/C$ inside is because $\mathbb{H}$ is non-quasicompact. Basically, computing pro-étale cohomology of $\mathbb{H}$ is the same as computing étale cohmologies of all affinoid coverings of $\mathbb{H}$, which have a lot of denominators. In particular, we could have just put pro-étale everywhere as for integral coefficients pro-étale cohomology and étale cohomology are the same.

(3) Even though the situation is nicer on the Drinfeld side, we have to cut out (pro-)étale cohomology as soon as one starts to climb up the Drinfeld tower. In particular, the integral pro-étale cohomology will be not admissible, even for a cyclic cover of degree > 1 prime to p over $\mathbb{H}$.

2. Completed cohomology of Drinfeld tower.

Ideally, in the hope that the p-adic cohomology of Drinfeld tower realizes the p-adic local Langlands correspondence, we would like to study the functor

$$V \mapsto \operatorname{Hom}_{W_{\mathbb{Q}_p}}(V, H^1_{\text{ét}}(\mathrm{LT}_\infty, \mathbb{Q}_p(1))) =: \mathcal{F}(V),$$

for a p-adic local Galois representation V. In particular, $\mathcal{F}(V)$ has an action of $G \times D^\times$, so the best hope is that this is just a tensor product of local Langlands correspondence and its Jacquet-Langlands transfer. This is what happens in the ℓ-adic world.

Remark 2.1. (1) We don't put pro-étale cohomology, because as we saw above, the pro-étale cohomology has a huge chunk inside in general.

(2) In fact, in the ℓ-adic case, $H^1_{\text{ét}}(\mathrm{LT}_\infty, \mathbb{Q}_\ell(1))$ is very close to being semisimple, and we have a complete description of the whole space. This can never happen in the p-adic case. Very informally and intuitively, $H^1_{\text{ét}}(\mathrm{LT}_\infty, \mathbb{Q}_p(1))$ is something like "$C^\infty(G \times D^\times, \mathbb{Q}_p)$."

Unfortuantely, this functor is very complicated, and every naive conjecture one might suggest seems to be wrong. But it is strongly believed that the functor should contain both p-adic local Langlands and p-adic Jacquet-Langlands.

Alternatively, one can try to study something smaller, e.g. the $D^\times$-smooth power, $\mathcal{F}(V)^{D^\times-\mathrm{sm}}$, which will be related to the study of Drinfeld tower, i.e. G-action on $H^1_{\text{ét}}(\text{“}\mathrm{LT}_\infty/(1+p^n\mathcal{O}_D)\text{”}, \mathbb{Q}_p(1))$, or similarly $\mathcal{F}(V)^{G-\mathrm{sm}}$, and study the analogous étale cohomology of Lubin-Tate tower. As we have illustrated that the Lubin-Tate side has more mysterious cohomology, we will stick to the Drinfeld tower side.

Remark 2.2. Of course, taking quotient here does not necessarily have any geometric meaning. If you want, one might try to use Scholze's theory of diamonds. However, in this specific case, $\mathrm{LT}_\infty/(1+p^n\mathcal{O}_D)$ is a (non-proper) rigid analytic space in the most classical sense.

We can even study these finite level stuffs at once, by considering

$$\left(\varinjlim_n H^1(\mathrm{LT}_\infty/(1+p^n\mathcal{O}_D), \mathbb{Z}_p(1))\right)^\wedge,$$

where here $\wedge$ means p-adic completion. This turns out to be much smaller than the cohomology of the perfectoid space $H^1_{\text{ét}}(\mathrm{LT}_\infty, \mathbb{Z}_p(1))$.

Now here comes a similarity to completed cohomology: at finite level, étale cohomology sees only "nice" representations, but after taking p-adic completion, one sees a lot more representations; just like completed cohomology sees much more representation than just those of global nature because of p-adic completion process.

Theorem 2.1 (Colmez-Dospinescu-Niziol)**.** *For V an absolutely irreducible $G_{\mathbb{Q}_p}$-representation of dimension ≥ 2, $\mathcal{F}(V)^{D^\times-\mathrm{sm}}$ is nonzero only if V is 2-dimensional, de Rham, Hodge-Tate weights 0,1 and $\mathrm{WD}(V)$ is irreducible.*

This tells us that the first étale cohomology of the rigid analytic space $\mathrm{LT}_\infty/(1+p^n\mathcal{O}_D)$ is in some sense "de Rham." This is very nontrivial, as we have seen that already the rigid open ball has p-adic étale cohomology that sees all sorts of Galois representations.

Remark 2.3. To be fair, the above result does not guarantee whether there is a non-de Rham **subquotient** of $H^1_{\text{ét}}(\mathrm{LT}_\infty/(1+p^n\mathcal{O}_D), \mathbb{Q}_p(1))$; it is only about (finite-dimensional) subrepresentations.

This is some sort of "classicality." On the other hand, we expect the following.

Conjecture 2.1 (Colmez-Dospinescu-Niziol)**.** *Let V be a 2-dimensional $G_{\mathbb{Q}_p}$-representation. Let*

$$\widehat{\mathcal{F}(V)^{D^\times-\mathrm{sm}}} := \mathrm{Hom}_{W_{\mathbb{Q}_p}}\left(V, \left(\varinjlim_n H^1(\mathrm{LT}_\infty/(1+p^n\mathcal{O}_D), \mathbb{Z}_p(1))\right)^\wedge\right).$$

Then,

$$\widehat{\mathcal{F}(V)^{D^\times-\mathrm{sm}}} \xrightarrow{\sim} \mathrm{JL}_p(V)\widehat{\otimes}\Pi(V)^*,$$

where $\mathrm{JL}_p(V)$ is an admissible unitary infinite-dimensional representation of $D^\times$ of finite length, and $\Pi(V)$ is the p-adic local Langlands correspondent.

In particular, $\mathrm{JL}_p(V)$ is hypothetical p-adic Jacquet-Langlands correspondence, whose smooth vectors realize the classical local Jacquet-Langlands correspondence. This expectation comes from the following main theorem, which is about uncompleted cohomology.

Theorem 2.2 (Colmez-Dospinescu-Niziol). *Let V be a 2-dimensional de Rham $G_{\mathbb{Q}_p}$-representation such that Hodge-Tate weights are 0,1 and* $\mathrm{WD}(V)$ *is irreducible. Then,*

$$\mathrm{Hom}_{W_{\mathbb{Q}_p}}(V, \varinjlim_n H^1_{\text{ét}}(\mathrm{LT}_\infty/(1+p^n\mathcal{O}_D), \mathbb{Q}_p(1))) = \mathrm{JL}(V) \otimes \Pi(V)^*,$$

where $\mathrm{JL}(V)$ *is now the finite-dimensional irreducible smooth representation of $D^\times$ attached by the local Jacquet-Langlands correspondence and the classical local Langlands correspondence on* $\mathrm{WD}(V)$.

Using this, for "nice" V (i.e. 2-dimensional, de Rham, $\cdots$), Colmez-Dospinescu-Niziol could prove that $\mathrm{JL}_p(V)$, which can be just defined as a Hom space from $\Pi(V)^*$ to $\widehat{\mathcal{F}(V)^{D^\times-\mathrm{sm}}}$, is nonzero and has smooth vectors equal to $\mathrm{JL}(V)$. This is thus some sort of local-global compatibility.

Main question. How to show $\mathrm{JL}_p(V) \neq 0$ **for all** 2-dimensional V?

Strategy. We can use Scholze's functor, which we denote it as H, which sends G-representations to $D^\times \times G_{\mathbb{Q}_p}$-representations, because for $G = \mathrm{GL}_2(\mathbb{Q}_p)$, his functor is "generically" just

$$H(\Pi) = \mathrm{Hom}_G(\Pi^*, H^1_{\text{ét}}(\mathrm{LT}_\infty, \mathbb{Q}_p(1))).$$

Remark 2.4. One conceptual explanation why we see Π^*'s, not Π's, is because what we really want is $\mathrm{Hom}(H^1_{\text{ét},c}, \Pi)$. Of course, we don't have Poincaré duality or anything general in this context, so we have to prove everything by hand.

Scholze showed that H preserves admissibility, and is "compatible with patching." Thus we can try to transfer the problem of showing that $H(\Pi) \neq 0$ to a "global statement," that the support of the patched module for $D^\times$ is the whole deformation space.

This is in general a very hard problem; even though the support contains every classical point, you don't know it is everything because the patched module is not of finite type over the deformation ring so the support is a priori not closed. Even in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$, the proof of this fact uses p-adic local Langlands correspondence. However, this fact will follow from the following conjecture, which is expected to hold in much greater generality.

Conjecture 2.2. *Let* (Sh_n) *be a tower (along p, with fixed tame level) of Shimura curves for a quaternion algebra over $\mathbb{Q}$ which is ramified at p and split at ∞. Let $X = \varinjlim_n H^1_{\text{ét}}(\mathrm{Sh}_n, \mathbb{F}_p)$. Then, the Gelfand-Kirillov dimension of all Hecke eigenspaces in X is 1.*

This is about asymptotic growth of dimension of Hecke eigenspaces.

- Emerton showed that this is true for modular curves, which is essentially the only known case.
- Paskunas showed that Conjecture 2.2 is true when you localize at Eisenstein maximal ideal, i.e. when the corresponding Galois representation is residually reducible. Thus, via this strategy, we get the desired result that $\mathrm{JL}_p(V) \neq 0$ for residually reducible Galois representations.

3. Drinfeld symmetric spaces.

We now study the étale cohomology of Drinfeld symmetric spaces. Fix a finite extension $K/\mathbb{Q}_p$, and let $C = \widehat{\overline{K}}$. From now on we set $G := \mathrm{GL}_{d+1}(K)$.

Definition 3.1. *The* ***Drinfeld symmetric space*** $\mathbb{H}^d_K$ *is* $\mathbb{P}^d_K \setminus \bigcup_{H \in \mathscr{H}} H$, *where $\mathscr{H}$ is the set of K-rational hyperplanes in $\mathbb{P}^d_K$. This is a rigid analytic space with a natural G-action.*

We have the following classical calculations.

Theorem 3.1 (Schneider-Stuhler). *(1) For $\ell \neq p$, there is a $G \times G_K$-equivariant isomorphism*

$$H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbb{Q}_\ell(r)) \cong \mathrm{Sp}_r(\mathbb{Z}_\ell)^* \otimes \mathbb{Q}_\ell,$$

where Sp_r is the "generalized Steinberg representation," and the RHS has a trivial G_K-action. Similarly, there is a $G \times G_K$-equivariant isomorphism

$$H^r_{\text{proét}}(\mathbb{H}^d_C, \mathbb{Q}_\ell(r)) \cong \mathrm{Sp}_r(\mathbb{Q}_\ell)^*,$$

where the RHS has a trivial G_K-action.

(2) For $\ell = p$, there is a G-equivariant isomorphism.

$$H^r_{\mathrm{dR}}(\mathbb{H}^d_K) \cong \mathrm{Sp}_r(K)^*.$$

Remark 3.1. We briefly recall how the (pro-)étale cohomology of $\mathbb{H}^d_C$ can be computed. As $\mathbb{H}^d_C$ is a Stein space (which uses that coherent cohomology is acyclic), there is a chain of affinoids $U_n \Subset U_{n+1}$. Then we have

$$R\Gamma_{\text{ét}}(\mathbb{H}^d_C, \mathbb{Q}_\ell) \cong (\mathrm{holim}_n\, R\Gamma_{\text{ét}}(U_n, \mathbb{Z}_\ell)) \otimes \mathbb{Q}_\ell,$$

$$R\Gamma_{\text{proét}}(\mathbb{H}^d_C, \mathbb{Q}_\ell) \cong \mathrm{holim}_n\, R\Gamma_{\text{ét}}(U_n, \mathbb{Q}_\ell).$$

4. **Generalized Steinberg representations.**

Before we proceed, we should know what generalized Steinberg representations are.

Definition 4.1. *For an abelian group A, let*

$$\mathrm{Sp}_r(A) := \frac{\mathrm{LC}(G/P_{\{1,\cdots,d-r\}}, A)}{\sum_{P' \supsetneq P_{\{1,\cdots,d-r\}}} \mathrm{LC}(G/P', A)},$$

where LC means the space of locally constant functions, and $P_{\{1,\cdots,s\}}$ is the parabolic subgroup corresponding to the partition $(s+1, 1, \cdots, 1)$ of $d+1$.

Remark 4.1. Let A be a field of characteristic either 0 or p.

1. The generalized Steinberg representation $\mathrm{Sp}_r(A)$ is irreducible (Grosse-Klönne for characteristic p case). Note that this is false for a field of characteristic $\ell \neq p$.
2. More generally, for any subset $I \subset \{1, \cdots, d+1\}$, a similar construction yields $\mathrm{Sp}_I(A)$, which is irreducible. All the irreducible constituents of $\mathrm{LC}(H/B, A)$ are of this form.
3. For any $I \subset \{1, \cdots, d+1\}$, $\mathrm{Sp}_I(\mathcal{O}_A)$ is, up to $A^\times$-homothety, the unique G-stable lattice in $\mathrm{Sp}_I(A)$.

Theorem 4.1 (Colmez-Dospinescu-Niziol). *(1) There is an exact sequence*

$$0 \longrightarrow \Omega^{r-1}(\mathbb{H}^d_C)/\ker d \xrightarrow{\text{Bloch-Kato exponential}} H^r_{\text{proét}}(\mathbb{H}^d_C, \mathbb{Q}_p(r)) \longrightarrow \mathrm{Sp}_r(\mathbb{Q}_p)^* \longrightarrow 0,$$

of Fréchet space representations of $G \times G_K$, and d is the de Rham differential.

(2) There is an $G \times G_K$-equivariant isomorphism

$$H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbb{Q}_p(r)) \cong \mathrm{Sp}_r(\mathbb{Z}_p)^* \otimes \mathbb{Q}_p.$$

(3) There is an $G \times G_K$-equivariant isomorphism

$$H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbb{Z}_p(r)) \cong \mathrm{Sp}_r(\mathbb{Z}_p)^*.$$

Remark 4.2. (1) The exact sequence of Theorem 4.1(1) has the following motivic meaning:

$$H^r_{\mathcal{M}}(\text{“}X_{\mathcal{O}_C}\text{ rel }X_{\mathcal{O}_C}/(p)\text{”}, \mathbb{Q}_p) \longrightarrow H^r_{\mathcal{M}}(X_{\mathcal{O}_C}, \mathbb{Q}_p) \longrightarrow H^r_{\mathcal{M}}(X_{\mathcal{O}_C}/(p), \mathbb{Q}_p),$$

for some formal model X of $\mathbb{H}^d_C$. Thus, $\mathrm{Sp}_r(\mathbb{Q}_p)^*$ in some sense “comes from the special fiber,” while $\Omega^{r-1}(\mathbb{H}^d_C)/\ker d$ is in some sense the “de Rham part” of the pro-étale cohomology.

(2) The $d = 1$ case of Theorem 4.1(2) is due to Drinfeld and Fresnel-van der Put. The proof uses Kummer theory and vanishing of Picard groups for $\{U_n\}$.

The proof of Theorem 4.1 uses p-adic Hodge theory:

(Step 1) Compute de Rham cohomology, Frobenius action and monodromy.

(Step 2) Pass to étale cohomology via comparison theorems; for (1) and (2), one uses Tsuji's comparison theorem on relating nearby cycles with syntomic complexes. To be more descriptive, let $j : \mathbb{H}^d_C \hookrightarrow X_{\mathcal{O}_C}$. Then, the nearby cycles functor $R\Psi = Rj_*$ in this context applied to $\mathbb{Z}/p^n\mathbb{Z}(r)$, for $r \geq 0$, is compared via

$$\tau_{\leq r} S_n(r) \overset{N(d,r)}{\cong} \tau_{\leq r} R\Psi \mathbb{Z}/p^n\mathbb{Z}(r),$$

where $\tau_{\leq r}$ is truncation, $S_n(r)$ is the “syntomic complex”, which should be thought as a Frobenius filtered eigenspace of absolute crystalline cohomology, and $\overset{N(d,r)}{\cong}$ means the complexes are quasi-isomorphic up to the constant $N(d, r)$ which depends on d, r, or more precisely, the cone has cohomology killed by $p^{N(d,r)}$.

For (3), one has to use Bhatt-Morrow-Scholze and its extension by Cesnavicius-Koshikawa and use A_{inf}-cohomology.

5. **Proof of Theorem 4.1.**

We will discuss the proof of Theorem 4.1 more detailedly.

Proof sketch of Theorem 4.1. Let $\mathfrak{X}$ be the standard semistable model for $\mathbb{H}^d_K$. We use the following

Theorem 5.1 (Grosse-Klönne). *Let $\Omega^\bullet_{\log}$ be the log de Rham complex. Then, $H^i(\mathfrak{X}, \Omega^j_{\log}) = 0$ for all $i > 0, j \geq 0$, while $H^0(\mathfrak{X}, \Omega^j_{\log})$ is killed by the de Rham differential d. We call such object **strongly ordinary**.*

This is proved by showing acyclicity of local systsems on pieces of Bruhat-Tits building. This implies that

$$H^r_{\mathrm{dR}}(\mathfrak{X}) \cong H^0(\mathfrak{X}, \Omega^r_{\log}).$$

Now to prove (2), we use the following

Theorem 5.2 (Colmez-Dospinescu-Niziol). *Recall that $\mathrm{Sp}_r(\mathbb{Z}_p)^*$ can be thought as a suitable quotient of $\mathbb{Z}_p$-valued measures on $\mathscr{H}^{r+1}$, the space of K-rational hyperplanes in K^{d+1}. Using this, one can construct the **Hodge-Tate regulator***

$$r_{\mathrm{HT}} : \mathrm{Sp}_r(\mathcal{O}_K)^* \longrightarrow H^0(\mathfrak{X}, \Omega^r_{\log}),$$

*and the **de Rham regulator***

$$r_{\mathrm{dR}} : \mathrm{Sp}_r(\mathcal{O}_K)^* \longrightarrow H^r_{\mathrm{dR}}(\mathfrak{X}).$$

Then, both regulators are isomorphisms, and they are compatible with the isomorphism we got from the Theorem of Grosse-Klönne,

$$H^r_{\mathrm{dR}}(\mathfrak{X}) \cong H^0(\mathfrak{X}, \Omega^r_{\log}).$$

To prove this, we recall the isomorphism of Schneider-Stuhler,

$$\alpha_S : H^r_{\mathrm{dR}}(X) \xrightarrow{\sim} \mathrm{Sp}_r(K)^*,$$

where we denote $\mathbb{H}^d_K$ as X. Iovita-Speiss made this isomorphism explicit; namely, there is an exact sequence

$$0 \longrightarrow \mathcal{D}(\mathscr{H}^{r+1}, K)_{\mathrm{deg}} \longrightarrow \mathcal{D}(\mathscr{H}^{r+1}, K) \longrightarrow \mathrm{Sp}_r(K)^* \longrightarrow 0,$$

where $\mathcal{D}$ is the space of distributions, and $\mathcal{D}(\mathscr{H}^{r+1}, K)_{\mathrm{deg}}$ is a certain subspace, so that there is a commutative diagram

$$\begin{tikzcd} \mathcal{D}(\mathscr{H}^{r+1}, K) \arrow[r, "r_{\mathrm{dR}}"] \arrow[dr] & H^r_{\mathrm{dR}}(X) \arrow[d, "\alpha_S"] \\ & \mathrm{Sp}_r(K)^* \end{tikzcd}$$

where r_{dR} is explicitly given by

$$r_{\mathrm{dR}}(\mu) = \int_{\mathscr{H}^{r+1}} \omega_{H_0,\cdots,H_r}\mu(H_0, \cdots, H_r),$$

where

$$\omega_{H_0,\cdots,H_r} = \mathrm{dlog}\,\frac{\ell_{H_1}}{\ell_{H_0}} \wedge \mathrm{dlog}\,\frac{\ell_{H_2}}{\ell_{H_0}} \wedge \cdots \wedge \mathrm{dlog}\,\frac{\ell_{H_r}}{\ell_{H_0}},$$

is the **de Rham symbol**, where ℓ_{H_i} is a linear form defining H_i. Using the explicit nature of this isomorphism (plus some integral and representation theory computations) to lift this isomorphism α_S to $\mathcal{O}_K$.

This also gives

$$H^r(\mathfrak{X}_0, W\Omega^\bullet_{\mathfrak{X}_0/\mathcal{O}_F^0}) \cong \mathrm{Sp}_r(\mathcal{O}_F)^*,$$

where $\mathfrak{X}_0$ is the special fiber of $\mathfrak{X}$, $\mathcal{O}_F = W(k)$, $\mathcal{O}_F^0$ is the log scheme $\operatorname{Spec}\mathcal{O}_F$ with log structure $1 \mapsto 0$, and $W\Omega$ is the de Rham-Witt complex, so that the LHS computes the Hyodo-Kato cohomology. Similarly, one has

$$H^0_{\text{ét}}(\mathfrak{X}_0, W\Omega^r_{\log}) \xrightarrow{\sim} H^0_{\text{ét}}(\mathfrak{X}_{\overline{k}}, W\Omega^r_{\log}) \cong \mathrm{Sp}_r(\mathbb{Z}_p)^*,$$

where now one uses the log de Rham-Witt complex. In fact, there is an isomorphism

$$H^r_{\text{ét}}(\mathfrak{X}_0, \mathbb{Z}_p(r)) \cong H^0_{\text{ét}}(\mathfrak{X}_{\overline{k}}, W\Omega^r_{\log}).$$

From this, we use Artin-Schreier theory. Recall that A_{inf} can be defined as

$$A_{\mathrm{inf}} = W(\mathcal{O}_C^\flat),$$

and that it comes equipped with $\theta : W(\mathcal{O}_C^\flat) \longrightarrow \mathcal{O}_C$ whose kernel is a principal ideal, say generated by ξ. Then, on $(X_C)_{\text{proét}}$, consider the Artin-Schreier sequence

$$0 \longrightarrow \widehat{\mathbb{Z}}_p \longrightarrow \mathbb{A}_{\mathrm{inf}} \xrightarrow{1-\varphi} \mathbb{A}_{\mathrm{inf}} \longrightarrow 0.$$

Now also consider $\nu : (X_C)_{\text{proét}} \longrightarrow (\mathfrak{X}_{\mathcal{O}_C})_{\text{ét}}$. The nearby cycles through this gives

$$R\nu_*\widehat{\mathbb{Z}}_p \cong (R\nu_*\mathbb{A}_{\mathrm{inf}})^{\varphi=1}.$$

There is a twisted Artin-Schreier sequence on $(X_C)_{\text{proét}}$,

$$0 \longrightarrow \widehat{\mathbb{Z}}_p(r) \longrightarrow \mathbb{A}_{\mathrm{inf}}\{r\} \xrightarrow{1-\varphi^{-1}} \mathbb{A}_{\mathrm{inf}}\{r\} \longrightarrow 0,$$

where $\mathbb{A}_{\inf}\{r\} = \frac{1}{\mu^r}\mathbb{A}_{\inf}(r)$, $\mu = [\varepsilon] - 1$ and $\varepsilon = (1, \zeta_p, \cdots)$. This gives after the nearby cycles functor

$$R\nu_*\widehat{\mathbb{Z}}_p(r) \cong (R\nu_*\mathbb{A}_{\inf}\{r\})^{\varphi^{-1}=1}.$$

Now the $A_{\inf}$-cohomology of Bhatt-Morrow-Scholze(+Cesnavicius-Koshikawa) defines

$$A\Omega_{\mathfrak{X}_{\mathcal{O}_C}} = L\eta_*R\nu_*\mathbb{A}_{\inf} \in D^{\geq 0}((\mathfrak{X}_{\mathcal{O}_C})_{\text{ét}}, A_{\inf}),$$

which can give every other known cohomology. For example, the de Rham cohomology can be extracted via

$$A\Omega_{\mathfrak{X}_{\mathcal{O}_C}}/\xi \cong \Omega^*_{\mathfrak{X}_{\mathcal{O}_C}/\mathcal{O}_C}.$$

Note the following

Theorem 5.3 (Bhatt-Morrow-Scholze)**.**

$$\tau_{\leq r}(\tau_{\leq r}A\Omega_{\mathfrak{X}_{\mathcal{O}_C}}\{r\})^{\varphi^{-1}=1} \cong \tau_{\leq r}R\nu_*\widehat{\mathbb{Z}}_p(r).$$

Using this, we have an exact sequence

$$0 \longrightarrow H^{r-1}_{\text{ét}}(\mathfrak{X}_{\mathcal{O}_C}, A\Omega_{\mathfrak{X}_{\mathcal{O}_C}}\{r\})/(1-\varphi^{-1}) \longrightarrow H^r_{\text{ét}}(X_C, \mathbb{Z}_p(r)) \longrightarrow H^r_{\text{ét}}(\mathfrak{X}_{\mathcal{O}_C}, A\Omega_{\mathfrak{X}_{\mathcal{O}_C}}\{r\})^{\varphi^{-1}=1} \longrightarrow 0.$$

From this, everything boils down to calculating $H^r_{\text{ét}}(\mathfrak{X}_{\mathcal{O}_C}, A\Omega_{\mathfrak{X}_{\mathcal{O}_C}}\{r\})$ with the φ^{-1}-action. This is done in the following

Theorem 5.4 (Colmez-Dospinescu-Niziol)**.** *There exists a natural isomorphism, compatible with φ^{-1}-action,*

$$r_{\inf} : A_{\inf}\widehat{\otimes}\,\mathrm{Sp}_r(\mathbb{Z}_p)^* \xrightarrow{\sim} H^r_{\text{ét}}(\mathfrak{X}_{\mathcal{O}_C}, A\Omega_{\mathfrak{X}_{\mathcal{O}_C}}\{r\}).$$

□

The paradigm is that cohomology of arithmetic groups is interesting. We will today study the homotopy theory and K-theory of arithmetic groups.

1. **Moduli of abelian varieties.**

Let us remember what is a polarizable abelian variety over $\mathbb{C}$. It is uniformized by $\mathbb{C}^g$, so it is $\mathbb{C}^g/\Lambda$ for some lattice Λ with basis say $\langle e_1, \cdots, e_g, f_1, \cdots, f_g\rangle$. As it has a complex structure, we have a relation

$$\begin{pmatrix} f_1 \\ \cdots \\ f_g \end{pmatrix} = \tau \begin{pmatrix} e_1 \\ \cdots \\ e_g \end{pmatrix},$$

for $\tau \in M_{g\times g}(\mathbb{C})$. That it is polarizable translates into $\tau = \tau^T$ and $\operatorname{im}\tau > 0$. Thus, τ is parametrized by $\mathbb{H}_g = \{\tau \in M_{g\times g}(\mathbb{C}) \mid \tau = \tau^T, \operatorname{im}\tau > 0\} =: \mathbb{H}_g$. As there is $\mathrm{Sp}_{2g}(\mathbb{Z})$ much ambiguity in choosing bases, we see that

$$\mathcal{A}_g(\mathbb{C}) = \mathbb{H}_g / \mathrm{Sp}_{2g}(\mathbb{Z}).$$

Thus, $H^*(\mathcal{A}_g(\mathbb{C})) = H^*(\mathrm{Sp}_{2g}(\mathbb{Z}))$ (even though $\mathcal{A}_g(\mathbb{C})$ is not the classifying space of $\mathrm{Sp}_{2g}(\mathbb{Z})$, i.e. $\mathrm{Sp}_{2g}(\mathbb{Z})$-action is not free, it is finite order). As $\mathcal{A}_{g,\mathbb{C}}$ admits a $\mathbb{Q}$-algebraic variety structure, we see that the group cohomology of $\mathrm{Sp}_{2g}(\mathbb{Z})$ admits $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action via comparison isomorphism.

2. **Algebraic K-theory.**

Let's review how algebraic K-theory is defined. For example, $K_i(\mathbb{Z}) = \pi_i(B\,\mathrm{GL}_\infty(\mathbb{Z})^+)$ for $i > 0$, where $\mathrm{GL}_\infty(\mathbb{Z}) = \varinjlim \mathrm{GL}_n(\mathbb{Z})$, and the plus construction is a general construction $X \longrightarrow X^+$ for perfect normal subgroup $N \trianglelefteq \pi_1(X)$ such that $\pi_1(X^+) = \pi_1(X)/N$; in this case we choose $N = \mathrm{SL}_\infty(\mathbb{Z}) = [\mathrm{GL}_\infty(\mathbb{Z}), \mathrm{GL}_\infty(\mathbb{Z})]$. A fact is that $X \longrightarrow X^+$ gives an isomorphism on homology.

Now we can define $K_i^{\mathrm{Sp}}(\mathbb{Z}) := \pi_i(B\,\mathrm{Sp}_\infty(\mathbb{Z})^+)$, with $N = [\mathrm{Sp}_\infty(\mathbb{Z}), \mathrm{Sp}_\infty(\mathbb{Z})]$. It is plausible to believe that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $K_i^{\mathrm{Sp}}(\mathbb{Z})$.

Algebraic K-theory of $\mathbb{Z}$ is closely related to number theory. For example, there is the following

Theorem 2.1 (Mazur-Wiles, Rost-Voevodsky)**.**

$$\zeta(1-2i) = \pm 2^i \frac{\#K_{4i-2}(\mathbb{Z})}{\#K_{4i-1}(\mathbb{Z})},$$

for $i \geq 1$.

Also, the Vandiver's conjecture is equivalent to $K_{4i}(\mathbb{Z}) = 0$ for all $i > 0$.

Now we have a map

$$K_i^{\mathrm{Sp}}(\mathbb{Z}) = \pi_i(B\,\mathrm{Sp}_\infty(\mathbb{Z})^+) \longrightarrow H_i(B\,\mathrm{Sp}_\infty(\mathbb{Z})) = H_i(\mathrm{Sp}_\infty(\mathbb{Z})).$$

Note that $H_*(\mathrm{Sp}_\infty(\mathbb{Z}))$ has algebra structure coming from $\mathrm{Sp}_{2g} \times \mathrm{Sp}_{2g'} \longrightarrow \mathrm{Sp}_{2(g+g')}$. How are the two different? In the case of rational coefficients, $K_i^{\mathrm{Sp}}(\mathbb{Z}) \otimes \mathbb{Q}$ maps to the set of **primitive elements** of the Hopf algebra $H_*(\mathrm{Sp}_\infty(\mathbb{Z})) \otimes \mathbb{Q}$. As Hopf algebra is determined by primitive elements, they encode the same information. On the other hand, for p-adic coefficient case, $K_i^{\mathrm{Sp}}(\mathbb{Z}, \mathbb{Z}_p) \longrightarrow H_i(\mathrm{Sp}_\infty(\mathbb{Z}), \mathbb{Z}_p)$ gives an isomorphism

$$K_i^{\mathrm{Sp}}(\mathbb{Z}, \mathbb{Z}_p) \xrightarrow{\sim} H_i/\text{decomposables},$$

for $i < 2p-2$ (decomposable into low-degree terms, involving Massey products).

3. **Main result.**

Recall that there is the Hodge bundle ω_g over $\mathcal{A}_g$. Taking the i-th Chern class, we get $\mathrm{ch}_i(\omega_g) \in H^{2i}(\mathcal{A}_g, \mathbb{Z}_p(i))$. This gives a map $H_{2i}(\mathcal{A}_g, \mathbb{Z}_p) \longrightarrow \mathbb{Z}_p(i)$, which is viable for limit-taking process. Thus, we get what's called **Hodge map**

$$K_{2i}^{\mathrm{Sp}}(\mathbb{Z}, \mathbb{Z}_p) \longrightarrow H_{2i}(B\,\mathrm{Sp}_\infty(\mathbb{Z}), \mathbb{Z}_p) \longrightarrow \mathbb{Z}_p(i),$$

which turns out to be Galois-equivariant.

Theorem 3.1 (Feng-Galatius-Venkatesh)**.** *The Hodge map is the* ***universal extension*** *of* $\mathbb{Z}_p(2i-1)$ *as* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-representations, which has splitting at* p*.*

Here, the **universal extension** is the initial object of the category of $\pi : U \twoheadrightarrow M$ of Γ-modules together with splitting of $\pi|_H$, for Γ profinite, $H \trianglelefteq \Gamma$, and M a Γ-module.

Remark 3.1. The kernel of the universal map $K_{4i-2}^{\mathrm{Sp}}(\mathbb{Z}, \mathbb{Z}_p) \longrightarrow \mathbb{Z}_p(2i-1)$ is $K_{4i-2}(\mathbb{Z}, \mathbb{Z}_p)$ (which is a consequence of $\mathrm{GL}_g \hookrightarrow \mathrm{Sp}_{2g}$ via Levi), which is related to $L(1-2i)$.

Remark 3.2. In proving $R = \mathbb{T}$ theorems, one shows that cohomology of arithmetic groups is also "universal".

An application of this theorem is that this gives an obstruction to existence of families of abelian varieties. Namely, the "volume" of the $(2i-1)$-dimensional support of family of abelian varieties should be divisible by primes dividing $\#K_{4i-2}$.

4. **Idea of proof.**

We will only talk about $i < 2p-2$. The basic idea is to write down enough classes on which Galois action can be computed explicitly. The group structure of $H_*(\mathrm{Sp}_{2g}(\mathbb{Z}))$ can be mostly exploited by using maps of form $\mu_q \hookrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$, and we know well about homology of cyclic group.

To understand the Galois action, we need to upgrade this to algebraic geometry, i.e. need to understand maps of form $B\mu_q \longrightarrow \mathcal{A}_g$, or abelian varieties with CM by μ_q. The Galois action by $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the pushforward of a class in homology of cyclic group by CM abelian variety A is the pushforward by A^σ of the same class, so we need to use main theorem of CM.

1. **Reinterpreting the Serre's conjecture.**

Recall that the classical Serre's conjecture is the following.

Conjecture 1.1 (Serre's conjecture, weak form). *Suppose* $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *is continuous irreducible odd Galois representation. Thne, there is a cuspidal eigenform* f *such that* $\overline{\rho} \cong \overline{\rho}_f$.

On the other hand, the **strong form of Serre's conjecture** gives the explicit recipe for the minimum weight $k(\overline{\rho})$ and level $N(\overline{\rho})$ that you can take for f. The minimum level is easy: $N(\overline{\rho})$ is prime-to-p Artin conductor of $\overline{\rho}$. In particular, it only depends on $\overline{\rho}|_{G_\ell}$ for $\ell \neq p$.

The minimum weight recipe on the other hand is more complicated. We will just mention that $2 \le k(\overline{\rho}) \le p+1$ (under Serre's recipe weight 1 is not considered) and it depends only on $\overline{\rho}|_{I_p}$.

Example 1.1. If $\overline{\rho}|_{I_p} \sim \begin{pmatrix} \omega^a & * \\ 0 & 1 \end{pmatrix}$ for $2 \le a \le p-3$, where ω is the cyclotomic character, then $k(\overline{\rho}) = a+1$.

The weak form is a theorem of Khare-Wintenberger and Kisin. Interestingly, that the weak form implies the strong form is shown by Edixhoven, Gross, Ribet, Coleman-Voloch much earlier (in '90s). Not only it is of independent interest, this is used in the proof of weak form.

We are interested in generalizing this to other groups and fields. Firstly we would have to contemplate how to even generalize the notions, because for example there is no notion of "minimal weight"; generally a weight is given by not just one integer but some tuple of integers.

Definition 1.1. *A* ***Serre weight*** *is an irreducible representation of* $\mathrm{GL}_2(\mathbb{F}_p)$ *over* $\overline{\mathbb{F}}_p$.

It is easy to see that any Serre weight is of the form

$$\mathrm{Sym}^a(\overline{\mathbb{F}}_p^2) \otimes \det{}^b,$$

where $0 \le a \le p-1$ and $0 \le b \le p-2$.

Definition 1.2. *Define*

$$W(\overline{\rho}) = \{V \text{ Serre weight} \mid \exists \text{Hecke eigenclass } \alpha \neq 0 \text{ in } H^1_{\mathrm{et}}(Y_1(N), \mathcal{V}) \text{ associated to } \overline{\rho} \text{ for some } (N,p)=1,$$
$$\text{where } \mathcal{V} \text{ is the local system associated to } V\}.$$

Then the strong Serre conjecture is about explicit description of $W(\overline{\rho})$.

Building on work of Buzzard-Diamond-Jarvis, Schein, $\cdots$, the work of Gee-Herzig-Savitt defines a set $W^?(\overline{\rho})$ "obtained combinatorially." For example, if $\overline{\rho}|_{I_p}$ is semisimple, then $W^?(\overline{\rho})$ is related to Deligne-Lusztig representations.

Example 1.2. If $\overline{\rho}|_{I_p} = \begin{pmatrix} \omega^a & 0 \\ 0 & 1 \end{pmatrix}$ for $2 \le a \le p-3$, then $W^?(\overline{\rho}) = \{\mathrm{Sym}^{a-1}, \mathrm{Sym}^{p-2-a} \otimes \det^a\}$. The recipe is

$$\text{“}\mathcal{R}(\mathrm{JH}(\overline{\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_p)}([\omega^a] \boxtimes 1)}))\text{”}$$

That the strong Serre conjecture is implied by the weak Serre conjecture in this setting is

$$W(\overline{\rho}) = W^?(\overline{\rho}).$$

This is useful as this is viable for generalization to other fields and groups.

2. **Totally real fields.**

Let now F be a totally real field. The weak form of Serre's conjecture for Hilbert modular forms should be that any continuous, irreducible, totally odd $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular. There is also a recipe for $W^?(\overline{\rho})$.

The weak form, that $W(\overline{\rho}) \neq \emptyset$, is out of reach, but we can still ask if weak implies strong.

Theorem 2.1 (Gee, $\cdots$)**.** *If $\overline{\rho}$ is modular, and if $\overline{\rho}$ satisfies Taylor-Wiles condition, then $W(\overline{\rho}) = W^?(\overline{\rho})$.*

3. **Unitary groups.**

Let F again be a totally real field, p be unramified in F, and $\tilde{F}$ be a CM extension of F such that $\tilde{F}/F$ is unramified everywhere, and for all $v \mid p$ in F, v is inert in $\tilde{F}$. Let U_2/F be a unitary group defined with respect to $\tilde{F}/F$, compact at infinity, quasisplt at every v, and is a p-adic unitary group at every $v \mid p$.

Galois representations we will be considering is of form

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \longrightarrow {}^C\,\mathrm{U}_2(\overline{\mathbb{F}}_p),$$

where ${}^C\,\mathrm{U}_2(\overline{\mathbb{F}}_p)$ is the "C-group" in the sense of Buzzard-Gee. It is of form $(\mathrm{GL}_2(\overline{\mathbb{F}}_p)\times\overline{\mathbb{F}}_p^{\times})\rtimes\mathrm{Gal}(\tilde{F}/F)$. The analogue for oddness here is that $\overline{\rho}$ has cyclotomic determinant (where determinant is just the projection to the second factor, "the $\mathbb{G}_m$-factor").

Remark 3.1. One has a similar setting in this case too.

- ${}^C\,\mathrm{U}_2$ is an enhancement of ${}^L\,\mathrm{U}_2$.
- Serre weights in this case are representations of $\prod_{v|p}\mathrm{U}_2(k_{\tilde{v}}/k_v)$ where $\tilde{v}$ is the place of $\tilde{F}$ above v.
- One can define $W(\overline{\rho})$, using cohomology of symmetric space associated to U_2.
- $W^?(\overline{\rho})$ can be defined combinatorially.

Theorem 3.1 (Koziol-Morra)**.** *Suppose $W(\overline{\rho})$ is nonempty, $\overline{\rho}$ is semisimple and "generic" at places above p, $\overline{\rho}$ satisfies some strengthening of Taylor-Wiles hypothesis, plus some technical assumption. Then, $W(\overline{\rho}) = W^?(\overline{\rho})$.*

1. **Constructing supercuspidal representations.**

Let G be a connected reductive group over a local field F with residue characteristic $p > 0$. We want to consturct all irreducible smooth representations of $G(F)$, over $\mathbb{C}$, or $\overline{\mathbb{F}}_\ell$ for $\ell \neq p$. Everything is built out of (super)cuspidal representations. We know how to construct (super)cuspidal representations for GL_n (Bushnell-Kutzko), classical groups, inner forms of GL_n, $\cdots$.

For a general reductive group, Moy-Prasad invented the notion of **depth** of representations, and in particular classified all depth 0 representations. Many more representations of greater depth were constructed by J. K. Yu, and Ju-Lee Kim proved that all representations are obtained by this way if p is very large and $\mathrm{char}\, F = 0$. This does not exhaust every representation, because Reeder-Yu constructed all of the so-called **epipelagic reprsentations** (~ small depth) that give some new representations when p is small.

Theorem 1.1 (Fintzen)**.** *Suppose G splits over a tamely ramified extension of F. Assume that $p \nmid |W|$. Then, Yu's construction yields all supercuspidal representations.*

So p does not have to be "too large," given the following table.

Type	$\lvert W\rvert$
A_n	$(n+1)!$
B_n	$2^n \cdot n!$
C_n	$2^n \cdot n!$
D_n	$2^{n-1} \cdot n!$
E_6	$2^7 \cdot 3^4 \cdot 5$
E_7	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$
E_8	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
F_4	$2^7 \cdot 3^2$
G_2	$2^2 \cdot 3$

Remark 1.1. (1) The work above exhibits types for all irreducible representations.
(2) It is expected that $p \nmid |W|$ is optimal in general.

Theorem 1.2 (Fintzen)**.** *An analogous construction yields all irreducible cuspidal $\overline{\mathbb{F}}_\ell$-representations.*

2. **Moy-Prasad filtration.**

How do we construct supercuspidal representations?

(1) We pick a compact mod center open subgroup K of $G(F)$,
(2) and then construct a representation of $G(F)$ out of this.

Example 2.1. Let $G = \mathrm{SL}_2(F)$. Then we can take

$$K = \begin{pmatrix} 1+t\mathcal{O} & t\mathcal{O} \\ \mathcal{O} & 1+t\mathcal{O} \end{pmatrix} \times \{\pm 1\}.$$

Now K has a filtration whose subquotients are finite groups of Lie type. For example the first filtration is

$$\begin{pmatrix} 1+t\mathcal{O} & t\mathcal{O} \\ \mathcal{O} & 1+t\mathcal{O} \end{pmatrix} \times \{\pm 1\} \supset \begin{pmatrix} 1+t\mathcal{O} & t^2\mathcal{O} \\ t\mathcal{O} & 1+t\mathcal{O} \end{pmatrix} \times \{\pm 1\},$$

and the quotient is $\begin{pmatrix} 0 & \mathbb{F}_q \\ \mathbb{F}_q & 0 \end{pmatrix}$.

In general, a **Moy-Prasad filtration** for SL_2, starting from $\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}$, is

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} \supset \begin{pmatrix} 1+t\mathcal{O} & t\mathcal{O} \\ \mathcal{O} & 1+t\mathcal{O} \end{pmatrix} \supset \begin{pmatrix} 1+t\mathcal{O} & t^2\mathcal{O} \\ t\mathcal{O} & 1+t\mathcal{O} \end{pmatrix} \supset \begin{pmatrix} 1+t^2\mathcal{O} & t^2\mathcal{O} \\ t\mathcal{O} & 1+t^2\mathcal{O} \end{pmatrix} \supset \begin{pmatrix} 1+t^2\mathcal{O} & t^3\mathcal{O} \\ t^2\mathcal{O} & 1+t^2\mathcal{O} \end{pmatrix}.$$

In general, a Moy-Prasad filtration is constructed out of the Bruhat-Tits building. Namely, one fixes a point in the building, and one gets filtration by taking the stabilizer of a ball of growing radius around the point. In particular, the example we had started from depth 1/2. In general $G_{x,0}/G_{x,0+}$ is a finite group of Lie type, whereas $G_{x,r}/G_{x,r+}$ is abelian, where r is a radius > 0.

Theorem 2.1 (Fintzen)**.** *We have a description of this quotient in terms of Weyl modules, Vinberg-Levy representations and special fibers of a global object.*

1. **Congruence of modular forms.**

Congruence of modular forms has a lot of arithmetic applications.

Example 1.1. Greenberg-Vatsal proved that if two modular forms are congruent modulo p, then the Iwasawa Main Conjectures for two modular forms are equivalent.

Ribet used cusp forms congruent to Eisenstein series to construct abelian extensions of $\mathbb{Q}(\mu_{691})$.

Question. Given a cusp newform f of level $\Gamma_0(N)$ and weight k, how "many" eigenforms congruent to f modulo p exist in $S_k(\Gamma_0(N))$?

To understand the problem, we would like to reformulate the notion of congruence of modular forms. Let F_λ be a finite extension of $\mathbb{Q}_p$ generated by Fourier coefficients of f, after choice of embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Suppose further that the residual Galois representation $\overline{\rho}_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{F}_\lambda)$, where $\mathbb{F}_\lambda$ is the residue field of $\mathcal{O}_{F_\lambda}$.

Associated to a newform f is an algebra homomorphism $\pi_f : \mathbb{T}_N \twoheadrightarrow \mathcal{O}_\lambda$ where $\mathbb{T}_N$ is the Hecke algebra in $\mathrm{End}(S_k(\Gamma_0(N), \mathbb{Z}_p))$. Then, Galois conjugacy classes of f are in one-to-one correspondence with $\mathfrak{p}_f = \ker(\pi_f)$ height 1 prime ideals of $\mathbb{T}_N$. Now, $f \equiv g (\mathrm{mod}\, p)$ means $\mathfrak{p}_f$ and $\mathfrak{p}_g$ lie both in the same maximal ideal $\mathfrak{m}$ of $\mathbb{T}_N$ (which is equivalent to $\overline{\rho}_f \cong \overline{\rho}_g$). Motivated from this observation, we can try to define the following.

Definition 1.1. *Let $\Phi_f = \mathfrak{p}_f/\mathfrak{p}_f^2$ be the cotangent space of $\mathbb{T}_N/\mathfrak{p}_f$. Then, the* ***congruence ideal*** *$\eta_f(N)$ is defined by $\pi_f(\mathrm{Ann}_{\mathbb{T}_N}(\mathfrak{p}_f)) \subset \mathcal{O}_\lambda$.*

The congruence ideal is the gadget that precisely measure congruences of the above form.

2. **A variant.**

One can try to further restrict to the problem of detecting congruences between **newforms**. Let $N = N^+N^-$, where N^- is assumed to be square-free. Then, consider $\mathbb{T}_N^{N^-}$, the N^--new quotient of $\mathbb{T}_N$. Then, the homomorphism $\pi_f : \mathbb{T}_N \longrightarrow \mathcal{O}_\lambda$ factors through $\mathbb{T}_f^{N^-} : \mathbb{T}_N^{N^-} \longrightarrow \mathcal{O}_\lambda$.

Definition 2.1. *The N^- congruence ideal $\eta_f(N^+, N^-)$ is defined by*

$$\eta_f(N^+, N^-) = \pi_f^{N^-}(\mathrm{Ann}_{\mathbb{T}_N^{N^-}}(\mathfrak{p}_f^{N^-})),$$

where $\mathfrak{p}_f^{N^-} = \ker(\pi_f^{N^-})$.

The difference between $\eta_f(N^+, N^-)$ and $\eta_f(N)$ is that $\eta_f(N^+, N^-)$ quantifies level. In particular, it captures information about level-lowering at $\ell \mid N^-$.

3. **Level lowering.**

Conjecture 3.1 (Pollack-Weston)**.** *For a squarefree $N = aqb$ where q is prime, and if $k = 2$, then*

$$\mathrm{ord}_\lambda(\eta_f(n/b, b)) = t_f(q) + \mathrm{ord}_\lambda(\eta_f(\frac{n}{bq}, bq)),$$

where $t_f(q)$ is the "Tamagawa exponent," namely the largest integer t such that $A_f[\lambda^t]$ is unramified at q (so that $A_f[\lambda^{t+1}]$ is ramified at q), where $A_f = V_f/T_f$ is the divisible Galois representation.

Theorem 3.1 (Pollack-Weston)**.** *Assuming the following technical asusmptions, the above conjecture holds:*

- *$\overline{\rho}$ has a "big image," i.e. it contains $\mathrm{SL}_2(\mathbb{F}_p)$,*

- *if $\ell \mid qb$ and $\ell \equiv \pm 1 \pmod p$, then $\overline{\rho}$ is ramified at ℓ,*
- *and $\overline{\rho}$ is ramified at at least 2 primes.*

Their proof uses a method of Ribet-Takahashi, which compares degrees of parametrizations of elliptic curve by modular curve and Shimura curve. In particular, the method does not work if $k > 2$.

Such theorem has several interesting arithmetic consequences.

(1) If N^- is a product of odd numbers of primes (plus some technical assumptions, regarding "mod p multiplicity one principle" for Hecke modules), then one can explicitly compute the difference between the so-called **Gross period** of the definite quaternion algebra ramified at N^- and Hida-Shimura's canonical period attached to f. Specifically,

$$\mathrm{ord}_\lambda\left(\frac{\Omega_f^{N^-}}{\Omega_f}\right) = \sum_{q \mid N^-} t_q(f),$$

where $\Omega_f^{N^-}$ is the Gross period and Ω_f is Hida's canonical period.

(2) In a similar situation, if f is ordinary, as well as $a_p(f) \not\equiv \pm 1 \pmod p$, then the theorem can be used to prove the μ-part of the anticyclotomic Iwasawa Main Conjecture, for an imaginary quadratic field K under certain assumptions (more precisely, $(\mathrm{disc}(K), Np) = 1$, $\ell \mid N^-$ inert in K, $\ell \mid N^+$ split in K).

(3) If N^- is even, under technical assumptions, one can explicitly compute the "level lowering congruences," namely for B the quaternion algebra corresponding to N^-,

$$\mathrm{ord}_\lambda\left(\frac{\langle f, f\rangle_{\Gamma_0(N)}}{\langle f_B, f_B\rangle_\Gamma}\right) = \sum_{q \mid N^-} t_q(f).$$

(4) Kato's Kolyvagin systems can be "primitive", which has implications towards cyclotomic Iwasawa Main Conjecture.

Theorem 3.2 (Kim-Ota)**.** *Let p be odd, $f \in S_k(\Gamma_0(N))$, $N = N^+N^-$, as before.*

(1) $\overline{\rho}|_{G_{\mathbb{Q}(\sqrt{p^*})}}$ is absolutely irreducible, where $p^* = (-1)^{\frac{p-1}{2}} p$.
(2) $2 \le k \le p-1$, i.e. Fontaine-Laffaille range.
(3) p, N^+, N^- are pairwise relatively prime.
(4) N^- is squarefree.
(5) If $q \equiv \pm 1 \pmod p$, $q \mid N^-$, then $\overline{\rho}$ is ramified at q.

Then, $\mathrm{ord}_\lambda(\eta_f(N)) = \sum_{q \mid N^-} t_q(f) + \mathrm{ord}_\lambda(\eta_f(N^+, N^-))$.

Proof. There is no direct geometric argument, so we need an $R = \mathbb{T}$ theorem by Diamond-Flach-Guo (Dimitrov for Hilbert modular forms). Very classical computation is that

$$\#\,\mathrm{Sel}(\mathbb{Q}, \mathrm{ad}^0(\rho_f) \otimes F_\lambda/\mathcal{O}_\lambda) = \#\Phi_f = \#\mathcal{O}_\lambda/\eta_f(N) = \frac{L(\mathrm{ad}^0(f), 1)}{\Omega_f^+ \Omega_f^-}.$$

To compute the size of $\eta_f(N^+, N^-)$, one might try to change a local Selmer condition to reflect $\ell | N^-$ and $\ell \nmid N(\overline{\rho})$, but this is not good, as unioning two local conditions is not a good deformation condition (Steinberg at ℓ and $\cdots$). To avoid this, we use a simple trick to "send the problem to the L-value side", by comparing Euler factors of adjoint L-functions. $\square$

1. What we know about supersingular representations.

Let $G = \mathrm{GL}_2(F)$ for a p-adic field $F/\mathbb{Q}_p$, $K = \mathrm{GL}_2(\mathcal{O}_F)$ and consider the Iwahori subgroup $I = \left\{ \begin{pmatrix} a & b \\ \varpi c & d \end{pmatrix} \in K \right\}$ and the pro-p Iwahori subgroup $I_1 = \left\{ \begin{pmatrix} 1+\varpi a & b \\ \varpi c & 1+\varphi d \end{pmatrix} \in K \right\}$, the pro-$p$-Sylow subgroup of I. It is a maximal open pro-p subgroup of G.

Recall that **weights** V are irreducible K-representations. If $k = \mathbb{F}_p$ is the residue field of F, then weights are of form $V_{a,b} = \det^b \otimes \mathrm{Sym}^{a-b} \overline{\mathbb{F}}_p^2$, for $0 \le a-b \le p-1$ and $0 \le b < p-1$.

Recall also that any irreducible admissible representation of G is a quotient of

$$\frac{\mathrm{ind}_K^G V}{(\tau_{1,0} - \lambda, \tau_{1,1} - \alpha)}.$$

As α accounts for central character action, we can safely twist by an unramified character to assume from now on that $\alpha = 1$.

Theorem 1.1.
- *(Barthel-Livné, 1994) Let π be* **any** *irreducible smooth representation of G. Then, π is a quotient of $\mathrm{ind}_{KZ}^G V/(\tau_{1,0} - \lambda)$. If $\lambda \neq 0$, then π is a principal series. If $\lambda = 0$, then π is supersingular, and also is not a subquotient of principal series, or "supercuspidal."*
- *(Breuil, 2003) If $F = \mathbb{Q}_p$, then $\pi_{a,b} = \mathrm{ind}_{KZ}^G V_{a,b}/(\tau_{1,0})$ are all irreducible.*
- *(Berger, 2011) If $F = \mathbb{Q}_p$, then every irreducible smooth representation has central character.*

Breuil proves the theorem by brute force. In particular, he computes $\dim_{\overline{\mathbb{F}}_p} \pi_{a,b}^{I_1} = 2$, which implies that $\pi_{a,b}$ is admissible and any nonzero I_1-invariant vector generates.

Corollary 1.1 (Mod p local Langlands correspondence). *We have $\pi_{a,b} \cong \pi_{p-1-b,a}$, and these are the only isomorphisms between $\pi_{a,b}$. Thus, there is a 1-1 correspondence*

$$\left\{ \textit{Modular weights } \{V_{a,b}, V_{p-1-b,a}\} \right\} \leftrightarrow \left\{ \textit{Irreducible } \rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p) \right\}.$$

Unfortunately this does not generalize. What do we know?

- Morra computed K-socle filtrations of $\pi_{a,b}$, namely $\mathrm{soc}_1 = \mathrm{soc}_{KZ}(\mathrm{soc}_{i-2} / \mathrm{soc}_{i-1})$.
- Paskunas (2007) computed that, in the category of representations with central character, $p \ge 5$, $\mathrm{Ext}^1(\tau, \pi_{a,b}) = 0$ for irreducible τ, unless $\tau = \pi_{a,b}$, and $\dim \mathrm{Ext}^1(\pi_{a,b}, \pi_{a,b}) = 3$. It happens that $\mathrm{ind}_{KZ}^G V_{a,b}/\tau_{1,0}^2$ and $\mathrm{ind}_{KZ}^G V_{p-1-b,a}/\tau_{1,0}^2$ are linearly independent.
- Anandavardhanan-Borisagar (2013) showed that the space generated by the two linearly independent elements $\mathrm{ind}_{KZ}^G V_{a,b}/\tau_{1,0}^2$ and $\mathrm{ind}_{KZ}^G V_{p-1-b,a}/\tau_{1,0}^2$ are of form $\mathrm{ind}_{IZ}^G \chi/(\lambda_1^{-1} U + \lambda_2^{-1} V)$; U, V are the two operators generating the Iwahori-Hecke algebra which happens to be commutative for characters not factoring through the determinant. Here, the Iwahori-Hecke-algebra $\mathscr{H}_{I_1}^G(\mathbb{1}) = \mathrm{End}(\mathrm{ind}_{I_1}^G(\mathbb{1}))$, which makes sense from $\pi^{I_1} = \mathrm{Hom}_{I_1}(\mathbb{1}, \pi|_{I_1}) = \mathrm{Hom}_G(\mathrm{ind}_{I_1}^G \mathbb{1}, \pi)$. Ollivier proved that $\pi \mapsto \pi^{I_1}$ gives an equivalence of categories

$$\left\{ \begin{array}{c} G\text{-representations} \\ \text{generated by } \pi^{I_1} \end{array} \right\} \leftrightarrow \{\mathscr{H}_{I_1}^G(\mathbb{1})\text{-modules}\},$$

  for $G = \mathrm{GL}_2(\mathbb{Q}_p)$ still. This shows that the "third basis" should be found from induction from not I_1 but something smaller.

- For $G = \mathrm{SL}_2(\mathbb{Q}_p)$, R. Abdellatif-Cheng (2013) showed that there are p supersingular representations σ_i such that
$$\pi_{a,b}|_{\mathrm{SL}_2(\mathbb{Q}_p)} = \sigma_{a-b} \oplus \sigma_{p-1-(a-b)}.$$
- Koziol and Koziol-Xu dealt rank 1 unitary groups.
- Peskin dealt a metaplectic case, $\widetilde{G} = \widetilde{\mathrm{SL}_2}(\mathbb{Q}_p) \cong \mathrm{SL}_2(\mathbb{Q}_p) \rtimes \{\pm 1\}$, where in this case a maximal compact subgroup is $\widetilde{K} = \mathrm{SL}_2(\mathbb{Z}_p) \rtimes \{\pm 1\}$. Then similarly there are p supersingular representations, then $\mathrm{ind}_{\widetilde{K}Z}^{\widetilde{G}}(V_{a,b} \otimes \mathrm{sgn})/\tau_{1,0}$ also has a supersingular subrepresentation whose quotient is also supersingular, simlar to $\mathrm{SL}_2(\mathbb{Q}_p)$-case (but not split).

We know virtually nothing except these.

2. **How to approach supersingular representations.**
From now on, let $F \neq \mathbb{Q}_p$, $G = \mathrm{GL}_2(F)$.

Proposition 2.1 (Schein, Hendel)**.** *A representation* $\mathrm{ind}_{KZ}^G V/\tau_{1,0}$, *for a weight* V, *is not admissible. Furthermore,* $(\mathrm{ind}_{KZ}^G V/\tau_{1,0})^{I_1}$ *is explicitly computed. Thus,* $\mathrm{End}(\mathrm{ind}_{KZ}^G V/\tau_{1,0}) = \overline{\mathbb{F}}_p$.

It has an irreducible subrepresentation just because of Zorn's lemma, but one does not know if there is an admissible. One sometimes try to find such one by taking an image of some part of endomorphism algebra, so this result is disappointing.

Proposition 2.2 (Schraen, 2012)**.** *If $F \neq \mathbb{Q}_p$, then for supersingular representations, the "Euler characteristic is infinite." In particular, $[F : \mathbb{Q}_p] = 2$ (so that cohomology is concentrated in one degree), then supersingular representations of G are never finitely presented.*

Remark 2.1. One can try to take socle and kill all weights that are expected to not arise from Serre's conjecture, but then even though one knows that the resulting represntation is admissible, we do not know if it is nonzero.

We thus keep things explicit, and consider the following consideration of Paskunas. Let π be an irreducible admissible G-representation. Then, we have a map
$$\pi^{I_1} \hookrightarrow K\pi^{I_1},$$
where π^{I_1} is a representation of $N_G(I_1) = \left\langle I, \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \right\rangle$, and $K\pi^{I_1}$ is a representation of KZ. We call such object a **diagram**, namely a map $r : D_1 \longrightarrow D_0$ of IZ-representations such that D_1 is a finite-dimensional $N_G(I_1)$-representation and D_0 is a finite-dimensional KZ-representation. Then, for $G = \mathrm{GL}_2(\mathbb{Q}_p)$,
$$\left\{ \begin{matrix} \text{Irreducible} \\ \text{representations of } G \end{matrix} \right\} = \{\text{Irreducible diagrams}\}.$$
Paskunas proved that in a more general setting
$$\left\{ \begin{matrix} \text{Coefficient systems} \\ \text{on Bruhat-Tits tree} \end{matrix} \right\} = \{\text{Diagrams}\}.$$
Then one has a hope of constructing supersingular representations geometricallly out of coefficient systems on Bruhat-Tits tree (using cohomology complexes built out of coefficient systems).

What this leads to however is that (Breuil-Paskunas), for $F \neq \mathbb{Q}_p$ unramified, even though you get supersingular representations from this construction, this turns out to form an infinite family, and one actually needs infinite number of parameters to parametrize the family (Y. Hu). This means that there are way too many supersingular representations in the automorphic side to be matched bijectively.

Our goal is to study families of p-adic Galois representations of p-adic families parametrized by a variety over $\mathbb{Q}_p$, from p-adic Hodge-theoretic viewpoint.

1. p-adic Hodge theory.

Consider a Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \mathrm{GL}(V)$ for a finite-dimensional $\mathbb{Q}_p$-vector space V.

Example 1.1. (1) The p-adic cyclotomic character $\chi : G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$ can be defined by $g\zeta_{p^n} = \zeta_{p^n}^{\chi(g)}$.

(2) Given a smooth proper algebraic variety Y over $\mathbb{Q}_p$, we can consider $V := H^n_{\text{ét}}(Y_{\mathbb{Q}_p}, \mathbb{Q}_p)$, which has an action of $G_{\mathbb{Q}_p}$ by functoriality. Such p-adic representations have special properties.

Theorem 1.1 (C_{dR}-conjecture; Tsuji, Faltings, Niziol)**.** *There is a natural $G_{\mathbb{Q}_p}$-isomorphism*

$$V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong H^n_{\mathrm{dR}}(Y/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}},$$

where B_{dR} is Fontaine's de Rham period ring.

In other words, V is **de Rham**.

Remark 1.1. There are a lot of non-de Rham Galois representations. For example, χ^α, a one-dimensional p-adic Galois representation, for $\alpha \in \mathbb{Z}_p$, is de Rham if and only if $\alpha \in \mathbb{Z}$. Another example is a Galois representation associated to a non-classical overconvergent modular forms.

Thus, even though de Rham representations are nice, one might want to study a family of Galois representations where non-de Rham representations are involved. To study a general p-adic local Galois representation, one might want to use Sen's theory. Namely, Sen associated to $\rho : G_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_r(\mathbb{Q}_p)$ the **generalized Hodge-Tate weights** $\alpha_1, \cdots, \alpha_r \in \overline{\mathbb{Q}}_p$.

Example 1.2.
- If $\rho = \chi^\alpha$, α is the generalized Hodge-Tate weight of ρ.
- If $V = H^n(Y_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$, then the generalized Hodge-Tate weights are $h^{n,0}$ 0's, $h^{n-1,1}$ -1's, $\cdots$, $h^{0,n}$ $-n$'s, where $h^{p,q} = \dim H^q(\Omega^p)$.

Remark 1.2. One can also try to associate the generalized Hodge-Tate weights using the theory of (φ, Γ)-modules.

Question. How do these concepts behave in a "geometric" family?

2. Geometric families of Galois representations.

What we mean by a "geometric family" is that it is association of Galois representation of residue field for every point of an algebraic variety.

Example 2.1. Let $\pi : E \longrightarrow Y(N)$ be the universal elliptic curve. Then, for any point $y \in Y(N)$, T_pE_y is a Galois representation of $G_{k(y)}$.

This is on the other hand equivalent to the notion of **étale local system**. Namely, in the above example,

$$(R^1\pi_*\mathbb{Z}_p)_{\overline{y}} \cong (T_pE_y)^\vee.$$

Recall that a rank r $\mathbb{Q}_p$-étale local system on a connected scheme Y is the same as a continuous homomorphism $\pi_{1,\text{ét}}(Y) \longrightarrow \mathrm{GL}_r(\mathbb{Q}_p)$, so it recovers the usual notion of Galois representation when Y is Spec of a field. Thus, it is reasonable to believe that $\mathbb{Q}_p$-étale local systems are the right objects to study.

3. **Results.**

Let $k/\mathbb{Q}_p$ be a finite-extension, X be a smooth geometrically connected algebraic variety over k (or a smooth geometrically connected rigid analytic variety over k), and $\mathbb{L}$ be a $\mathbb{Q}_p$-étale local system on X.

Theorem 3.1 (Liu-Zhu). *If $\mathbb{L}_{\overline{x}}$ is de Rham as a Galois representation of $G_{k(x)}$ for a single classical point x (i.e. $k(x)$ is a finite extension of $\mathbb{Q}_p$), then $\mathbb{L}$ is de Rham at every point.*

Theorem 3.2 (Shimizu). *The multiset of generalized Hodge-Tate weights of $\mathbb{L}_{\overline{x}}$ is constant on X.*

Compare this with arithmetic families of Galois representation:

(1) within a Hida/Coleman family, generalized Hodge-Tate weights vary as weights of modular forms vary,
(2) and a family of Galois representations arising from Galois deformation, even after imposing nice local conditions at p such as crystallinity, it is not the case that every point has a de Rham Galois representation.

This suggests that, quite surprisingly, the notion of geometric family of Galois representations is much more rigid that the analogous notion of arithmetic family of Galois representations.

4. **Idea of proof.**

Let $k_\infty = k(\mu_{p^\infty})$, $K = \widehat{k_\infty}$, $\Gamma = \mathrm{Gal}(k_\infty/k)$, which can be realized as an open subgroup of $\mathbb{Z}_p^\times$ via the cyclotomic character χ.

How do we get generalized Hodge-Tate weights, e.g. how do we get α from χ^α? Naively we would like to perform "derivation". Given a Galois representation $\rho : G_k \longrightarrow \mathrm{GL}(V)$, Sen's theory gives "infinitesimal action of Γ", $\phi_V \in \mathrm{End}_K(\mathcal{H}(V))$, where $\mathcal{H}(V) = (V \otimes_{\mathbb{Q}_p} \widehat{\overline{k}})^{\mathrm{Gal}(\overline{k}/k_\infty)}$ is a K-vector space with Γ-action. Then, the generalized Hodge-Tate weights are obtained as the eigenvalues of ϕ_v.

Example 4.1. Given the case of $\chi^\alpha : \Gamma \longrightarrow \mathbb{Z}_p^\times$, we have $d\chi^\alpha : \mathrm{Lie}\,\Gamma \longrightarrow \mathrm{Lie}\,\mathbb{Z}_p^\times$ which sends 1 to α.

To prove the constancy of Hodge-Tate weights, we use **p-adic Riemann-Hilbert correspondence**, and a local variant of the Fargues-Fontaine curve. Informally, we would like to add "arithmetic direction" by adding the local variant of the Fargues-Fontaine curve, which should be $(B_{\mathrm{dR}})^{\mathrm{Gal}(\overline{k}/k_\infty)}$, which is known to be a discrete valued field with residue field K and with explicit uniformizer t. Thus, if we forget the Galois action, we can regard it as $K((t))$, and geometrically "adding an arithmetic direction" is to consider "$X\widehat{\otimes}K[[t]] = X\widehat{\otimes}K((t)) \coprod X_K$".

Theorem 4.1 (Liu-Zhu, Shimizu). *There exists a natural functor*

$$\mathcal{RH} : \left\{ \mathbb{Q}_p\text{-local systems on } X \right\} \to \left\{ \begin{array}{c} \text{Filtered vector bundles with} \\ \text{integrable connections on} \\ \text{“}X\widehat{\otimes}K((t))\text{”} \end{array} \right\},$$
$$\mathbb{L} \mapsto (\mathcal{RH}(\mathbb{L}), \nabla, \mathrm{Fil}^*),$$

where

$$\nabla = \nabla^{\mathrm{geom}} + \phi_{\mathrm{dR}} \otimes \frac{dt}{t} : \mathcal{RH}(\mathbb{L}) \longrightarrow \mathcal{RH}(\mathbb{L}) \otimes_{\mathcal{O}_X} (\Omega_X^1 \oplus \mathcal{O}_X \otimes \frac{dt}{t}),$$

such that

(1) $\mathrm{rk}\,\mathcal{RH}(\mathbb{L}) = \mathrm{rk}\,\mathbb{L}$,

(2) $(\mathcal{RH}(\mathbb{L}), \nabla^{\mathrm{geom}})$ *has a* Γ*-action, so that* $D_{\mathrm{dR}}(\mathbb{L}) = \mathcal{RH}(\mathbb{L})^{\Gamma}$ *is a vector bundle on* X *of rank* $\leq \operatorname{rk} \mathbb{L}$*,*

(3) $\mathcal{H}(L) = \operatorname{gr}^0 \mathcal{RH}(\mathbb{L})$ *is a vector bundle on* X_K *of rank* $\operatorname{rk} \mathbb{L}$ *with* $\phi_{\mathbb{L}} \in \operatorname{End}_{X_K}(\mathcal{H}(\mathbb{L}))$*, generalizing the classical* $D_{\mathrm{dR}}(V)$*.*

Now (3) plus the formal connection theory gives the constancy of eigenvalues of $\phi_{\mathbb{L}}$.

We are interested in the mod p local Langlands correspondence for $\mathrm{GL}_2(L)$, for a finite unramified extension $L/\mathbb{Q}_p$.

1. **History of the case of** $\mathrm{GL}_2(\mathbb{Q}_p)$**.**

Let $\mathbb{F}$ be a characteristic p finite field. Breuil proved the semisimple version of mod p local Langlands of $\mathrm{GL}_2(\mathbb{Q}_p)$; namely, to a **semisimple** Galois representation $\overline{\rho} : G_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_2(\mathbb{F})$ one can attach a semisimple $\pi(\overline{\rho})$ of $\mathrm{GL}_2(\mathbb{Q}_p)$. To be more precise, if $\overline{\rho}$ is of form say $\begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$, $\pi(\overline{\rho}) = \begin{pmatrix} \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_2 \otimes \chi_1\omega^{-1} & 0 \\ 0 & \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\omega^{-1} \end{pmatrix}$, and if $\overline{\rho}$ is irreducible, $\pi(\overline{\rho})$ is supersingular.

In general, if $\overline{\rho}$ is reducible non-semisimple, then correspondingly there must be an extension between the two characters appearing in the above expression of $\pi(\overline{\rho})$. Colmez constructed a functor that attaches $\pi(\overline{\rho})$ "functorially" (but works only for generic case). And Emerton proved the local-global compatibility in the following sense: given an irreducible odd $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{F})$ satisfying some technical conditions (e.g. Taylor-Wiles condition), we have

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(\overline{\rho}, \varinjlim_N H^1(Y(N)_{\overline{\mathbb{Q}}}, \mathbb{F})) = {\bigotimes_{\ell}}' \pi(\overline{\rho}|_{G_{\mathbb{Q}_\ell}}).$$

2. **Buzzard-Diamond-Jarvis conjecture.**

Motivated by the local-global compatibility of Emerton, Buzzard-Diamond-Jarvis conjectured the local-global compatibility in the case of totally real field. Namely, given a totally real field F where p is unramified, and a quaternion algebra D/F split at only one infinite place and split at p, if $\overline{r} : G_F \longrightarrow \mathrm{GL}_2(\mathbb{F})$ is totally odd irreducible, then one expects that

$$\mathrm{Hom}_{G_F}(\overline{r}, \varinjlim_U H^1(\mathrm{Sh}_{U,\overline{F}}, \mathbb{F})) = {\bigotimes_{w \text{ finite}}}' \pi_w^D(\overline{r}),$$

where $\pi_w^D(\overline{r})$ is a representation of $(D \otimes_F F_w)^\times$, such that each $\pi_w^D(\overline{r})$ corresponds to $\overline{r}|_{G_{F_w}}$ via the "mod w local Langlands correspondence." If w does not divide p, this is the mod w local Langlands correspondence of Vigneras and Emerton, and if w divides p, then even though we do not know the mod p local Langlands, we have an expectation of the Serre weights appearing in it, which is the same as describing the $\mathrm{GL}_2(\mathcal{O}_{F_w})$**-socle of** $\pi_w^D(\overline{r})$. We use $D(\overline{r}|_{G_{F_w}})$ to denote the explicit set of expected Serre weights.

We however do not know much about $\pi_w^D(\overline{r})$. Is $\pi_w^D(\overline{r})$ finite length? It is finitely generated? Before we proceed, we fix some notations:

- fix $w \mid p$, and let $\overline{\rho} = \overline{r}|_{G_{F_w}}$,
- $G = \mathrm{GL}_2(F_w)$, $K = \mathrm{GL}_2(\mathcal{O}_{F_w})$, $K_1 = 1 + pM_2(\mathcal{O}_{F_w})$, I_1 be the upper-triangular Iwahori subgroup,
- $\Gamma = K/K_1$, $f = [F_w : \mathbb{Q}_p]$.

We summarize what we know about $\pi(\overline{\rho})$.

1. We know the weight part of BDJ conjecture, i.e. we know which Serre weights occur in the K-socle of $D(\overline{\rho})$ (Gee; Gee-Liu-Savitt).
2. We know the mod p multiplicity one result by Emerton-Gee-Savitt, i.e. each Serre weight occurs exactly once in $\mathrm{soc}_K\, \pi(\overline{\rho})$.

(3) We know $D_0(\overline{\rho}) := \pi(\overline{\rho})^{K_1}$ as Γ-representation. This is first done for semisimple case by Le-Morra-Schraen and Hu-Wang, and the general case is done by Le.

We are interested in $D_0(\overline{\rho})$ because prior to these works Breuil-Paskunas described K_1-invariant of conjectural mod p local Langlands correspondant, which turns out to be the largest Γ-representation such that the K-socle is the sum of expected Serre weights, with multiplicity one for each Serre weight.

Example 2.1. If $G = \mathrm{GL}_2(\mathbb{Q}_p)$, then if $\overline{\rho} = \begin{pmatrix} \omega^{r+1} & * \\ 0 & 1 \end{pmatrix}$ is a nonsplit extension, then we know (under genericity assumption) that $D(\overline{\rho}) = \{\mathrm{Sym}^r \mathbb{F}^2\}$. By definition, $\mathrm{soc}\, D_0(\overline{\rho}) = \bigoplus_{\sigma \in D(\overline{\rho})} \sigma$ implies that, by universal property, $D_0(\overline{\rho})$ should sit inside $\mathrm{Inj}_\Gamma \mathrm{Sym}^r \mathbb{F}^2$, the injective envelope of $\mathrm{Sym}^r \mathbb{F}^2$. This can be computed by the socle filtration:

$$\begin{tikzcd}
& \mathrm{Sym}^{p-1-r}\mathbb{F}^2 \otimes \det^r \arrow[dl, dash] \arrow[dr, dash] & \\
\mathrm{Sym}^r \mathbb{F}^2 \arrow[dr, dash] & \bigoplus & \mathrm{Sym}^r \mathbb{F}^2 \arrow[dl, dash] \\
& \mathrm{Sym}^{p-3-r}\mathbb{F}^2 \otimes \det^{r+1} &
\end{tikzcd}$$

Here, bars are nonsplit extensions. This diagram means that the socle filtration is consisted of three steps, where the socle is $\mathrm{Sym}^r \mathbb{F}^2$, the next graded piece is $\mathrm{Sym}^{p-1-r}\mathbb{F}^2 \otimes \det^r \oplus \mathrm{Sym}^{p-3-r}\mathbb{F}^2 \otimes \det^{r+1}$, and the last graded piece, which is the cosocle, is $\mathrm{Sym}^r \mathbb{F}^2$. By multiplicity one condition, $D_0(\overline{\rho})$ is the subrepresentation corresponding to the first two pieces of the filtration.

That this expectation actually coincides with $\pi(\overline{\rho})^{K_1}$ requires a lot of work, e.g. by using potentially Barsotti-Tate Galois deformation rings with fixed tame type (or even multiple of tame types).

Obviously $\pi(\overline{\rho})$ contains the G-representation generated by $D_0(\overline{\rho})$. But it turns out that this subrepresentation is everything for some cases.

Theorem 2.1 (Wang-Hu). *Suppose we have some global conditions on $\overline{\rho}$. Namely, assume that $\overline{\rho}$ is reducible non-split (so that $D_0(\overline{\rho}) = \{\sigma_0\}$), and assume that the **Gelfand-Kirillov dimension** (or **GK-dimension**) of $\pi(\overline{\rho})$ is f (=$\dim G/B$). Then,*

(1) $\pi(\overline{\rho})$ is the G-representation generated by $D_0(\overline{\rho})$,

(2) and when $f = 2$, $\pi(\overline{\rho})$ has length 3. To be more precise, if $\overline{\rho} = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$, then there is a filtration of $\pi(\overline{\rho})$ with graded pieces being

$$\pi_0 - \pi_1 - \pi_2,$$

where

- $\pi_0 = \mathrm{Ind}(\chi_2 \otimes \chi_1 \omega^{-1})$,
- π_1 *is supersingular,*
- $\pi_2 = \mathrm{Ind}(\chi_1 \otimes \chi_2 \omega^{-1})$.

Remark 2.1. (1) The GK-dimension of π, for a smooth admissible representation of K, is determined by the growth of $\dim \pi^{K_n}$, where K_n is the n-th principal congruence subgroup.

Namely, if $\dim \pi^{K_n} = \lambda p^{nc} + O(p^{n(c-1)})$ for $\lambda \neq 0$, then c is the GK-dimension of π. This is always well-defined, and is known to be an integer. For example, if π is a principal series, then we know that the GK-dimension is f. A supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ also has the GK-dimension 1.

The condition "GK-dimension = f" is coming from the works of Emerton and Gee-Newton.
(2) Finite generation does not necessarily imply finite length.
(3) It is hoped that the condition "GK-dimension = f" implies finite length property. The difficulty lies in giving a lower bound on the GK-dimension of supersingular representation.
(4) Breuil-Paskunas conjectured that
   - if $\overline{\rho}$ is irreducible, then $\pi(\overline{\rho})$ is irreducible,
   - if $\overline{\rho}$ is reducible non-split, then $\pi(\overline{\rho})$ has a filtration of length $f+1$ with graded piece

$$\pi_0 - \pi_1 - \cdots - \pi_f,$$

   where π_0, π_f are principal series, and all other graded pieces are supersingular.

We have several corollaries of the Theorem 2.1.

Corollary 2.1. *We keep the same condition as Theorem 2.1.*

(1) $\mathrm{End}_G(\pi(\overline{\rho})) = \mathbb{F}$*, which matches* $\mathrm{End}_{G_{F_w}}(\overline{\rho}) = \mathbb{F}$.
(2) We have the surjectivity of $R_\infty \longrightarrow \mathrm{End}_G(M_\infty)$*, where* M_∞ *is the universal patched module constructed in the 6-author papers, and* $R_\infty = R_{\overline{\rho}}^{\mathrm{univ}}[[x_1, \cdots, x_h]]$*. Note that the injectivity is already proved in general by Hellmann-Schraen, Emerton-Paskunas.*
(3) If $f = 2$*, then* $\mathrm{Ext}^1_G(\pi(\overline{\rho}), \pi(\overline{\rho})) \geq \mathrm{Ext}^1_{G_{F_w}}(\overline{\rho}, \overline{\rho})$.

Remark 2.2. Recall that Paskunas proved that in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$, given a mod p local Langlands correspondence $\overline{\rho} \leftrightarrow \pi(\overline{\rho})$, the "universal deformations" ρ^{univ} and π^{univ} of $\overline{\rho}$ and $\pi(\overline{\rho})$ correspond to each other via Colmez's functor. One can hope that the similar thing holds for $\mathrm{GL}_2(L)$, and therefore one hopes to have $\mathrm{Ext}^1_G(\pi(\overline{\rho}), \pi(\overline{\rho})) = \mathrm{Ext}^1_{G_{F_w}}(\overline{\rho}, \overline{\rho})$. Note that computing this is very difficult, as so far we do not know how to compute extensions between two supersingular representations.

3. **Proof of Theorem 2.1.**

(1) We first prove a preliminary result which does not depend on the condition of GK-dimension. Let $J \subset \mathbb{F}[[K_1]]$ be a maximal ideal, so that $\pi(\overline{\rho})^{K_n} = \pi(\overline{\rho})[J^{p^{n-1}}]$. Then one proves the following

Theorem 3.1 (Wang-Hu)**.** $\pi(\overline{\rho})[J^2]$ *is the largest* $\tilde{\Gamma}$*-representation, where* $\tilde{\Gamma} = \mathbb{F}[[K]]/J^2$*, satisfying that the* K*-socle is just* σ_0.

Proof of this uses the notion of **ordinary part of** $\pi(\overline{\rho})$ (Emerton) and a result of Le on $\pi(\overline{\rho})[J]$.
(2) Now the condition "GK-dimension = f" implies that, by a result of Gee-Newton, M_∞ is flat over R_∞. Thus, $\pi(\overline{\rho})^\vee$, which is the $\mathfrak{m}_\infty$-reduction of M_∞, is self-dual up to twist. This implies that the socle π_0 and the cosocle π_f of $\pi(\overline{\rho})$ are principal series.
(3) We use the following criterion for $W \subset \pi(\overline{\rho})$, a finite-dimensional sub-K-representation, to generate the whole $\pi(\overline{\rho})$:

Lemma 3.1. *If, for some i, the composition*

$$\mathrm{Ext}^i_K(\sigma_0, W) \xrightarrow{\beta_i} \mathrm{Ext}^i_K(\sigma_0, \pi(\overline{\rho})) \xrightarrow{\alpha_i} \mathrm{Ext}^i_K(\sigma_0, \pi_f),$$

is nonzero, then $\pi(\overline{\rho})$ is generated by W.

This is because the G-representation generated by W, if it is not the whole $\pi(\overline{\rho})$, must lie in $\ker \alpha_i$.

(4) One proves that if $i = 2f$, α_{2f} is an isomorphism, and by Theorem 3.1, which implies that β_1 is surjective, we know β_{2f} is surjective.

1. **Recipe for the set of Serre weights $W(\overline{\rho})$.**

Let p be an odd prime, and let $K/\mathbb{Q}_p$ be a finite extension with residue field k. To each $\overline{\rho} : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, one associates a set of Serre weights, i.e. irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k)$, called $W(\overline{\rho})$. There are many recipes and many known to be equivalent by work of many people.

We spell this out for $K = \mathbb{Q}_p$. In this case, Serre weights are

$$\sigma_{s,t} = \det{}^t \otimes \mathrm{Sym}^s \overline{\mathbb{F}}_p^2,$$

for $0 \leq s \leq p-1$, $t \in \mathbb{Z}$. Let $\omega = \overline{\varepsilon}|_{I_p}$.

(1) **The explicit description of** $W(\overline{\rho})$**.** (cf. Serre '87) If $\overline{\rho}$ is reducible, then $\sigma_{s,t} \in W(\overline{\rho})$ if and only if $\overline{\rho}|_{I_p} \cong \begin{pmatrix} \omega^{s+t+1} & * \\ 0 & \omega^t \end{pmatrix}$, with an exception below. In particular,

- If $* \neq 0$, then $W(\overline{\rho}) = \{\sigma_{s,t}\}$ for $s \neq 0, p-1$, except when $s = 0$ or $p-1$, $W(\overline{\rho}) = \{\sigma_{0,t}, \sigma_{p-1,t}\}$, **except** when $\overline{\rho}$ is **très ramifiée**. This is defined as follows: say
$$\overline{\rho} = \begin{pmatrix} \chi\overline{\varepsilon} & * \\ 0 & \chi \end{pmatrix},$$
then $\mathrm{Ext}^1(\chi, \chi\overline{\varepsilon}) \cong (\mathbb{Q}_p^\times)/(\mathbb{Q}_p^\times)^p$ by Kummer theory, which is 2-dimensional $\mathbb{F}_p$-vector space, and there is a canonical line, **peu ramifiée line** L, which is characterized by $L = \{x \mid p \mid \mathrm{val}_p\, x\}$. We say $\overline{\rho}$ is très ramifiée if the extension class does not belong to L. Only in this case the recipe is slightly different, namely $W(\overline{\rho}) = \{\sigma_{p-1,t}\}$.
- If $* = 0$, then not only that $\sigma_{s,t} \in W(\overline{\rho})$, $\sigma_{p-3-s,s+t+1} \in W(\overline{\rho})$ as one can exchange sub and quotient.

(2) **Crystalline lifts description of** $W(\overline{\rho})$**.** (cf. Gee '11) We say $\sigma_{s,t} \in W(\overline{\rho})$ if and only if $\overline{\rho}$ lifts to a crystalline representation $\rho : G_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ with Hodge-Tate weights $\{t, s+t+1\}$.

(3) **Breuil-Mézard conjecture description of** $W(\overline{\rho})$**.** In particular, this works for any K. Let $R_{\overline{\rho}}^{\mathrm{univ}}$ be the univesral (framed) deformation ring of $\overline{\rho}$. Let $\tau : I_K \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be an inertial type. There is a reduced p-torsion free quotient $R_{\overline{\rho}}^{\tau}$ of $R_{\overline{\rho}}^{\mathrm{univ}}$ which parametrizes potentially Barsotti-Tate representations of G_K of type τ.

Let $\sigma(\tau)$ be the representation of $\mathrm{GL}_2(\mathcal{O}_K)$ corresponding to τ under the **inertial local Langlands correspondence**.

Theorem 1.1 (Gee-Kisin, Emerton-Gee)**.** *For each Serre weight σ, there is a cycle $Z'(\sigma) \subset R_{\overline{\rho}}^{\mathrm{univ}}/p$ such that, for all inertial types τ,*
$$Z(R_{\overline{\rho}}^{\tau}/p) = \sum_{\sigma} n(\sigma, \tau) Z'(\sigma),$$
where $n(\sigma,\tau)$ is the multiplicty of σ in the mod p reduction of $\sigma(\tau)$.

Then, we define $W(\overline{\rho}) = \{\sigma \mid Z'(\sigma) \neq 0\}$.

We have the globalization of the Breuil-Mézard description.

Theorem 1.2 (Caraiani-Emerton-Gee-Savitt + Emerton-Gee)**.** *There is an algebraic stack Z of finite type over k, a finite field, and closed substacks Z^τ for each tame type τ, such that*

- *Z is reduced, has normal irreducible components, and is of equidimension $[K : \mathbb{Q}_p]$,*
- *for a finite extension $\mathbb{F}/k$, the $\mathbb{F}$-points of Z are exactly $\overline{\rho} : G_K \longrightarrow \mathrm{GL}_2(\mathbb{F})$,*

- *irreducible components of Z are in bijection with Serre weights, $Z(\sigma) \leftrightarrow \sigma$ such that,*
  - $\overline{\rho} \in Z(\sigma)(\mathbb{F})$ *if and only if* $\sigma \in W(\overline{\rho})$, *if* $\sigma \neq \chi \otimes \mathrm{St}$, *where* $\mathrm{St} = \mathrm{Sym}^{p-1}$,
  - $\overline{\rho} \in Z(\chi)(\mathbb{F})$ *or* $Z(\chi \otimes \mathrm{St})(\mathbb{F})$ *if and only if* $\chi \otimes \mathrm{St} \in W(\overline{\rho})$,
- *if $\sigma \neq \chi$, then $Z(\sigma)$ has a dense set of points $\overline{\rho}$ with $W(\overline{\rho}) = \{\sigma\}$,*
- $Z^\tau = \bigcup_{\sigma \in \mathrm{JH}(\overline{\sigma(\tau)})} Z(\sigma)$, *such that $\overline{\rho} \in Z^\tau$ then $R^\tau_{\overline{\rho}}/p$ is a versal ring to Z^τ at $\overline{\rho}$.*

Example 1.1. The component for the trivial Serre weight is generally consisted of Galois representations of form
$$\begin{pmatrix} \mathrm{un}_a \overline{\varepsilon} & * \\ 0 & \mathrm{un}_b \end{pmatrix}, a \neq b,$$
where un_a means the Frobenius is sent to a. On the other hand, the component for $\mathrm{Sym}^{p-1} \overline{\mathbb{F}}_p^2$ has points generically of form
$$\begin{pmatrix} \mathrm{un}_a \overline{\varepsilon} & * \\ 0 & \mathrm{un}_a \end{pmatrix}, * \text{ is très ramifiée.}$$
Then the two components meet at points where $*$ is peu ramifiée.

The dimension of the trivial Serre weight component is 2 (unramified characters) + 1 (extension class) - 1 (endomorphism of representation) - 1 (extension giving rise to same representation) = 1, and the dimension of the St Serre weight component is 1 + 2 - 1 - 1 = 1.

2. Building the stack.

How do we build this stack? A family of Galois representations cannot have the phenomenon of reducible representaitons specializing to an irreducible representation, so we need to use p-adic Hodge theory. For a $\mathbb{Z}/p^a\mathbb{Z}$-algebra A, let
$$\mathfrak{S}_A = (W(k) \otimes_{\mathbb{Z}_p} A)[[u]],$$
equipped with $\varphi : \mathfrak{S}_A \longrightarrow \mathfrak{S}_A$, which is Frobenius on $W(k)$, linear on A, and $u \mapsto u^p$.

Definition 2.1. *An* ***étale φ-module*** *with A-coefficients is a pair (M, φ) where M is a projective $\mathfrak{S}_A[1/u]$-module, $\varphi : M \longrightarrow M$ is a φ-semilinear map such that $\varphi^* M = \mathfrak{S}_A \otimes_\varphi M \xrightarrow{1 \otimes \varphi} M$ is an isomorphism.*

We choose a uniformizer $\pi \in \mathcal{O}_K$, and a sequence $(\pi^{1/p^n})_n$, and let $K_\infty = K(\pi^{1/p^n}, n \geq 1)$. Let $E(u)$ be the Eisenstein polynomial for π.

Theorem 2.1 (Fontaine)**.** *If A is a finite algebra, then there is an equivalence of categories*
$$\left\{ \begin{array}{c} \text{étale } \varphi\text{-modules} \\ \text{with } A\text{-coefficients} \end{array} \right\} = \left\{ A\text{-representations of } G_{K_\infty} \right\}.$$

Definition 2.2. *A* ***Breuil-Kisin module*** *of height $\leq h$ with A-coefficients is a $\mathfrak{S}_A$-lattice $\mathfrak{M}$, invariant under φ, in an étale φ-module M such that* $\mathrm{coker}(\mathfrak{S}_A \otimes_\varphi \mathfrak{M} \longrightarrow \mathfrak{M})$ *is killed by $E(u)^h$.*

Theorem 2.2 (Kisin)**.**
$$\left\{ \begin{array}{c} \text{Breuil-Kisin modules of height} \\ \leq 1 \text{ with } \mathbb{Z}/p\mathbb{Z}\text{-coefficients} \end{array} \right\} = \left\{ \begin{array}{c} \text{finite flat group} \\ p\text{-torsion group} \\ \text{schemes over } \mathcal{O}_K \end{array} \right\}.$$

On the other hand in general this only gives an information over K_∞, so one cannot distinguish powers of cyclotomic character. Thus one needs to develop a version with (tame) descent data, characterizing those being Barsotti-Tate after base-changing to K'/K (tamely ramified).

Definition 2.3. *Define the following étale stacks over $\mathbb{Z}/p\mathbb{Z}$,*

$$\mathscr{C}(A) = \left\{ \begin{array}{c} \textit{rank 2 Breuil-Kisin modules with} \\ \textit{A-coefficients with descent data } K'/K, \\ \textit{with height} \leq 1 \end{array} \right\},$$
$$\mathscr{R}(A) = \{\textit{rank 2 étale } \varphi\textit{-modules}\}.$$

There is a natural map

$$\mathscr{C} \longrightarrow \mathscr{R},$$

which is obtained just by inverting u.

Theorem 2.3 (Pappas-Rapoport)**.** *$\mathscr{C}$ is an algebraic stack of finite type over $\mathbb{Z}/p\mathbb{Z}$.*

Even though $\mathscr{R}$ is some horrible (ind-)stack, one has the following phenomenal

Theorem 2.4 (Emerton-Gee)**.** *The natural map $\mathscr{C} \longrightarrow \mathscr{R}$ has a "scheme-theoretic image" which is an algebraic stack of finite type over $\mathbb{Z}/p\mathbb{Z}$.*

Definition 2.4. *Let Z be the scheme-theoretic image of $\mathscr{C} \longrightarrow \mathscr{R}$.*

1. The Hodge-Tate period morphism.

Let (G, X) be a Shimura datum of PEL type, where G is a reductive group over $\mathbb{Q}$ and X is the set of $G(\mathbb{R})$-conjugacy classes of homomorphisms $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$, where $\mathbb{S}$ is the Deligne torus. Let $K \subset G(\mathbb{A}_f)$ be a neat level. Then, there is a Shimura variety X_K/E for G of level K, for the reflex field E, the field of definition of the conjugacy class of the Hodge cocharacter $\mu = (h \times_{\mathbb{R}} \mathbb{C})|_{\mathbb{G}_{m},\text{1st factor}} : \mathbb{G}_{m,\mathbb{C}} \longrightarrow G_{\mathbb{C}}$, which is a number field, in the sense that $X_K(\mathbb{C}) = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f))/K$. This gives the grading on $\mathrm{Rep}_{\mathbb{C}}\, G$, and we can define two different filtrations on $V_{\mathbb{C}} \in \mathrm{Rep}_{\mathbb{C}}\, G$.

- The **standard filtration** (or the **Hodge-de Rham filtration**) $\mathrm{Fil}^p_{\mathrm{std}}\, V_{\mathbb{C}} = \bigoplus_{p' \geq p} V_{\mathbb{C}}^{(p',q')}$,
- the **Hodge-Tate filtration** $\mathrm{Fil}^p\, V_{\mathbb{C}} = \bigoplus_{p' \leq p} V_{\mathbb{C}}^{(p',q')}$.

Both filtrations have the same graded pieces, but the flags are rather different; unlike the stabilizer of the standard filtration $P_{\mu}^{\mathrm{std}} \subset G_{\mathbb{C}}$, the stabilizer of the Hodge-Tate filtration $P_{\mu} \subset G_{\mathbb{C}}$, and the flag variety $\mathrm{Fl}_{G,\mu} := G_{\mathbb{C}}/P_{\mu}$, are defined over E, thus more apt for p-adic picture.

Mimicing the classical complex picture of Borel embedding

$$X \hookrightarrow \mathrm{Fl}^{\mathrm{std}}_{G,\mu}(\mathbb{C}),$$

which is $G(\mathbb{R})$-equivariant, holomorphic (but not algebraic), we have the p-adic analogue for p-adic analytic spaces $\mathcal{X}_K$, which are the adifications of the localization of X_K's to $E_{\mathfrak{p}}$ for a choice of $\mathfrak{p} \mid p$ of E (or its minimal/toroidal compactification).

Theorem 1.1 (Scholze, Pilloni-Stroh (on toroidal version), Caraiani-Scholze (on extending to PEL case))**.** *There exists a perfectoid space*

$$\mathcal{X}^{(*,\mathrm{tor})}_{K^p} \text{“}\sim\text{”} \varprojlim_{K_p \subset G(\mathbb{Q}_p)} \mathcal{X}^{(*,\mathrm{tor})}_{K^pK_p}.$$

On this level, there is the p-adic analogue of Borel embedding, ***the Hodge-Tate period morphism****,*

$$\pi_{\mathrm{HT}} : \mathcal{X}^*_{K^p} \longrightarrow \mathscr{F}\ell_{G,\mu},$$

which is $G(\mathbb{Q}_p)$-equivariant. There is also the toroidal version $\pi^{\mathrm{tor}}_{\mathrm{HT}} : \mathcal{X}^{\mathrm{tor}}_{K^p} \longrightarrow \mathscr{F}\ell_{G,\mu}$ which factors through π_{HT}.

Here $*$ is the minimal compactification, and this is analogue of the Borel embedding in the sense that, as the Borel embedding is about just remembering the Hodge filtration, on the level of $(C, \mathcal{O}_C)$-points, π_{HT} is

$$A/C \longmapsto \mathrm{Lie}\, A \subset T_pA \otimes_{\mathbb{Z}_p} C,$$

where C is a perfectoid field of characteristic 0.

Definition 1.1. *The* ***good reduction locus*** *$\mathcal{X}^{\circ}_{K^p} \subset \mathcal{X}_{K^p}$ is defined as the preimage of $\mathcal{X}^{\circ}_{K^pG(\mathbb{Z}_p)} \subset \mathcal{X}_{K^pG(\mathbb{Z}_p)}$, where $G(\mathbb{Z}_p)$ is a hyperspecial level at p, and $\mathcal{X}^{\circ}_{K^pG(\mathbb{Z}_p)}$ is the adic generic fiber of formal completion of integral model of $X_{K^pG(\mathbb{Z}_p)}$ (canonical integral model, which makes sense as it is of hyperspecial level for a PEL Shimura variety) along its special fiber.*

Theorem 1.2 (Caraiani-Scholze). *Let F/F^+ be an imaginary CM extension with $F^+ \neq \mathbb{Q}$. Let X_K be a (G) $\mathrm{U}(n,n)$-Shimura variety. Let $\mathfrak{m} \subset \mathbb{T}_K$ be a maximal ideal corresopnding to $\overline{\rho}_{\mathfrak{m}} : G_F \longrightarrow \mathrm{GL}_{2n}(\overline{\mathbb{F}}_{\ell})$, where $\mathbb{T}_K$ is the Hecke algebra acting on $H^*_{(c)}(X_K, \mathbb{F}_{\ell})$. Further assume the following.*

(1) There exists $p \neq \ell$, splitting completely in F, such that $\overline{\rho}_{\mathfrak{m}}|_{\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})}$ is "decomposed generic" at every $\mathfrak{p} \mid p$ (cf. Caraiani-Scholze on compact Shimura varieties; any lift corresponds to a generic principal series representation via local Langlands).

(2) $\mathfrak{m}$ is non-Eisenstein, i.e. $\overline{\rho}_{\mathfrak{m}} = \overline{\rho}_1 \oplus \overline{\rho}_2$, where $\overline{\rho}_1, \overline{\rho}_2$ are irreducible ant n-dimensional.

Then, $H^i_c(X_K, \mathbb{F}_\ell)_{\mathfrak{m}} = 0$ unless $i \leq \dim_{\mathbb{C}} X_K$, and $H^i(X_K, \mathbb{F}_\ell)_{\mathfrak{m}} = 0$ unless $i \geq \dim_{\mathbb{C}} X_K$.

We will study this via the geometry of π_{HT}. It will turn out that as π_{HT} is "affinoid", $R\pi_{\mathrm{HT}*}\mathbb{F}_\ell$ behaves like a perverse sheaf. We will not talk about the details of the proof, but it uses

- computation of cohomology of Igusa varieties due to Shin,
- Pink's formula,
- construction of torsion Galois representations to Hecke eigenclass of cohomology of locally symmetric spaces of GL_n (!!),
- $\cdots$.

The theorem is important for various applications, e.g.

- Mantovan's product formula,
- local-global compatibility of torsion classes,
- Fargues' geometrization conjecture,
- $\cdots$.

2. **The geometry of π_{HT}.**

Let $K = K^pK_p$ be of hyperspecial level for large enough p. Let $\overline{X}_K$ be the special fiber of the integral model over $\overline{\mathbb{F}}_p$. Then both special fiber of Shimura variety and the flag variety have **Newton stratifications**,

$$\overline{X}_K = \coprod_{b \in B(G,\mu^{-1})} \overline{X}^b_K,$$

$$\mathscr{F}\ell_{G,\mu} = \coprod_{b \in B(G,\mu^{-1})} \mathscr{F}\ell^b_{G,\mu},$$

where each stratum is locally closed, and π_{HT} respects the stratifications.

Remark 2.1. Note that closure relations are reversed.

Example 2.1. For the modular curve case, $G = \mathrm{GL}_{2,\mathbb{Q}}$, $\mathscr{F}\ell_{G,\mu} = \mathbb{P}^{1,\mathrm{ad}}$, and the newton stratifications are

$$\begin{tikzcd} \mathcal{X}^\circ_{K^p} \arrow[r, equal] \arrow[d, "\pi^\circ_{\mathrm{HT}}"'] & \mathcal{X}^{\circ,\mathrm{ord}}_{K^p} \arrow[d, "\pi^\circ_{\mathrm{HT}}"'] & \coprod & \mathcal{X}^{\circ,\mathrm{ss}}_{K^p} \arrow[d, "\pi^\circ_{\mathrm{HT}}"'] \\ \mathscr{F}\ell_{G,\mu} \arrow[r, equal] & \mathbb{P}^1(\mathbb{Q}_p) & \coprod & \Omega \end{tikzcd}$$

which is respected by π°_{HT} "on rank 1 points" (not literally respected, because closure relations are reversed).

The fibers of π°_{HT} over the ordinary locus are Igusa curves, while the fibers of π°_{HT} over the supersingular locus are profinite sets.

To study the geometry of PEL Shimura varieties in general, it is not enough to just study Newton strata. Instead, one studies **isomorphism classes** of p-divisible groups, which gives the notion of **Oort foliation**. Namely, given an isocrystal b, let $\mathbb{X}_b/\overline{\mathbb{F}}_p$ be a p-divisible group with extra structures corresponding to b. Then, the **Oort central leaf** corresponding to $\mathbb{X}_b$ is defined by

$$\mathcal{C}_{\mathbb{X}_b} := \{x \in \overline{X}_K^b \mid A^{\mathrm{univ}}[p^\infty]_x \cong \mathbb{X}_b \times_{\overline{\mathbb{F}}_p} \kappa(x)\} \subset \overline{X}_K^b.$$

It is a closed subset which becomes a smooth scheme when endowed with the reduced subscheme structure.

Let $\mathscr{G}$ be the restriction of $A^{\mathrm{univ}}[p^\infty]$ to $\mathcal{C}_{\mathbb{X}_b}$. When $\mathbb{X}_b$ is "completely slope divisible," i.e. if $\mathbb{X}_b$ is a direct sum of isoclinic pieces defined over finite fields, $\mathscr{G}$ admits a slope filtration

$$0 \subset \mathscr{G}_1 \subset \cdots \subset \mathscr{G}_r = \mathscr{G},$$

where $\mathscr{G}^i = \mathscr{G}_i/\mathscr{G}_{i-1}$ is isoclinic of slope λ_i.

Example 2.2. In the case of modular curves, where there is only one leaf inside each Newton stratum, on the ordinary locus the slope filtration is just the connected component $\mathscr{G}^0 \subset \mathscr{G}$.

Definition 2.1 (Mantovan)**.** *The **Igusa variety** $\mathcal{I}g^b$ is the pro-finite étale scheme over $\mathcal{C}_{\mathbb{X}_b}$ that parametrizes trivializations of graded pieces of slope filtration.*

Suppose that we have two non-isomorphic isogenous completely slope divisible p-divisible groups $\alpha : \mathbb{X}_b \longrightarrow \mathbb{X}_b'$. What are the relations between the two Igusa varieties $\mathcal{I}g^b$ and $(\mathcal{I}g^b)'$ corresponding to the leaves $\mathcal{C}_{\mathbb{X}_b}$ and $\mathcal{C}_{\mathbb{X}_b'}$? The key idea is that, if you take pullback through sufficiently high power (say $N \gg 0$) of Frobenius, then the slope filtration of $\mathbb{X}_b$ splits, which means that the trivialization of graded pieces of slope filtration gives the trivialization of $\mathbb{X}_b[p^N]$ (or something like this). Also, if $N \gg 0$, $\mathbb{X}_b[p^N]$ determines $\mathbb{X}_b$. Thus, $\ker \alpha$ becomes a global object over $(\mathcal{I}g^b)^{(p^{-N})}$, and you can quotient this out to get an actual map $(\mathcal{I}g^b)^{(p^{-N})} \longrightarrow \mathcal{C}_{\mathbb{X}_{b'}}$. Thus, this gives a finite-to-finite correspondence

$$\begin{tikzcd} & (\mathcal{I}g^b)^{(p^{-N})} \arrow[dl] \arrow[dr] & \\ \mathcal{C}_{\mathbb{X}_b} & & \mathcal{C}_{\mathbb{X}_b'} \end{tikzcd}$$

From this, one sees that the isomorphism class of the perfection $(\mathcal{I}g^b)^{\mathrm{perf}}$ does not depend on the choice of isomorphism class $\mathbb{X}_b$. Even for $\mathbb{X}_b$ that is not necessarily completely slope divisible, one can take a more naive version of Igusa variety, namely the moduli that parametrizes the trivialization of the whole $\mathbb{X}_b$, and the perfection still gives an isomorphic perfect scheme.

How do we relate this to the geometry of adic spaces? Note that, by Scholze-Weinstein, for C a perfectoid field of characteristic zero, giving $x \in \mathscr{F}\ell_{G,\mu}$ is the same as giving a p-divisible group $X/\mathcal{O}_C$ with $\alpha : T_pX \cong \mathbb{Z}_p^{2g}$ (plus some PEL extra structure). One can consider the canonical lift $\mathcal{I}g^b_{\mathcal{O}_C}$, a formal scheme over $\operatorname{Spf} \mathcal{O}_C$, of $(\mathcal{I}g^b)^{\mathrm{perf}}/\overline{\mathbb{F}}_p$ such that there is an abelian variety $\mathcal{A}_X$ over $\mathcal{I}g^b_{\mathcal{O}_C}$ which gives an isomorphism

$$\rho : \mathcal{A}_X[p^\infty] \cong X \times \mathcal{I}g^b_{\mathcal{O}_C}.$$

A construction of such $\mathcal{A}_X$ can be done by applying Witt vector construction.

Now, let $\mathcal{I}g^b_C$ be the adic generic fiber of $\mathcal{I}g^b_{\mathcal{O}_C}$, which is a perfectoid space over $\operatorname{Spa}(C, \mathcal{O}_C)$.

Theorem 2.1 (Caraiani-Scholze)**.** *(1) There is an isomorphism $\mathcal{I}g^b_C \cong (\pi^\circ_{\mathrm{HT}})^{-1}(x)$.*

(2) *There is a partial toroidal compactification* $Ig_C^{b,\mathrm{tor}}$ *of* Ig_C^b *towards only the boundary, and there is an isomorphism* $Ig_C^{b,\mathrm{tor}} \cong (\pi_{\mathrm{HT}}^{\mathrm{tor}})^{-1}(x)$.

(3) *There is an isomorphism* $Ig_C^{b,*} \cong \pi_{\mathrm{HT}}^{-1}(x)$ *of affinoid perfectoid spaces.*

Proof. For (1), one gets a map from Ig_C^b to the infinite-level Shimura variety by just using the universal abelian variety $\mathcal{A}_X/Ig_{\mathcal{O}_C}^b$ and taking the adic generic fiber. One then checks that the map factors through $(\pi_{\mathrm{HT}}^{\circ})^{-1}(x)$, and that this is a bijection on points.

For (2), one extends the works of Mantovan and Oort, and uses extended Serre-Tate theory for semiabelian schemes. You get (3) from (2) by just taking global sections. $\square$

1. **Statement and conjecture.**

Let $\mathscr{A}$ be the category of smooth $G = \mathrm{GL}_2(\mathbb{Q}_p)$-representations over $\overline{\mathbb{Z}}_p$ on which p acts locally nilpotently, with some fixed central character $\zeta : \mathbb{Q}_p^\times \longrightarrow \overline{\mathbb{Z}}_p^\times$. We choose to not work with admissible representations as we would like to have "family of representations" in the category.

Example 1.1. A typical example of smooth non-admissible representation is

$$\mathrm{cInd}_{KZ}^G \sigma,$$

for a Serre weight σ with central character $\overline{\zeta}|_{\mathbb{F}_p^\times}$, where $\overline{\zeta} : \mathbb{F}_p^\times \times p^{\mathbb{Z}} \longrightarrow \overline{\mathbb{F}}_p^\times$ comes from the identification $\mathbb{Q}_p^\times = \{\text{1-units}\} \times \mathbb{F}_p^\times \times p^{\mathbb{Z}}$, $K = \mathrm{GL}_2(\mathbb{Z}_p)$, Z is the center of $\mathrm{GL}_2(\mathbb{Q}_p)$ and σ as a representation of KZ is regarded to have central character ζ.

Indeed, the above example is some form of "family of admissible representations" over $\mathrm{Spec}\,\mathscr{H}(\sigma) = \mathrm{Spec}\,\overline{\mathbb{F}}_p[T]$, where each stalk is an admissible representation (at a nonzero point: Barthel-Livne, at zero: Breuil)

Ultimately we would like to understand something like the Bernstein center of $\mathscr{A}$. On the other hand, the Bernstein center in this setting will be an infinitesimal thickening of a projective scheme, which is different from the classical story where the Bernstein center was affine.

We consider a chain of $\frac{p\pm1}{2}$ $\mathbb{P}^1$'s, where adjacent $\mathbb{P}^1$'s are glued at a point, and ±1 is determined by $-\zeta(-1) = (\zeta\varepsilon)(-1)$. We justify the construction later.

Theorem 1.1 (Dotto-Emerton-Gee)**.** *For each open subset $U \subset Z$, there is a category $\mathscr{A}_U$, a certain "localization" of $\mathscr{A}$, such that $\{\mathscr{A}_U\}$ form a stack ("sheaf of categories") over Z.*

This is analogous to the classical story of Bernstein center. Recall that if the Bernstein center is just a commutative ring, then every object in the category has a canonical action by the ring, and by considering the object as a quasicoherent sheaf on the Spec of the ring, one can think of localization at prime ideals of the ring. Of course in the classical story there is no need to do this process, but as our Bernstein center is no longer "affine," we instead need this perspective.

Conjecture 1.1. *There is a formal thickening of Z, say $\widehat{Z}$, such that $\mathscr{A}$ is a sheaf of $\mathcal{O}_{\widehat{Z}}$-categories.*

Remark 1.1. We kind of know what $\widehat{Z}$ should be; closed points of Z are semisimple Galois representations

$$G_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p),$$

and the completion of $\widehat{Z}$ at each point should give the pseudo-deformation ring of the corresponding semisimple Galois representation.

2. **More about Z.**

By twisting, we can assume that

$$\overline{\zeta}|_{\mathbb{F}_p^\times} = \begin{cases} 1 & \overline{\zeta} \text{ even} \\ x \mapsto x^{-1} & \overline{\zeta} \text{ odd} \end{cases}$$

The various $\mathbb{P}^1$'s are labelled by pairs of Serre weights.

- Even case: $(\mathrm{Sym}^0, \mathrm{Sym}^{p-3} \otimes \det)$, $(\mathrm{Sym}^2 \otimes \det^{-1}, \mathrm{Sym}^{p-5} \otimes \det^3)$, $\cdots$, $(\mathrm{Sym}^{p-3} \otimes \det^{\frac{p+1}{2}}, \mathrm{Sym}^0 \otimes \det^{\frac{p-1}{2}})$.

- Odd case: $(\mathrm{Sym}^{p-2}, \text{“}\mathrm{Sym}^{-1}\text{”}), (\mathrm{Sym}^{p-4}\otimes\det, \mathrm{Sym}^1\otimes\det^{-1}), \cdots, (\mathrm{Sym}^1\otimes\det^{\frac{p-3}{2}}, \mathrm{Sym}^{p-4}\otimes\det^{\frac{p+1}{2}})$, $(\text{“}\mathrm{Sym}^{-1}\otimes\det^{\frac{p-1}{2}}\text{”}, \mathrm{Sym}^{p-2}\otimes\det^{\frac{p-1}{2}})$.

Note that reducible semisimple mod p Galois representation has two Serre weights, and fixing the weights, one has a freedom of choosing unramified character. Thus basically we can identify this with $\mathbb{G}_m$, and we manually put the two Serre weights to 0 and ∞ of $\mathbb{P}^1$. Then these Serre weights arise as another pair of Serre weight of irreducible residual Galois representation. Thus this explains the geometry of chain of $\mathbb{P}^1$'s, and that $Z(\overline{\mathbb{F}}_p)$ is in 1-1 correspondence with semisimple Galois representations of $G_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ with matching central character.

Remark 2.1. The only exception is when the Serre weight in consideration is Sym^{p-2}. Then a typical reducible semisimple Galois representation is of form

$$\begin{pmatrix} \mathrm{un}_\alpha & 0 \\ 0 & \mathrm{un}_{\alpha^{-1}} \end{pmatrix},$$

and this is parametrized by not $\alpha \neq 0$ but $\alpha + \alpha^{-1}$. Thus this can be regarded as $\mathbb{A}^1$ with with one marked irreducible point compactifying $\mathbb{A}^1$, which is also $\mathbb{P}^1$. This is why we paired twists of Sym^{p-2} with some nonexistent weights Sym^{-1} (as there is only one marked point).

Paskunas proved on the other hand that the locally admissible subcategory $\mathscr{A}^{\mathrm{loc.adm}} \subset \mathscr{A}$ decomposes into a product of blocks, labelled by semisimple $G_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$. Thus this is an instance of mod p local Langlands.

Definition 2.1. *If $Y \subset Z$ is closed, define*

$\mathscr{A}_Y := \{\pi \in \mathscr{A} \mid$ *each irreducible subquotient of π lives in one of the blocks labelled by $\overline{\mathbb{F}}_p$-points of Y*$\}$,

a full subcategory of $\mathscr{A}$.

Example 2.1. If Y is a single point $\{y\}$, then $\mathscr{A}_Y$ is the block of $\mathscr{A}^{\mathrm{loc.adm}}$ labelled by y.

Example 2.2. If Y is the component of Z with σ as one of its labels, then $\mathrm{cInd}_{KZ}^G \sigma \in \mathscr{A}_Y$.

Definition 2.2. *If $U \subset Z$ is open, then define $\mathscr{A}_U$ to be the Serre quotient $\mathscr{A}/\mathscr{A}_Y$ for $Y = Z - U$.*

From this definition, it is obvious that there is a transition map, so it gives rise to a prestack over Z.

Why does it give a stack over Z?

- We want to know that things can be glued. Note that there is a fairly general construction of abelian categories which gives the right adjoint, "$j_* : \mathscr{A}_U \longrightarrow \mathscr{A}$", to the restriction to U. This is basically the direct limit over all objects of $\mathscr{A}$ that restricts to the given object in $\mathscr{A}_U$.
- Now we want to control j_*. As $\mathscr{A}$ is "built out of" $\mathrm{cInd}_{KZ}^G \sigma$'s, we want to compute $j_* \mathrm{cInd}_{KZ}^G \sigma$.

  Theorem 2.1 (Key computation)**.**

  $$j_*\left((\mathrm{cInd}_{KZ}^G \sigma)|_U\right) = (\mathrm{cInd}_{KZ}^G \sigma)[1/g],$$

  where if you consider the preimage of U by the map $\mathrm{Spec}\,\mathscr{H}(\sigma) \longrightarrow Z(\sigma) \hookrightarrow Z$*, it is* $\mathrm{Spec}\,\mathscr{H}(\sigma)[1/g]$.

Note that the map $\operatorname{Spec} \mathscr{H}(\sigma) \longrightarrow Z(\sigma)$ is a nice map, either $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ or a quasi-finite generic 2-1 map (only in the case of Sym^{p-2}; $\alpha \mapsto \alpha + \alpha^{-1}$).

- To prove this Theorem, we only need to show that $(\mathrm{cInd}_{KZ}^G \sigma)[1/g]$ is in the image of j_* by Frobenius reciprocity. To show this, one has to show that

$$\mathrm{Ext}^i(\text{anything supported in } Y, \mathrm{cInd}_{KZ}^G \sigma[1/g]) = 0,$$

  for $i = 0, 1$. The most tricky computation is Ext^1. One knows easily that this is of countable dimension. On the other hand, as the cokernel of $T - \alpha$ on $\mathrm{cInd}_{KZ}^G \sigma[1/g]$ lies in a different block from any irreducible supported on Y from Paskunas' work, this induces an invertible map on

$$\mathrm{Ext}^1(\text{irreducible supported on } Y, \mathrm{cInd}_{KZ}^G \sigma[1/g]).$$

  Thus, the whole function field $\overline{\mathbb{F}}_p(T)$ acts on Ext^1, and any nonzero such module must be of uncountable dimension over $\overline{\mathbb{F}}_p$, which is a contradiction.

Remark 2.2. The conjectural $\widehat{Z}$ should be the "coarse moduli space" of the corresponding Emerton-Gee stack. Although the Emerton-Gee stack is not a Deligne-Mumford stack but an Artin stack, there is still a notion of associated moduli space. This is why the versal ring to the Emerton-Gee stack is the universal deformation ring whereas the versal ring to $\widehat{Z}$ is the universal pseudo-deformation ring.

Remark 2.3. One expects that the Bernstein center for $\mathscr{A}_U$ is $\mathcal{O}(\mathscr{A}_U)$.

Remark 2.4. The justification for $\mathbb{P}^1$ comes from $\mathbb{G}_m \backslash \left(\mathbb{A}^2 - \{(0,0)\}\right)$, where the horizontal axis parametrizes "α", the vertical lines parametrize the extension, and the vertical axis parametrizes irreducible representations (and the operation $\alpha \mapsto \alpha^{-1}$).

1. Mod p local-global compatibility for definite unitary groups.

Let $F_w/\mathbb{Q}_p$ be a finite extension, and let k_w be the residue field of $\mathcal{O}_{F_w}$. We are interested in developing the conjectural mod p local Langlands correspondence,

$$\left\{\begin{array}{c}\text{continuous}\\ \text{representation}\\ G_{F_w} \longrightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_p)\end{array}\right\} \text{ vs. } \left\{\begin{array}{c}\text{admissible}\\ \overline{\mathbb{F}}_p\text{-representations}\\ \text{of } \mathrm{GL}_n(F_w)\end{array}\right\}.$$

The only known case is when $n = 2$ and $F_w = \mathbb{Q}_p$, where there is a map satisfying local-global compatibility via completed cohomology of modular curves. We are interested in generalizing the situation to higher dimensional cases.

A particularly more accessible case is when the group involved is a definite unitary group, so that the global geometry is very simple. Let F/F^+ be a totally imaginary quadratic CM extension over a totally real field F^+, and let's assume that all places of F^+ over p split. FIx preferred places $w|v|p$ where v is a place of F^+ and w is a place of F (so that $F_w \cong F^+_v$). Let G be a definite unitary group over F^+, such that $G_F \cong \mathrm{GL}_{n,F}$ and $G(F^+_\infty)$ is compact. Because v splits in F, $G(F^+_v) \cong G(F_w) \cong \mathrm{GL}_n(F_w)$.

Definition 1.1. *Let $U \subset G(\mathbb{A}^\infty_{F^+})$ be a compact open subset. If $A = \overline{\mathbb{Q}}_p$ or $\overline{\mathbb{F}}_p$, we define the space of* ***automorphic forms on*** *G with level U and coefficients in A to be*

$$S(U, A) = \{f : G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U \longrightarrow A\}.$$

Remark 1.1.
1. We have no conditions on functions as the **double quotient is finite** exactly because G is compact at infinity.
2. Upon fixing an isomorphism $\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C}$, $S(U, \overline{\mathbb{Q}}_p)$ is identified with the space of classical automorphic forms on G with trivial infinity type, level U.
3. We know how to attach Galois representations $r_{\pi,\iota} : G_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ to these automorphic forms for which we know local-global compatibility.

Definition 1.2. *Fix a tame level $U^v \subset G(\mathbb{A}^{\infty,v}_{F^+})$, a compact open subset. The space of* ***mod*** *p* ***automorphic forms*** *on G with tame level U^v is defined by*

$$S(U^v) = \varinjlim_{U_v \subset G(F^+_v) \text{ compact open}} S(U^v U_v, \overline{\mathbb{F}}_p).$$

This has a natural action of $G(F^+_v) \cong \mathrm{GL}_n(F_w)$.

Now we have a candidate for a mod p local Langlands:

Definition 1.3. *Given a mod p Galois representation $\overline{r} : G_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, we define*

$$S(\overline{r}) = \varinjlim_{U_v \subset G(F^+_v) \text{ compact open}} S(U^v U_v, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}],$$

where $\mathfrak{m}_{\overline{r}} \subset \mathbb{T}$ is the maximal ideal corresponding to $\overline{r}$.

This $S(\overline{r})$ is an admissible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_n(F_w)$! The question is however,

does $S(\overline{r})$ depend only on $\overline{r}|_{G_{F_w}}$?

This is very difficult, as this is almost justifying the mod p local Langlands correspondence, which is of course far from being proved. Our goal is more modest, to find enough information inside $S(\overline{r})$ to determine $\overline{r}|_{G_{F_w}}$.

2. **Serre weights and semisimple Galois representations.**

Definition 2.1. *Given $\overline{r}$ as above, define*

$$W(\overline{r}) = \{\textit{Serre weights } V \mid V \hookrightarrow S(\overline{r})|_{\mathrm{GL}_n(\mathcal{O}_{F_w})}\},$$

the set of ***Serre weights of*** $\overline{r}$.

Remark 2.1. The set of Serre weights of $\overline{r}$, $W(\overline{r})$, is understood for $\mathrm{GL}_2(F_w)$ and $\mathrm{GL}_3(F_w)$, when F_w is unramified and $\overline{r}|_{G_{F_w}}$ is semisimple (Le-Le Hung-Levin-Morra).

In general, if $\overline{r}|_{G_{F_w}}$ is semisimple, then there is an expected set of Serre weights $W^?(\overline{r}|_{I_{F_w}})$ (Gee-Herzig-Savitt), which only depends on the restriction of $\overline{r}|_{G_{F_w}}$ to inertia. Rather than defining $W^?(\overline{r}|_{I_{F_w}})$, we do one example.

Example 2.1. Let $n = 3$ and $F_w = \mathbb{Q}_p$. Suppose that $\overline{r}|_{G_{F_w}}$ is irreducible. Then, $\overline{r}|_{I_{F_w}}$ is of form

$$\overline{r}|_{I_{F_w}} \sim \begin{pmatrix} \omega_3^m & & \\ & \omega_3^{pm} & \\ & & \omega_3^{p^2 m} \end{pmatrix},$$

where $\omega_3 : I_{F_w} \longrightarrow \overline{\mathbb{F}}_p^\times$ where $m \in \mathbb{Z}$ and $1+p+p^2$ does not divide m. Then, $\overline{r}|_{G_{F_w}} = \mathrm{Ind}_{G_{\mathbb{Q}_{p^3}}}^{G_{\mathbb{Q}_p}}(\omega_3^m \mathrm{un}_\lambda)$ for $\lambda \in \overline{\mathbb{F}}_p^\times$. Thus, to recover $\overline{r}|_{G_{F_w}}$, we need to know m and λ. Roughly in this case (when $\overline{r}|_{G_{F_w}}$ is "generic"), $\overline{r}|_{I_{F_w}}$ can be recovered from $W^?(\overline{r}|_{I_{F_w}})$. The remaining λ can be recovered as an eigenvalue of the Hecke operator with support $\mathrm{GL}_3(\mathcal{O}_{F_w})\begin{pmatrix} p & & \\ & p & \\ & & p \end{pmatrix}$, from the Hecke action $\mathcal{H}(V)$ actin on $\mathrm{Hom}_{\mathrm{GL}_3(\mathcal{O}_{F_w})}(V, S(\overline{r}))$, for $V \in W(\overline{r})$.

3. **Non-semisimple Galois representations.**

From now on we assume $n = 3$.

Example 3.1. Again, let $F_w = \mathbb{Q}_p$. In contrast to the semisimple case, Galois representations in question are of form

$$\overline{r}|_{G_{F_w}} \sim \begin{pmatrix} \omega^a \mathrm{un}_{\lambda_a} & * & * \\ 0 & \omega^b \mathrm{un}_{\lambda_b} & * \\ 0 & 0 & \omega^c \mathrm{un}_{\lambda_c} \end{pmatrix}.$$

We restrict to a case when it is in Fontaine-Laffaille range (and generic), namely suppose $a - b, b - c > 2$ and $a - c < p - 3$. Then, by normalizing the corresponding Fontaine-Laffaille module, Herzig-Le-Morra defined the **Fontaine-Laffaille invariants** $\mathrm{FL}(\overline{r}|_{G_{F_w}}) \in \mathbb{P}^1(\overline{\mathbb{F}}_p)$ which uniquely determines the extensions.

Works of Herzig-Le-Morra and Le-Le Hung-Levin-Morra show that there is a finite set $X \subset \mathbb{P}^1(\overline{\mathbb{F}}_p)$ containing 0 and ∞, such that if $\mathrm{FL}(\overline{r}|_{G_{F_w}}) \notin X$, then there is only one Serre weight $W(\overline{r}) = \{V_0\}$. Then a, b, c can be read off from the height weight of V_0, and $\lambda_a, \lambda_b, \lambda_c$ can be read off from Hecke eigenvalues as before.

Theorem 3.1 (Herzig-Le-Morra for $f = 1$, Enns for $f > 1$)**.** *Let $n = 3$, and $F_w/\mathbb{Q}_p$ be the unramified extension of degree f. Assume*

- *$\overline{r}|_{G_{F_w}}$ is upper triangular, Fontaine-Laffaille and sufficiently generic,*
- *$W(\overline{r})$ is a singleton, $\{V_0\}$,*

- $\overline{r}, G, F, U^v$ *satisfy a Taylor-Wiles assumption.*

Then, from $S(\overline{r})$*, one can recover* $\overline{r}|_{G_{F_w}}$ *by the following explicit recipe.*

(1) *Information for the semisimplification is as before (i.e. can be read off from* V_0*). Thus, we need to determine extensions.*
(2) *Choose "well" (i.e. there is an explicit recipe) a collection of* $3f-2$ *principal series inertial types* $\tau = I_{F_w} \longrightarrow T_3(\overline{\mathbb{Z}}_p)$ *(e.g.* $\tau = \widetilde{\omega}^x \oplus \widetilde{\omega}^y \oplus \widetilde{\omega}^z$*; tildes mean Teichmuller lifts).*
(3) *For each inertial type* τ*, define* $\mathrm{FL}_\tau \in \overline{\mathbb{F}}_p^\times$*, which determine the extension classes.*
(4) *For all* τ*, one has explicit* $S_\tau, S'_\tau \in \overline{\mathbb{F}}_p[\mathrm{GL}_3(k_w)]$*.*
(5) *Then, there is an* ***equality*** *of* ***injective*** *operators*
$$S'_\tau \circ \Pi = \mathrm{FL}_\tau\, S_\tau \;:\; S(\overline{r})^{\mathrm{Iw}=\overline{\tau}} \longrightarrow S(\overline{r})^{T_3(\mathcal{O}_{F_w})=\psi_{V_0}},$$
where $\mathrm{Iw} \subset \mathrm{GL}_3(\mathcal{O}_{F_w})$ *is the Iwahori subgroup,* $\overline{\tau} : \mathrm{Iw} \longrightarrow \overline{\mathbb{F}}_p^\times$ *via left action,* ψ_{V_0} *is the character of* $T_3(\mathcal{O}_{F_w})$ *coming from* V_0*, and* $\Pi = \begin{pmatrix} & 1 & \\ & & 1 \\ p & & \end{pmatrix}$ *which normalizes* Iw*. From these one can read* FL_τ*'s, thereby determining extension classes.*

Example 3.2. If $f = 1$, there is only one Fontaine-Laffaille invariant to recover. If say $\tau = \widetilde{\omega}^{-b+1} \oplus \widetilde{\omega}^{-c+1} \oplus \widetilde{\omega}^{-a+1}$, the recipe for S_τ for example is
$$S_\tau = \sum_{x,y,z \in k_w} x^{p-(a-c)} z^{p-(a-b)} \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

Idea of proof. We try to use characteristic 0 local-global compatiblity. The point is that if one restricts to $\mathrm{Iw} = \overline{\tau}$ and its natural lift $\mathrm{Iw} = \tau$, automorphic forms appearing have v-local part principal series, so one can study the actions of Teichmuller lifts of S_τ and S'_τ very explicitly. By using Le-Le Hung-Morra-Levin's explicit description of $R_{\overline{\rho}}^{(2,1,0),\tau}$, one shows the desired relation lifted to characteristic zero. We reduce mod p and here we need Taylor-Wiles patching. □

Remark 3.1. These are extremely complicated and there is no conceptual explanation for group algebra operators. There are many related works, e.g. Park-Qian on $\mathrm{GL}_n(\mathbb{Q}_p)$, but the exact relations are unclear.

We want to find analytic $\mathbb{Z}$-forms of relevant object of p-adic Hodge theory.

1. **Abstract analytic geometry.**

The basic idea is that, given a closed monoidal symmetric category C, we can "define" the category of "affine schemes over C" Aff_C as

$$\mathrm{Aff}_C = \mathrm{Comm}(C)^{\mathrm{op}},$$

where $\mathrm{Comm}(C)$ is the category of commutative algebras over C. If you put an appropriate Grothendieck topology on Aff_C, then one gets geometry.

Let K be a valued field, and consider the category of locally convex spaces over K, which is unfortunately not closed monoidal category. Instead, the category SN_K of seminorms of K is a closed monoidal category. It embeds into the opposite category of functors $\mathrm{Func}(\mathrm{SN}_K, \mathrm{Sets})^{\mathrm{op}}$ by the co-Yoneda lemma. Now the category of locally convex spaces of K is the same as $\mathrm{Pro}^{\mathrm{ess}}(\mathrm{SN}_K)$, the pro-$\mathrm{SN}_K$ where morphisms are essential epimorphisms. The problem of the category of locally convex spaces not being closed monoidal can be explained by the following phenomenon:

For a closed monoidal category C, $\mathrm{Pro}(C)$ is usually not closed monoidal.

However, the Ind-category $\mathrm{Ind}(C)$ is always closed monoidal. Thus, we would instead want to work with inductive systems.

Let Ban_R be the category of Banach spaces for a Banach ring R. Then, Ban_R is a closed monoidal category. By the Yoneda lemma, we can embed this into $\mathrm{Func}(\mathrm{Ban}_R^{\mathrm{op}}, \mathrm{Sets})$, and then we can make sense of Ind-category $\mathrm{Ind}(\mathrm{Ban}_R) \subset \mathrm{Func}(\mathrm{Ban}_R^{\mathrm{op}}, \mathrm{Sets})$ in this optic. Similarly, we can form $\mathrm{Ind}^{\mathrm{ess}}(\mathrm{Ban}_R)$, the category of ind-systems of objects in Ban_R where transition maps are essentially monomorphisms. This category is equivalent to **complete bounded modules**, or **bornological modules**, Born_R.

Remark 1.1.
- Born comes from "borné," which means bounded in French.
- If $R = K$, then this coincides with the classical definition.
- There is an adjunction

$$\mathrm{Born}_K \leftrightarrow \mathrm{lcv}_K,$$

which restricts to an equivalence of categories on a subcategory containing "all reasonable objects."

Proposition 1.1.
- Born_R *is a closed monoidal category.*
- Born_R *is a "quasi-abelian category"; it is a derived equivalent to an abelian category.*
- Born_R *has all limits and colimits.*

Definition 1.1. *Define the category of* ***affine analytic spaces*** *over R,* Aff_R*, to be* $\mathrm{Comm}(\mathrm{Born}_R)^{\mathrm{op}}$.

There is a notion of base change for a map between bornological rings.

2. **Basic examples.**

The ring of integers $\mathbb{Z}$ with its Euclidean norm $|-|_\infty$ is initial in the category of Banach rings. Thus, it is initial in $\mathrm{Born}_{\mathbb{Z}}$.

Definition 2.1. *Let R be a Banach ring (or a bornological ring). Then the* ***Berkovich spectrum*** $M(R)$ *is defined by*

$$M(R) = \{ |-| : R \longrightarrow \mathbb{R}_{\geq 0} \text{ bounded multiplicative seminorms}\},$$

with the weakest topology such that, for all $r \in R$, $|-| \mapsto |r|$ is continuous.

The Berkovich spectrum $X = M(\mathbb{Z}, |-|_\infty)$ is a tree branched out of the trivial seminorm $|-|_0$, where there is one line segment for each valuation v of $\mathbb{Z}$, where

- if v is a finite prime, then the line segment corresponds to $|-|_v^\varepsilon$, $0 \leq \varepsilon \leq \infty$, with another endpoint being the valuation v,
- and if $v = \infty$, then the line segment corresponds to $|-|_\infty^\varepsilon$, $0 \leq \varepsilon \leq 1$, with another endpoint being the Euclidean norm $|-|_\infty$.

The topology is very coarse. In particular, any open set containing $|-|_0$ is a union of almost every branches.

Let $U = X - \{|-|_0\}$. Then, $\mathcal{O}_U(U) = \mathbb{R} \times \prod_p \mathbb{Z}_p$. For the open embedding $j : U \hookrightarrow X$, one has

$$(j_*\mathcal{O}_U)|_{|-|_0} = \mathbb{A}_{\mathbb{Q}}.$$

Consider the power series ring

$$T_{\rho,\mathbb{Z}} = \{\sum_{i\in\mathbb{N}} a_i t^i \in \mathbb{Z}[[t]] \mid \sum_i |a_i|_\infty \rho^i < \infty\},$$

which we call a **Tate algebra over** $\mathbb{Z}$.

Remark 2.1. Note that $T_{1,\mathbb{Z}}\widehat{\otimes}_{\mathbb{Z}}\mathbb{Q}_p$ is not a Tate algebra in the usual sense.

Thus we define the following.

Definition 2.2. *Let $T_{\mathbb{Z}}^\circ = \varprojlim_{\rho<1} T_{\rho,\mathbb{Z}}$, which we call* ***analytic fundamental open disc****.*
Similarly, let $T_{\mathbb{Z}}^T = \varinjlim_{\rho>1} T_{\rho,\mathbb{Z}}$.

This is a more sensible notion, as $T_{\mathbb{Z}}^\circ\widehat{\otimes}\mathbb{Q}_p$ and $T_{\mathbb{Z}}^\circ\widehat{\otimes}\mathbb{C}$ are usual open discs.

Definition 2.3. *Let* $\mathrm{Int}(\mathbb{Z})$ *be the ring of integrally valued polynomials, i.e.* $\{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subset \mathbb{Z}\}$.

For example, $\binom{x}{n}$ is in $\mathrm{Int}(\mathbb{Z})$ that is not in $\mathbb{Z}[X]$.

Proposition 2.1.

$$\left\{\binom{x}{n}\right\}_{n\in\mathbb{N}},$$

forms a $\mathbb{Z}$-basis for $\mathrm{Int}(\mathbb{Z})$.

On the other hand, the multiplication is quite complicated,

$$\binom{x}{i}\binom{x}{j} = \sum_{k\leq \min\{i,j\}} \binom{i+j-k}{i-k, j-k, k}\binom{x}{i+j-k}.$$

In particular, the coefficients can have big $|-|_\infty$ values.

Definition 2.4. *Let* $\mathrm{Int}(\mathbb{Z}, |-|_0) = \{a_n\binom{x}{n} \mid |a_n|_0 \longrightarrow 0\}$.

Proposition 2.2.

- $(\mathrm{Int}(\mathbb{Z}, |-|_0), |-|_{1,\max})\widehat{\otimes}_{\mathbb{Z}}\mathbb{Q}_p = C^0(\mathbb{Z}_p)$*, via Mahler expansion.*
- $\varprojlim_{\rho>1}(\mathrm{Int}(\mathbb{Z}), |-|_{\rho,\max})\widehat{\otimes}_{\mathbb{Z}}\mathbb{Q}_p = \mathrm{LA}(\mathbb{Z}_p)$*. We thus define* $\mathrm{LA}(\mathbb{Z}) := \varprojlim_{\rho>1}(\mathrm{Int}(\mathbb{Z}, |-|_0), |-|_{\rho,\max})$.

Remark 2.2. Locally analytic functions over whole affine line (as opposed to unit disc) can be captured by the following.

$$\left(\varprojlim_{\rho\to\infty}\left(\mathrm{Int}(\mathbb{Z},|-|_\infty),|-|_\rho\right)\right)\widehat{\otimes}_{\mathbb{Z}}\mathbb{Q}_p \Rightarrow \mathcal{O}^{\mathrm{an}}(\mathbb{A}^1_{\mathbb{Q}}),$$

$$\left(\varprojlim_{\rho\to\infty}\left(\mathrm{Int}(\mathbb{Z},|-|_\infty),|-|_\rho\right)\right)\widehat{\otimes}_{\mathbb{Z}}\mathbb{C} \Rightarrow \mathcal{O}^{\mathrm{an}}(\mathbb{A}^1_{\mathbb{C}}).$$

3. **Non-basic examples.**

Now we take the **Robba ring over** $\mathbb{Z}$, $R_{\mathbb{Z}}$,

$$R_{\mathbb{Z}} = \{\sum_{i\in\mathbb{Z}} a_i t^i \mid \sum_{i\in\mathbb{Z}} |a_i|\rho^i < \infty \text{ for all } \rho < 1,$$
$$\text{and there exists } \rho > 1 \text{ such that } \sum_{i\in\mathbb{N}} |a_i|\rho^i < \infty\}.$$

For an analytic group scheme G over $\mathbb{Z}$, we have an action $G \times \mathrm{Spec}(R_{\mathbb{Z}}) \longrightarrow \mathrm{Spec}(R_{\mathbb{Z}})$ which is given by the coaction map $\psi : R_{\mathbb{Z}} \longrightarrow R_{\mathbb{Z}}\widehat{\otimes}_{\mathbb{Z}}\mathcal{O}(G)$, defined by

$$\psi(t) = (1+t)^\lambda - 1 = \sum_{n>0} t^n \binom{\lambda}{n}.$$

Now you pick G such that $\mathcal{O}(G) = \widehat{\mathrm{Int}(\mathbb{Z})} = \mathrm{LA}(\mathbb{Z})$, whose comultiplication map is given by

$$\Delta_*\left(\binom{x}{n}\right) = \binom{xy}{n}.$$

Proposition 3.1. *The map ψ is well-defined. If you base change to $\mathbb{Q}_p$, you get the usual coaction map.*

1. Unlikely intersections.

Example 1.1. What are the examples of $f \in \mathbb{C}[x, y]$, such that $f(\zeta, \zeta') = 0$ for infinitely many roots of 1 ζ, ζ'? For example one can try to use $f(x, y) = x^a - y^b$, or $x^a y^b - 1$.

Theorem 1.1 (Lang, Ihara, Tate)**.** *If $f(1, 1) = 0$ and if f is irreducible monic, the above examples are the only possible examples.*

The above situation generalizes to the following.

Definition 1.1. *Consider $A = \mathbb{G}_m^n$, or more generally an abelian variety. Then,* ***special points*** *are torsion points. A* ***special subvariety*** *is a translate of subtorus/abelian sub-variety by a torsion point.*

Then the above example can be thought as a special case of the following

Theorem 1.2 (Manin-Mumford; Raynaud)**.** *An irreducible component of the Zariski closure of a set of special points is special.*

Theorem 1.3 (Strengthening of Manin-Mumford; Laurent, Hindry, Faltings, McQuillen, Vojta, $\cdots$)**.** *Let G be a semi-abelian variety, and $\Gamma \subset G(\mathbb{C})$ be a finitely gneerated subgroup. Consider the divisible hull of Γ,*

$$\Gamma^{\text{div}} = \{g \in G(\mathbb{C}) \mid g^k \in \Gamma \textit{ for some } k > 0\}.$$

Then, $X \cap \Gamma^{\text{div}}$ is ***finite****, unless X contains a translate of a positive dimensional subgroup.*

Why is this useful?

Corollary 1.1 (Mordell conjecture)**.** *For a curve C over a number field K of genus $g \geq 2$, $|C(K)| < \infty$.*

Proof. Suppose $|C(K)| \neq 0$. Then we choose a rational point and get an embedding $i : C \hookrightarrow \mathrm{Jac}(C)$ into a g-dimensional abelian variety. As $\Gamma = \mathrm{Jac}(C)(K)$ is a finitely generated abelian group, $|i(C) \cap \Gamma| < \infty$, unless $i(C)$ contains a translate of a positive dimensional subtorus. Thus C has to contain an elliptic curve which is impossible because of genus ≥ 2 condition. □

2. p-adic unlikely intersections on tori.

We want to study if we can say anything about finitely generated torsion $\mathbb{Z}_p[[T_1, \cdots, T_n]] \cong \mathbb{Z}_p[[\mathbb{Z}_p^n]]$-modules in a similar vein. But one cannot just easily do this, as first of all we cannot evaluate at roots of unity because of convergence issue. But the following variant holds.

Proposition 2.1. *Let $\Phi \in \mathcal{O}_E[[X, Y]]$ where $E/\mathbb{Q}_p$ is finite. Suppose Φ is irreducible and $\Phi(0, 0) = 0$. If $\Phi(\zeta - 1, \zeta' - 1) = 0$ for infinitely many p-power roots of unity ζ, ζ', then $\Phi(X, Y) = (X+1)^A - (Y+1)$ for $A \in \mathbb{Z}_p$, up to multiplication by units and swapping X, Y.*

Theorem 2.1 (Serban)**.** *Let $S = (\widehat{\mathbb{G}}_m)_m^{n,\text{tor}}/\mathcal{O}_E \subset \mathbb{B}^n_{\mathbb{C}_p}(0)$. An irreducible component of the Zariski closure of a subset of S is* ***special****, namely it is the translate of a formal subtorus by a torsion point in S.*

Remark 2.1.
1. There is Tate-Voloch conjecture on stronger result, which needs distances between points.
2. This generalizes to the case where one uses a Lubin-Tate formal group instead of $\widehat{\mathbb{G}}_m$.

Proof sketch. Let $\Sigma_\Phi = \{\vec{x} \in \mathfrak{m}_{\mathbb{C}_p}^n \mid \Phi(\vec{x}) = 0\}$. Then,

$$\min_{1\le i<n}\left(\frac{v_p(x_i)}{v_p(x_{i+1})}, \frac{1}{v_p(x_n)}\right) \le C_\Phi,$$

on Σ_Φ. We then parametrize $\Sigma_\Phi \cap S$ and use compactness argument to get a finite cover by sets S_{τ_j} that after twist by $\tau_j \in \mathrm{End}(\widehat{\mathbb{G}}_m^n)$, $v_p(x_i)/v_p(x_{i+1}) \longrightarrow \infty$ is unbounded on S_{τ_j}. From this and the above inequality, we see that $\mathrm{proj}_n(S_{\tau_j})$ is a finite set, and we induct on this. ☐

3. **Applications and Generalizations.**

We mention two applications of Theorem 2.1.

(1) Let K be a number field. Let $\mathscr{H}$ be a (nearly) ordinary family of cuspidal p-adic automorphic forms over $\mathrm{GL}_{2,K}$, over the weight space $\mathcal{W}^0 = \mathrm{Spec}(\mathbb{Z}_p[[\mathbb{Z}_p^n]])$ (i.e. Hida family). In general, if K is not totally real, then Hida conjectured that $\mathscr{H}$ is torsion over $\mathcal{W}^0$ of codimension r_2. Now "classical points" form a proper subset of

$$S = \{(1+p)^{\vec{k}}\vec{\zeta} - 1 \mid \vec{k} \in \mathbb{Z}^n, \vec{\zeta} \in \mu_{p^\infty}\}.$$

We then study the density of classical points on $\mathscr{H}$. If we have such density, then $\mathscr{H}$ must be "very special"; in general, density of classical points on $\mathscr{H}$ is expected to be a very rare phenomenon. For example, if K is imaginary quadratic field (so that codimension is 1), then we can prove finiteness in this way.

(2) Now for a 2-variable p-adic L-function $L_p(x, y)$ interpolating $L^{\mathrm{alg}}(f_k, \chi, s)$ for $1 \le s \le k-1$, there is a conjecture of Greenberg.

Conjecture 3.1 (Greenberg)**.** *The vanishing along the line $s = k/2$ should be of order 0 or 1 according to the sign of the functional equation.*

From this, one expects that there are finite amount of vanishing not explained by the functional equation. From a joint work of Rob Pollack, by using explicit form of 2-variable p-adic L-function, one can check that the expectation holds for some specific elliptic curves.

We finish the talk with a generalization of Theorem 2.1 to abelian varieties (or more precisely analytifications of abelian varieties).

Theorem 3.1 (Serban)**.** *Let $F = \mathrm{Spf}(\mathcal{O}_E[[x_1, \cdots, x_n]])$ be a p-adic formal Lie group of dimension n. Let $X \subset F$ be a closed formal subscheme. Then either of the following holds.*

(1) There is $\varepsilon > 0$ such that only finitely many torsion points $Q \in F[p^\infty]$ satisfy $d_p(Q, \chi) < \varepsilon$ for some good notion of distance.

(2) There are infinitely many torsion points on χ and χ contains the translate of a formal subgroup of F.

This theorem is useful in studying p-adic dynamical systems.

1. **Background.**

Let $F/\mathbb{Q}_p$ be a finite extension, $G = \mathrm{GL}_2(F)$ and D/F be the quaternion division algebra. The **Drinfeld upper half plane** is defined by

$$\Omega := \mathbb{P}^{1,\mathrm{an}} - \mathbb{P}^1(F),$$

so that $\Omega(\mathbb{C}_p) = \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(F)$. It is not a simply connected space, and Drinfeld defined a tower

$$\cdots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M_0 = \Omega \times \mathbb{Z},$$

which is a tower of rigid spaces, and there is a G-action on the whole tower, such that

(1) the G-action on M_0 lifts to M_n,
(2) $M_n \longrightarrow M_0$ is a G-equivariant map for all $n \geq 0$,
(3) and $\mathrm{Gal}(M_n/M_0) \cong \mathcal{O}_D^\times/(1+\pi^n\mathcal{O}_D)$.

Theorem 1.1 (Harris-Taylor, Carayol, Faltings, Mieda). *For $\ell \neq p$, the ℓ-adic étale cohomology of Drinfeld tower, $M_\infty = \varprojlim_n M_n$, realizes the Jacquet-Langlands and local Langlands correspondence. Namely, there is a $D^\times \times G \times W_F$-equivariant isomorphism*

$$\varinjlim_n H^1_{\text{ét},c}(\Sigma_{n,\overline{F}}, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_{\rho\in\mathrm{Irr}(D^\times)} \rho \otimes \mathrm{JL}(\rho)^\vee \otimes \mathrm{rec}^{-1}(\mathrm{JL}(\rho)^\vee),$$

where $\Sigma_n = M_n/\begin{pmatrix} p & 0 \\ 0 & p\end{pmatrix}^{\mathbb{Z}}$.

What if $\ell = p$? There is a recent work of Colmez, Dospinescu and Niziol on p-adic (pro-)étale cohomology of the tower and they relate to the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. On the other hand, there is an expectation (Schneider-Teitelbaum) that **coherent cohomology** of the tower produces **admissible locally analytic representations** of G.

Conjecture 1.1 (Schneider-Teitelbaum). *For $n \geq 0$, $\mathcal{O}(\Sigma_n)^*$ is an admissible locally analytic representation of G. Furthermore, $\mathcal{O}(\Sigma_n)^*$ has finite length for all $n \geq 0$.*

There are some evidences:

(1) (Dospinescu-Le Bras, 2017) $\mathcal{O}(\Sigma_n)^*$ is an admissible locally analytic representation for $F = \mathbb{Q}_p$, $n \geq 0$.
(2) (Patel-Schmidt-Strauch, 2017) $\mathcal{O}(\Sigma_1)^*$ is an admissible locally analytic representation for any F.

Theorem 1.2 (Ardakov-Wadsley). *Let $\mathscr{L}$ be a G-equivariant line bundle on Ω with a G-equivariant connection. Suppose there is an isomorphism $\psi : \mathscr{L}^{\otimes d} \xrightarrow{\sim} (\mathcal{O}_\Omega, \nabla_{\mathrm{triv}})$ for minimal such $d \geq 1$ which satisfies $p \nmid d$. Then,*

(1) $\mathscr{L}(\Omega)$ is a coadmissible $\mathcal{D}(G)$-module,
(2) and if $d > 1$, then $\mathscr{L}(\Omega)$ is irreducible.

Remark 1.1. This implies that $\mathcal{O}(\Sigma_1)^*$ is finite length (a part of Conjecture 1.1) as the pushforward of $\mathcal{O}(\Sigma_1)^*$ to the base decomposes into a sum of line bundles via Kummer theory, as the Galois group of the first covering is abelian. To go up further the tower, we have to consider vector bundles with connection and the corresponding D-module theory will be considerably more involved.

2. **Algebraic situtation.**

Let $V \subset \mathbb{P}^1_{\mathbb{C}}$ be a Zariski open set. Let $\pi : V' \longrightarrow V$ be some finite étale covering.

Proposition 2.1. *As a $U(\mathfrak{sl}_2)$-module, $\mathcal{O}(V')$ is finitely generated.*

Proof. Let $j : V \hookrightarrow \mathbb{P}^1$. Note that $\mathcal{O}_{V'}$ is a holonomic $\mathcal{D}_V$-module. By Bernstein's theorem, there is a six-functor formalism for regular holonomic D-modules. Thus, $\pi_+\mathcal{O}_{V'}$ is a holonomic $\mathcal{D}_V$-module. As π is finite étale, this is $\pi_*\mathcal{O}_{V'}$ with Gauss-Manin connection (i.e. the pushforward comes from local system again). Applying Bernstein's theorem again, $j_+(\pi_*\mathcal{O}_{V'})$ is a holonomic $\mathcal{D}_{\mathbb{P}^1}$-module. As j is affine, this pushforward is also equal to $j_*(\pi_*\mathcal{O}_{V'})$. By Beilinson-Bernstein localization (in the setting of D-modules, it is due to Brylinski-Kashiwara), we know that

$$\{\text{coherent } \mathcal{D}_{\mathbb{P}^1}\text{-modules}\} \xrightarrow{\sim} \{\text{coherent } U(\mathfrak{sl}_2)\text{-modules with } (\ker\chi_0)M = 0\},$$

$$M \longmapsto \Gamma(\mathbb{P}^1, M),$$

and holonomic D-modules are automatically coherent, we see hat $\Gamma(\mathbb{P}^1, j_*(\pi_*\mathcal{O}_{V'})) = \mathcal{O}(V')$ is a finitely generated $U(\mathfrak{sl}_2)$-module. □

This is the core of the proof which will apply to the proof of Theorem 1.2.

3. **Rigid analytic situation.**

In our rigid analytic situation, we consider the following analogous situation: for $u \in \kappa(x)^\times$, $d \geq 1$,

$$Z(u,d) := \operatorname{Spec}\left(\frac{K[x][\frac{1}{x-a_i} : i = 1, \cdots, h][z]}{(z^d - u)}\right)^{\mathrm{an}} \xrightarrow{\pi} \mathbb{A}^{1,\mathrm{an}} - \zeta(u),$$

where $\zeta(u) := u^{-1}(\{0,\infty\}) = \{a_1, \cdots, a_h\}$, and $j : \mathbb{A}^{1,\mathrm{an}} - \zeta(u) \hookrightarrow \mathbb{A}^{1,\mathrm{an}}$. In particular,

$$\pi_*\mathcal{O}_{Z(u,d)} = \bigoplus_{m=0}^{d-1} \mathcal{O}_{\mathbb{A}^{1,\mathrm{an}} - \zeta(u)} z^m.$$

Definition 3.1. *Let $U = U(\zeta(u), t) := \{a \in \mathbb{A}^{1,\mathrm{an}} \mid |a - s| \geq |t| \text{ for all } s \in \zeta(u)\}$.*

Definition 3.2. *Let $\mathcal{M}(u,d) := j_*(\mathcal{O}_{\mathbb{A}^{1,\mathrm{an}} - \zeta(u)} z)$.*

This is a little nasty object, for example it attains an essential singularity around missing points. However, it is not so crazy:

Theorem 3.1 (Bode). *The object $\mathcal{M}(u,d)$ is a coadmissible $\widehat{\mathcal{D}}_{\mathbb{A}^{1,\mathrm{an}}}$-module.*

But for our purpose $\widehat{\mathcal{D}}$-module is not enough.

Definition 3.3. *For an abelian sheaf $\mathcal{F}$ over $\mathbb{A}^{1,\mathrm{an}}$, define*

$$\mathcal{F}_{\overline{U}} := i_* i^{-1} \mathcal{F}^{\mathrm{ad}},$$

where $i : \overline{U^{\mathrm{ad}}} \hookrightarrow \mathbb{A}^{1,\mathrm{ad}}$ which happens on the level of adic spaces.

A classical-minded person can think of this as the **sheaf of overconvergent sections** into U.

Proposition 3.1. *We have $\mathcal{F}_{\overline{U}}(X) = \operatorname{colim}_{|s|<|t|} \mathcal{F}(X \cap U(\zeta(u), s))$.*

Theorem 3.2. *Suppose $p \nmid d$ and $|t| \leq \min_{i \neq j} |a_i - a_j|$. Let*

$$R(u, d) := \prod_{i=1}^{h} (x - a_i) \left(\partial_x - \frac{1}{d} \operatorname{dlog}(u) \right) \in K[x, \partial_x].$$

Let $\mathcal{D}_t^\dagger := \operatorname{colim}_{|r|>|t|} \mathcal{D}_r$, where $\mathcal{D}_r(X) = \mathcal{O}(X)\langle \partial_x / r \rangle$. Let

$$\mathbb{A}_{\geq t} = \left\{ \textit{affinoid subdomains } X \subset \mathbb{A}^{1,\mathrm{an}} \mid |t| \geq |\partial_x|_{\mathrm{sp}, \mathcal{O}(X)} \right\}.$$

Then,

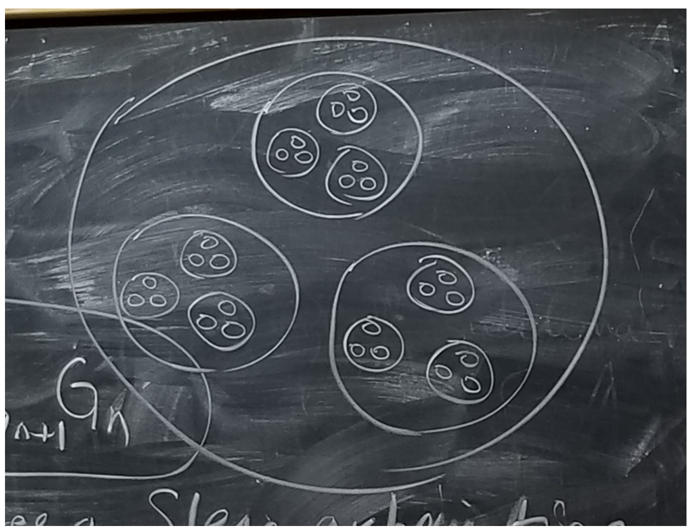
$$\mathcal{M}(u, d)_{\overline{U(\zeta(u), t)}} \cong \mathcal{D}_t^\dagger / \mathcal{D}_t^\dagger R(u, d),$$

as sheaves of $\mathcal{D}_t^\dagger$-modules on $\mathbb{A}_{\geq t}$.

Proof of Theorem 1.2(1). We want to show that $j_* \mathscr{L}$ is a coadmissible $\mathcal{D}(\mathbb{P}^1/G)$-module. It suffices to show locally, namely that $(j_*\mathscr{L})_{\mathbb{D}}$ is a coadmissible $\mathcal{D}(\mathbb{D}/G_1)$-module, where $\mathbb{D} = \operatorname{Sp} K\langle x \rangle$ and G_1 is the first congruence subgroup. It then suffices to prove that $\mathscr{L}(\mathbb{D} \cap \Omega)$ is a $\widehat{\mathcal{D}}(\mathbb{D}, G_1)$-module. Now we have

$$\widehat{\mathcal{D}}(\mathbb{D}, G_1) \cong \varprojlim \mathcal{D}_{\pi^n}^\dagger(\mathbb{D}) \rtimes_{G_{n+1}} G_n.$$

The set $\mathbb{D} \cap \Omega$ is a Cantor-like set which has a Stein exhaustion, so $\mathbb{D} \cap \Omega = \cup_{n \geq 0} U_n$,



so $\mathscr{L}(\mathbb{D} \cap \Omega) = \varprojlim \mathscr{L}^\dagger(U_n)$. Now Theorem 3.2 implies that

$$\mathscr{L}^\dagger(U_n) \cong \mathcal{D}_{\pi^n}^\dagger(\mathbb{D}) / \mathcal{D}_{\pi^n}^\dagger(\mathbb{D}) R(u_n, d).$$

Thus we have a system of $\mathcal{D}_{\pi^n}^\dagger(\mathbb{D})$'s and $\mathcal{D}_{\pi^n}^\dagger(\mathbb{D}) \rtimes_{G_{n+1}} G_n$'s with connecting maps

$$\begin{tikzcd} \mathcal{D}_{\pi^{n+1}}^\dagger(\mathbb{D}) \arrow[r] \arrow[d, hook] & \mathcal{D}_{\pi^n}^\dagger(\mathbb{D}) \arrow[d, hook] \\ \mathcal{D}_{\pi^{n+1}}^\dagger(\mathbb{D}) \rtimes_{G_{n+2}} G_{n+1} \arrow[r] & \mathcal{D}_{\pi^n}^\dagger(\mathbb{D}) \rtimes_{G_{n+1}} G_n \end{tikzcd}$$

We will get the desired conclusion that $\mathscr{L}(\mathbb{D} \cap \Omega)$ is a coadmissible $\widehat{\mathcal{D}}(\mathbb{D})$-module, if we prove that

$$\mathcal{D}_{\pi^n}^\dagger(\mathbb{D}) \otimes_{\mathcal{D}_{\pi^{n+1}}^\dagger(\mathbb{D})} \mathscr{L}^\dagger(U_{n+1}) \longrightarrow \mathscr{L}^\dagger(U_n),$$

is an isomorphism for all n. This is however false, because this is never injective. However, somehow one can prove that

$$\mathcal{D}^{\dagger}_{\pi^n}(\mathbb{D}) \rtimes_{G_{n+1}} G_n \otimes_{\mathcal{D}^{\dagger}_{\pi^{n+1}}(\mathbb{D}) \rtimes_{G_{n+2}} G_{n+1}} \mathscr{L}^{\dagger}(U_{n+1}) \longrightarrow \mathscr{L}^{\dagger}(U_n),$$

is an isomorphism for all $n \geq 0$. □

Theorem 1.2(2) is then proved by using the explicit nature of $R(u_n, d)$.

1. Algebraicity of L-values.

Let $K \subset \mathbb{C}$ be an imaginary quadratic field, $N \geq 5$ and suppose that the Heegner hypothesis is satisfied, so that there is $\mathfrak{n} \subset \mathcal{O}_K$ such that $\mathcal{O}_K/\mathfrak{n} \cong \mathbb{Z}/N\mathbb{Z}$. Let $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ be a character. Let f be an eigenform of level $\Gamma_1(N)$, nebentype ε, of weight $k \geq 2$. For example,

(I) f can be Eisenstein series (Katz),
(II) or f can be a cuspidal newform (Bertolini-Darmon-Prasanna, Mori).

Let $\chi : \mathbb{A}_K^\times/K^\times \longrightarrow \mathbb{C}^\times$ be an algebraic Hecke character such that

- conductor of χ divides $\mathfrak{n}$,
- $\chi|_{\mathcal{O}_K^\times} : \mathcal{O}_K^\times \longrightarrow \mathbb{C}^\times$ factors through $\varepsilon : (\mathcal{O}_K/\mathfrak{n})^\times \longrightarrow \mathbb{C}^\times$.

We are interested in interpolating central critical values of L-values of $f \otimes \chi^{-1}$. Note that in the case of (II) (i.e. cuspidal newforms) one has to also assume that χ is central critical for f.

The natures of L-values differ according to the infinity type of χ.

Definition 1.1. *We say that χ has **infinity type** $(m, n) \in \mathbb{Z}^2$ if*

$$\chi|_{(K\otimes\mathbb{R})^\times}(z \otimes 1) = z^{-m}\overline{z}^{-n}.$$

We would like $m + n = k$, so that we want to study algebraic properties of $L(f, \chi^{-1}, 0)$. Accordingly, the set of (critical) algebraic Hecke characters χ splits into three regions:

- $\Sigma^{(1)}$, those of infinity type $(m, n) = (k - 1 - j, 1 + j)$ for $0 \leq j \leq k - 2$,
- $\Sigma^{(2)}$, those of infinity type $(m, n) = (k + j, -j)$ for $j \in \mathbb{N}_{\geq 0}$,
- $\Sigma^{(2')}$, those of infinity type $(m, n) = (-j, k + j)$ for $j \in \mathbb{N}_{\geq 0}$.

This is important as the "transcendental periods" for $L(f, \chi^{-1}, 0)$ change according to the region of χ. We will talk about the case of $\chi \in \Sigma^{(2)}$.

Theorem 1.1. *There is the "algebraic part" $L_{\mathrm{alg}}(f, \chi^{-1}) \in \overline{\mathbb{Q}}$ of $L(f, \chi^{-1}, 0)$. Namely,*

- *in case (I),*
$$L_{\mathrm{alg}}(f, \chi^{-1}) = \frac{1}{|\mathcal{O}_K^\times|}\frac{(-1)(k + j - 1)!N^{k+1}\pi^j}{\mathfrak{g}(\varepsilon)\operatorname{disc}(K/\mathbb{Q})^j\Omega_p^{k+2j}},$$
where $\mathfrak{g}(\varepsilon)$ is the Gauss sum and $\Omega_p \in \mathbb{C}$ is some transcendental period (Damerell).
- *in case (II),*
$$L_{\mathrm{alg}}(f, \chi^{-1})^2 = (*)\frac{L(f, \chi^{-1}, 0)}{\Omega^{2(k+2j)}},$$
where $(*)$ is similar explicit factor which does not depend on f and χ, and $\Omega \in \mathbb{C}$ is also some transcendental period (Waldspurger).

More precisely,
$$L_{\mathrm{alg}}(f, \chi^{-1}) = \sum_{\mathfrak{a}\in\operatorname{Pic}\mathcal{O}_K} \chi_j^{-1}(\mathfrak{a})\delta_k^j(f)(\mathfrak{a} * (A_0, \omega_0, t_0)),$$
where
- $\chi_j = \chi \cdot \mathrm{Nm}^j$,
- *A_0 is an elliptic curve with CM by $\mathcal{O}_K$,*
- *t_0 is a generator of $A_0[\mathfrak{n}]$ (so that $(A_0, t_0) \in Y_1(N)$),*
- *ω_0 is the Néron differential of A_0,*
- $\mathfrak{a} * A_0 := A_0/A_0[\mathfrak{a}]$,

– *and* δ_k *is the* ***Shimura-Maass operator****, which is a* C^∞*-operator*

$$\delta_k(f) = \frac{1}{2\pi i}\left(\frac{\partial}{\partial z} - \frac{k}{z-\overline{z}}\right)(f).$$

Remark 1.1. The transcendental periods "Ω" come from the ratio between ω_0, the Néron differential, which is algebraic, and "dz", which comes from complex uniformization of elliptic curves.

2. **Interpreting the Shimura-Maass operator geometrically.**

The Shimura-Maass operator spits out something real analytic, so it is not at all clear from the formula that $L_{\mathrm{alg}}(f,\chi^{-1})$ is an algebraic number.

Question. Why is $\delta_k^j(f)(\mathfrak{a} * (A_0, \omega_0, t_0)) \in \overline{\mathbb{Q}}$?

We give an interpretation of δ_k using algebraic geometry (specifically, using Gauss-Manin connection and Kodaira-Spencer isomorphism). Consider the universal elliptic curve $\mathbb{E}$ over $Y_1(N)$. Then, the de Rham cohomology $H^1_{\mathrm{dR}}(\mathbb{E}/Y_1(N))$, which we will denote just as H^1_{dR} for simplicity, as a vector bundle over $Y_1(N)$, comes equipped with the Gauss-Manin connection ∇. This induces a connection on $\mathrm{Sym}^k H^1_{\mathrm{dR}}$.

On the other hand, the first piece of Hodge filtration is $\omega_{\mathbb{E}}$, the sheaf of invariant differentials (whose quotient is $H^1(\mathbb{E}, \mathcal{O}_{\mathbb{E}})$), and the Kodaira-Spencer isomorphism gives an isomorphism

$$\mathrm{KS} : \omega_{\mathbb{E}}^2 \xrightarrow{\sim} \Omega^1_{Y_1(N)}.$$

Thus, $\Omega^1_{Y_1(N)}$ can be thought as a subspace of $\mathrm{Sym}^2 H^1_{\mathrm{dR}}$, and thus ∇ can be thought as a map

$$\nabla : \mathrm{Sym}^k H^1_{\mathrm{dR}} \longrightarrow \mathrm{Sym}^k H^1_{\mathrm{dR}} \otimes \Omega^1_{Y_1(N)} \longrightarrow \mathrm{Sym}^{k+2} H^1_{\mathrm{dR}}.$$

Theorem 2.1 (Katz)**.** *The Shimura-Maass operator has an algebro-geometric interpretation,*

$$\delta_k^j(f) = (\text{Hodge–splitting}) \circ \nabla^j(f),$$

where (Hodge–splitting) *is the* C^∞ *Hodge-splitting of* $\omega^{k+2j} \hookrightarrow \mathrm{Sym}^{k+2j} H^1_{\mathrm{dR}}$.

Still both ∇ and (Hodge–splitting) are analytic, so it is not clear why the value at some specific elliptic curve is algebraic. On the other hand, K acts on the 2-dimensional vector space $H^1_{\text{ét}}(A_0)$, so it decomposes into a sum of 1-dimensional eigenspaces $H^1_{\mathrm{dR}}(A_0) = \omega \oplus \overline{\omega}$, where ω and $\overline{\omega}$ each correspond to complex embeddings of K into $\mathbb{C}$. The crucial fact is that this, after base-changing to $\mathbb{C}$, coincides with the Hodge decomposition of $H^1_{\mathrm{dR}}(A_0(\mathbb{C}))$. This gives that $L_{\mathrm{alg}}(f,\chi^{-1})$ is algebraic.

3. ***p*-adic *L*-function over $\Sigma^{(2)}$.**

Now that we have algebraic values, we are interested in congruences between them. After fixing a prime p, the map $w : \Sigma^{(2)} \xrightarrow{(k+j,-j)\mapsto j} \mathbb{Q}$ can be used to p-adically complete $\Sigma^{(2)}$, and get a continuous map $w : \widehat{\Sigma}^{(2)} \longrightarrow \mathbb{Q}_p$.

Question. Is it possible to define $L_p(f,\chi^{-1}) \in \mathbb{C}_p$ for $\chi \in \widehat{\Sigma}^{(2)}$, related to $L_{\mathrm{alg}}(f,\chi^{-1})$ for $\chi \in \Sigma^{(2)}$?

If p splits in K, then the works of Katz, Bertolini-Darmon-Prasanna, Mori establish this. This is particularly because $\mathfrak{a} * A_0$ is **ordinary** at primes above p, and over $Y_1(N)^{\mathrm{ord}}$, H^1_{dR} has a unit root splitting. This a priori has nothing to do with the C^∞-splitting, but this coincides with C^∞-splitting at CM points, so in particular at $\mathfrak{a} * (A_0, t_0)$. In particular, δ is now related to the θ-operator of Serre and Katz.

Our primary goal in interpolating the formula is to make sense of δ^j for a now p-adic weight j. As δ, being equal to θ, on the level of q-expansions has the effect of

$$\delta(\sum a_n q^n) = \sum n a_n q^n,$$

for j a natural number, $\delta^j(\sum a_n q^n) = \sum n^j a_n q^n$. Thus we want $\delta^j(\sum a_n q^n) = \sum j(n) a_n q^n$ for general p-adic weight j, but this only makes sense for $n \in \mathbb{Z}_p^\times$, or $p \nmid n$. Thus, if we take the p-**depletion** $f^{[p]} = (1 - V \circ U)(f)$, which precisely misses parts of q-expansion with $p|n$, then $\delta^j(f^{[p]})$ really makes sense with the expected formula, and Katz showed that the q-expansion really gives a Katz modular form. Now we can apply the same formula for this Katz modular form, and we get the interpolation.

Now if p does not split in K, then this strategy cannot work, as unit root splitting does not overconverge.

Theorem 3.1 (Andreatta-Iovita). *Even if p does not split in K, (if $p \geq 5$) we can construct a p-adic L-function*

$$L_p(f, -) : \widehat{\Sigma}^{(2)} \longrightarrow \mathbb{C}_p,$$

such that for $\chi \in \Sigma^{(2)} \subset \widehat{\Sigma}^{(2)}$, $L_p(f, \chi^{-1}) = (\text{Euler factor}) L_{\text{alg}}(f, \chi^{-1})$, and it also has Gross-Zagier type result for values at $\Sigma^{(1)} \subset \widehat{\Sigma}^{(2)}$.

Remark 3.1. Daniel Kriz defined δ^j purely on supersingular locus, at least for $k = 2$, giving a similar p-adic L-function, but there is no precise interpolation formula for it.

Strategy of proving Theorem 3.1.

(1) We can define a p-adic interpolation of $\omega^m \subset \operatorname{Sym}^m H^1_{\text{dR}}$, so that we obtain, for j an analytic p-adic weight, $\omega^j \subset \mathbb{W}_j$ over strict neighborhoods Y_r of $Y_1(N)^{\text{ord}}$, where $Y_r = \{|\operatorname{Ha}| \leq \frac{1}{r}\}$ (Andreatta-Iovita-Pilloni). The sheaf $\mathbb{W}_j$ is an infinite-dimensional Banach sheaf which contains $\operatorname{Sym}^m H^1_{\text{dR}}$ when you specialize at $m \in \mathbb{N}$. **This works for $r \geq 2$.**

(2) For every p-adic analytic weight j, we can justify

$$\nabla^j(f^{[p]}) \in H^0(Y_r, \mathbb{W}_{k+2j}),$$

for some $r \in \mathbb{Q}_{>0}$. **This works for $r \geq p(p+1)$.**

(3) We want to evaluate $\nabla^j(f^{[p]})$ at $\mathfrak{a} * (A_0, t_0, \omega_0)$ and use algebraic splitting of $\omega^{k+2j} \subset \mathbb{W}_{k+2j}$. The problem is that $\mathfrak{a} * (A_0, t_0, \omega_0)$ is not in Y_r for $r \geq p(p+1)$, but there is a way to get around this, namely one can use U_p-correspondence to move the point inside the desired region.

□

1. **Motivation.**

Let Π be a mod p^r smooth representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Let $B = TN \subset G$ be the upper triangular Borel, $B_+ := \begin{pmatrix} \mathbb{Z}_p\setminus\{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \mathbb{Q}_p^\times$, $N_0 = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$. Let F be the conjugation-by-$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{Z}/p^r\mathbb{Z}[[N_0]] \cong \mathbb{Z}/p^r\mathbb{Z}[[T]]$. This acts on T via $FT = ((T+1)^p - 1)F$. Even though T and F do not commute, for notational simplicity we will denote a non-commutative ring of F acting on $\mathbb{Z}/p^r\mathbb{Z}[[T]]$ as $\mathbb{Z}/p^r\mathbb{Z}[[T]][F]$.

Lemma 1.1 (Emerton, Colmez). *Let M be a subspace of Π satisfying the following conditions.*

(1) M is a finitely generated $\mathbb{Z}/p^r\mathbb{Z}[[T]][F]$-module.

(2) M is stable under $T_0 = \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & \mathbb{Z}_p^\times \end{pmatrix} \mathbb{Q}_p^\times$.

(3) M is admissible as an N_0-representation.

Then, $M^\vee[T^{-1}]$ is a (φ, Γ)-module over $\mathbb{Z}/p^r\mathbb{Z}((T))$ (plus a $\mathbb{Q}_p^\times$-action via the center), where $M^\vee$ is the Pontryagin dual of M.

Remark 1.1. The φ-action is not quite the F-action on $M^\vee[T^{-1}]$, but rather the so-called ψ-action is given by F.

Theorem 1.1 (Colmez). *If Π is admissible and finite length with a central charcater, then there is a maximal M satisfying the conditions of Lemma 1.1, denoted as $\Pi_{\max}$. The functor $D(\Pi) = M_{\max}^\vee[T^{-1}]$ is an exact functor.*

2. **Generalization of Colmez's functor to higher rank groups.**

Let $\mathbb{G}$ be a split connected reductive group over $\mathbb{Q}_p$ with connected center, and let $G = \mathbb{G}(\mathbb{Q}_p)$. Let $B = TN$ be a Borel subgroup, and let N_0 be a maximal compact subgroup of N which is furthermore totally decomposed in the sense that $N_0 = \prod_{\alpha\in\Phi^+}(N_0 \cap N_\alpha)$, where N_α is the root subgroup for positive root α. Fix an isomorphism $\iota_\alpha : \mathbb{Z}_p \xrightarrow{\sim} N_{0,\alpha} = N_0 \cap N_\alpha$. Consider a Whittaker character

$$\ell : N_0 \twoheadrightarrow N_0 / \prod_{\beta\in\Phi^+\setminus\Delta} N_{0,\beta} \xrightarrow{\sum_{\alpha\in\Delta}\iota_\alpha^{-1}} \mathbb{Z}_p,$$

and let $H_0 = \ker(\ell) \trianglelefteq N_0$. Let $\xi : \mathbb{Q}_p^\times \longrightarrow T$ be a cocharacter such that $\alpha\circ\xi = \mathrm{id}_{\mathbb{Q}_p^\times}$ for all $\alpha \in \Delta$.

Example 2.1. If $\mathbb{G} = \mathrm{GL}_n$, then one can take

- $N_0 = \begin{pmatrix} 1 & \mathbb{Z}_p & \mathbb{Z}_p \\ 0 & \cdots & \mathbb{Z}_p \\ 0 & 0 & 1 \end{pmatrix}$,
- $\ell((a_{i,j})) = a_{1,2} + \cdots + a_{n-1,n}$,
- $\xi(x) = \begin{pmatrix} x^{n-1} & & & \\ & x^{n-2} & & \\ & & \cdots & \\ & & & 1 \end{pmatrix}$.

Given a smooth N_0-representation Π over $\mathbb{Z}/p^r\mathbb{Z}$, $\mathbb{Z}_p$ acts on Π^{H_0}, as well as the Hecke operator

$$F : \Pi^{H_0} \xrightarrow{\xi(p)\text{-conjugation}} \Pi^{\xi(p)H_0\xi(p)^{-1}} \xrightarrow{\mathrm{Tr}} \Pi^{H_0}.$$

Then the same (noncommutative) ring $\mathbb{Z}/p^r\mathbb{Z}[[T]][F]$ acts on Π^{H_0}.

Lemma 2.1 (Breuil). *If $M \subset \Pi^{H_0}$ is a subspace satisfying the following conditions,*

(1) M is finitely generated over $\mathbb{Z}/p^r\mathbb{Z}[[T]][F]$,

(2) M is stable under $\xi(\mathbb{Z}_p^\times)Z(G)(\mathbb{Q}_p)$,

(3) M is admissible as a representation of N_0/H_0,

then $M^\vee[T^{-1}]$ is a (φ,Γ)-module.

One cannot expect similar admissibility result as in $\mathrm{GL}_2(\mathbb{Q}_p)$ case. Still, if M_1, M_2 satisfy the conditions of Lemma 2.1, then $M_1 + M_2$ satisfies them as well, so we can form a pro-object in the category of (φ,Γ)-modules via

$$D_\xi^\vee(\Pi) := \varprojlim_M M^\vee[T^{-1}].$$

Proposition 2.1 (Breuil)**.** *The functor $D_\xi^\vee$ satisfies the following properties.*

1. *It is compatible with parabolic induction and tensor products.*
2. *It is right exact.*
3. *It is exact on the subcategory of subquotients of principal series, denoted as* $\mathrm{SP}(\mathbb{Z}/p^r\mathbb{Z})$*.*

The obvious downside is that this functor loses information.

Example 2.2. For example, $D_\xi^\vee\left(\mathrm{Ind}\begin{pmatrix}\chi_1 & \cdots & \\ & & \chi_n\end{pmatrix}\right)$ only knows about $\chi_1^{n-1}\chi_2^{n-2}\cdots\chi_{n-1}$.

Also, there are many extensions on the automorphic side that cannot be captured by (φ,Γ)-modules, as $G_{\mathbb{Q}_p}$ has p-cohomological dimension 2.

3. **Multivariable (φ,Γ)-modules.**

A possible remedy to these problems is to use **multivariable (φ,Γ)-modules**. Namely, one does not need to use just H_0; consider

$$H_{0,\Delta} = \ker\left(N_0 \longrightarrow \prod_{\alpha\in\Delta} N_{0,\alpha}\right),$$

and for each $\alpha\in\Delta$, consider $\xi_\alpha : \mathbb{Q}_p^\times \longrightarrow T$ such that $\alpha\circ\xi_\alpha = \mathrm{id}_{\mathbb{Q}_p^\times}$ and $\beta\circ\xi_\alpha = 1$ for all other simple roots $\beta\neq\alpha$.

Example 3.1. If $G = \mathrm{GL}_n(\mathbb{Q}_p)$, then $\xi_\alpha(x)$ for $\alpha = e_i - e_{i+1}$ can be given by $\mathrm{diag}(x, x, \cdots, x, 1, 1, \cdots, 1)$, where there are i many x's.

Then, for a smooth $\mathbb{Z}/p^r\mathbb{Z}$-representation Π of N_0, $\Pi^{H_{0,\Delta}}$ has a Hecke action F_α of $\xi_\alpha(p)$.

Lemma 3.1 (Zabradi)**.** *If M is a subspace of $\Pi^{H_{0,\Delta}}$ such that*

1. *M is a finitely generated module over $\mathbb{Z}/p^r\mathbb{Z}[[N_0/H_{0,\Delta}]][F_\alpha \mid \alpha\in\Delta]$ (which is also noncommutative),*
2. *M is stable under $\xi_\alpha(\mathbb{Z}_p^\times)$ for all $\alpha\in\Delta$ and the central action $\mathbb{Q}_p^\times$,*
3. *and M is admissible as a $N_0/H_{0,\Delta}$-representation,*

then $M^\vee[T_\alpha^{-1} \mid \alpha\in\Delta]$ is a multivariable $(\varphi_\Delta,\Gamma_\Delta)$-module with $Z(G)$-action (i.e. there are $|\Delta|$-many variables; $N_0/H_{0,\Delta}\cong\prod_{\alpha\in\Delta}N_{0,\alpha}\cong\mathbb{Z}_p^{|\Delta|}$). If M_1, M_2 are two such subspaces, then M_1+M_2 also satisfies the above conditions.

Thus we can form a pro-object

$$D_\Delta^\vee(\Pi) = \varprojlim_M M^\vee[T_\alpha^{-1} \mid \alpha\in\Delta].$$

Proposition 3.1 (Zabradi)**.** *The functor $D_\Delta^\vee$ satisfies the following conditions.*

1. *It is compatible with parabolic induction and tensor products.*
2. *It is right exact.*
3. *It is exact on* $\mathrm{SP}(\mathbb{Z}/p^r\mathbb{Z})$*.*

(4) ***It is fully faithful on extensions of irreducible principal series.***

Proof. For $k \geq 1$, there is a natural normal subgroup $H_{k,\Delta} \trianglelefteq H_{0,\Delta}$ ("$p^k\mathbb{Z}_p \subset \mathbb{Z}_p$"), and one does the same thing for $H_{k,\Delta}$, and then one instead gets $H_{0,\Delta}/H_{k,\Delta}$-action. By sending $k \longrightarrow \infty$, one gets an action of $\mathbb{Z}/p^r\mathbb{Z}((N_0))$. A work of Schneider-Vigneras-Zabradi associates to this a sheaf $\mathbb{Y}$ on G/B (just a sheaf on the p-adic topological space). Then, $\Pi^\vee \hookrightarrow \mathbb{Y}(G/B)$, and one can describe its image. $\square$

4. **Galois side.**

What does a multivariable $(\varphi_\Delta, \Gamma_\Delta)$-module mean?

Theorem 4.1 (Zabradi, Carter-Kedlaya-Zabradi)**.** *There is an equivalence of categories*

$$\mathbb{V} : \left\{ \begin{array}{c} \textit{multivariable} \\ (\varphi_\Delta, \Gamma_\Delta)\textit{-modules} \\ \textit{with coefficients in} \\ \mathbb{Z}/p^r\mathbb{Z} \end{array} \right\} \xrightarrow{\sim} \left\{ \mathbb{Z}/p^r\mathbb{Z}\textit{-representations of } G_{\mathbb{Q}_p}^{|\Delta|} \right\},$$

respecting $Z(G)$-actions on both sides.

Theorem 4.2 (Pal-Zabradi)**.** *All multivariable $(\varphi_\Delta, \Gamma_\Delta)$-modules are overconvergent. Also, an analogue of Herr's complex computes $G_{\mathbb{Q}_p}^\Delta$-cohomology.*

5. **Conjectural global picture.**

Let $F/\mathbb{Q}$ be an imaginary quadratic field, and let p be a rational prime that splits in F. Let $\mathbb{G}$ be a unitary group over $\mathbb{Z}[1/N]$ which is compact at infinity and split at p. Let $K_f^p \leq \mathbb{G}(\mathbb{A}^{p,\infty})$ be a compact open subgroup. For $q = p^r$, let

$$S(K_f^p, \mathbb{F}_q) = \{f : \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}^\infty)/K_f^p \longrightarrow \mathbb{F}_q\}.$$

Let $\overline{\rho} : G_F \longrightarrow \mathrm{GL}_n(\mathbb{F}_q)$ be modular. The Hecke operators for ℓ away from Np gives a maximal ideal $\mathfrak{m}_{\overline{\rho}}$ of relevant Hecke algebra, which also acts on $S(K_f^p, \mathbb{F}_q)$. Also, $\mathbb{G}(\mathbb{Q}_p) = \mathrm{GL}_n(\mathbb{Q}_p)$ acts on it too.

Conjecture 5.1 (Mod p local-global compatibility)**.** *For all $d \geq 1$,*

$$\mathbb{V}\left(D_\Delta^\vee\left(S(K_f^p, \mathbb{F}_q)[\mathfrak{m}_{\overline{\rho}}]\right)\right)^\vee = \left(\left(\bigotimes_{i=1}^{n-1} \omega^{\frac{i^2-i}{2}} \otimes \wedge^i \overline{\rho}_p\right) \otimes \omega^{\frac{n^2-n}{2}} \otimes \wedge^n \overline{\rho}_p\right)^{\oplus d},$$

as $G_{\mathbb{Q}_p}^{n-1} \times \mathbb{Q}_p^\times$-representations, where ω is the mod p cyclotomic character and $\overline{\rho}_p = \overline{\rho}|_{G_{\mathbb{Q}_p}}$.

1. Trianguline representations.

Trianguline representations form a class of p-adic Galois representations containing the class of semistable Galois representations. This is defined by Colmez, and these are expected to be exactly the class of p-adic Galois representations coming from p-adic automorphic forms (similar to de Rham representations which encode Galois representaitons coming from geometry).

There are several equivalent definitions of trianguline representations:

- using (φ, Γ)-module over Robba ring
- using B-pairs
- using vector bundles on the Fargues-Fontaine curve.

We will use the second approach.

Definition 1.1. *Let $K/\mathbb{Q}_p$ (base field) and $E/\mathbb{Q}_p$ (coefficient field) be finite extensions, with $E/\mathbb{Q}_p$ Galois, and let $B_e = B_{\mathrm{cris}}^{\varphi=1}$. Then, a B_K^E-**pair** (or a E-B-**pair of** G_K) is a pair (W_e, W_{dR}^+) where W_e is a free $E \otimes_{\mathbb{Q}_p} B_e$-module of finite rank with a semilinear G_K-action, and W_{dR}^+ is a G_K-stable $E \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+$-lattice in $W_{\mathrm{dR}} := E \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \otimes_{E \otimes_{\mathbb{Q}_p} B_e} W_e$.*

Remark 1.1. Note that $E \otimes_{\mathbb{Q}_p} B_e$ is a PID, and G_K acts on it only via its action on B_e (and trivial action on E).

Example 1.1. (1) For an E-linear representation V of G_K,

$$W(V) = ((E \otimes_{\mathbb{Q}_p} B_e) \otimes_E V, (E \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+) \otimes_E V),$$

gives a B-pair. This realizes the category of p-adic representations of G_K as a full subcategory of B-pairs.

(2) For an E-linear filtered (φ, N)-module D over K,

$$W(D) = ((B_{\mathrm{st}} \otimes_{K_0} D)^{N=0, \varphi=\mathrm{id}}, \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K D_K)),$$

gives a B-pair. If $D = D_{\mathrm{st}}(V)$ for a semistable E-linear G_K-representation V, then $W(V) = W(D)$.

Definition 1.2. *Let W be a B_K^E-pair.*

- *W is **split trianguline** if it is a successive extension of objects of rank 1. We call these rank 1 objects the **parameters** of W.*
- *W is **trianguline** if there is a finite extension F/E such that $F \otimes_E W$ is split trianguline.*
- *W is **potentially trianguline** if W restricted to G_L for a finite extension L/K is trianguline.*

We say a p-adic representation V is split trianguline (trianguline, potentially trianguline, respectively) if the associated B-pair is split trianguline (trianguline, potentially trianguline, respectively).

Example 1.2. If V is semistable, then as $W(V) = W(D_{\mathrm{st}}(V))$, and as φ can be made upper triangular after an extension of coefficient field, V is trianguline.

Theorem 1.1 (Berger-Di Matteo). *If V, V' are two $G_{\mathbb{Q}_p}$-representations over a p-adic field E such that $V \otimes_E V'$ is trianguline, then V and V' are potentially trianguline.*

Definition 1.3. *A B_K^E-pair W is $\Delta(\mathbb{Q}_p)$-**trianguline** if it is trianguline and all parameters extend to $B_{\mathbb{Q}_p}^E$-pairs.*

Theorem 1.2 (Berger-Di Matteo). *If W, W' are two B_K^E-pairs such that $W \otimes_E W'$ is $\Delta(\mathbb{Q}_p)$-trianguline, then W and W' are potentially trianguline.*

2. **Ingredients of the proofs.**

Rank 1 B_K^E-pairs are classified by Colmez (for $K = \mathbb{Q}_p$) and Nakamura (for general K). Namely, there is a bijection

$$\{\text{characters } \delta : K^\times \longrightarrow E^\times\} \xrightarrow{\sim} \left\{\text{rank 1 } B_K^E\text{-pairs}\right\},$$
$$\delta \mapsto B(\delta).$$

Example 2.1. For $K = \mathbb{Q}_p$, a B_K^E-pair is the same as a (φ, Γ)-module. Given a character $\delta : \mathbb{Q}_p^\times \longrightarrow E^\times$, the associated rank 1 (φ, Γ)-module is just that φ acts by $\delta(p)$ and Γ acts by $\delta|_{\mathbb{Z}_p^\times}$.

However, Nakamura's construction is not as transparent as above example of $K = \mathbb{Q}_p$ case, and we need more explicit description.

Given a rank 1 B_K^E-pair, we want to first understand W_e, a rank 1 $E \otimes_{\mathbb{Q}_p} B_e$-representation of G_K. As $B_e^\times = \mathbb{Q}_p^\times$, rank 1 B_e-representations of G_K are just characters of form $G_K \longrightarrow \mathbb{Q}_p^\times$. For a general E, this strategy cannot be used because $E \otimes_{\mathbb{Q}_p} B_e$ has many units. Nevertheless, we have the following

Proposition 2.1. *Let $E_0 = \mathbb{Q}_{p^h}$ be the maximal unramified subfield of E. Let $\varphi_E = \mathrm{id} \otimes \varphi^h$ on $E \otimes_{E_0} B_{\mathrm{cris}}$. Then, the map*

$$E \otimes_{\mathbb{Q}_p} B_e \longrightarrow (E \otimes_{E_0} B_{\mathrm{cris}})^{\varphi_E = 1},$$

is an isomorphism.

Now we instead use $(E \otimes_{E_0} B_{\mathrm{cris}})^{\varphi_E = 1}$ and Lubin-Tate theory for E. Let π be a uniformizer of E, and let $\mathcal{F}$ be the Lubin-Tate group attached to π, with $[\pi](T) = P(T) = \pi T + T^p$. Let $\chi_\pi : \mathcal{O}_E \longrightarrow E^\times$ be the associated Lubin-Tate character.

Proposition 2.2. *There exists an elmenet $t_\pi \in E \otimes_{E_0} B_{\mathrm{cris}}$ such that $g(t_\pi) = \chi_\pi(g) t_\pi$ for all $g \in G_F$ and $\varphi_E(t_\pi) = \pi t_\pi$.*

Proof. Consider

$$\widetilde{\mathbb{E}}^+ = \mathcal{O}_{\mathbb{C}_p}^\flat = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/\pi,$$

and

$$\widetilde{\mathbb{A}}^+ = \mathcal{O}_E \otimes_{\mathcal{O}_{E_0}} W(\widetilde{\mathbb{E}}^+).$$

Then, given $\overline{u} \in \widetilde{\mathbb{E}}^+$, there is a unique lift $u \in \widetilde{\mathbb{A}}^+$ such that $\varphi_E(u) = P(u)$. Take $\overline{u} = (u_n)$ where $u_n \in \mathcal{F}[\pi^n]$, and let $t_\pi = \log_{\mathcal{F}}(u)$. $\square$

Let $\Sigma = \mathrm{Gal}(E/\mathbb{Q}_p)$. Then, $\tau \in \Sigma$ acts on $E \otimes_{E_0} B_{\mathrm{cris}}$ via $\tau \otimes \varphi^{n(\tau)}$, where $0 \le n(\tau) \le h - 1$ such that $\tau|_{\mathbb{F}_{p^h}} = \varphi^{n(\tau)}$. Let $t_\tau = (\tau \otimes \varphi^{n(\tau)})(t_\pi)$. This satisfies

- $g(t_\tau) = \tau(\chi_\pi(g)) t_\tau$, for $g \in G_F$,
- $\varphi_E(t_\tau) = \tau(\pi) t_\tau$.

Remark 2.1. The product $\prod_{\tau \in \Sigma} t_\tau$ is an $\widehat{E^{\mathrm{nr}}}$-multiple of the usual t of p-adic Hodge theory. This is because $N_{E/\mathbb{Q}_p}(\chi_\pi) = \chi_{\mathrm{cyc}} \eta$ for some unramified character $\eta : G_E \longrightarrow \mathbb{Z}_p^\times$.

The above remark implies that $t_\tau^{-1} \in E \otimes_{E_0} B_{\mathrm{cris}}$.

Now we take any $\underline{n} = (n_\tau)_{\tau \in \Sigma}$, each $n_\tau \in \mathbb{Z}$, such that $\sum_{\tau \in \Sigma} n_\tau = 0$. Then,

$$\varphi_E\left(\prod_{\tau \in \Sigma} t_\tau^{n_\tau}\right) = \prod_{\tau \in \Sigma} \tau(\pi)^{n_\tau} \prod_\tau t_\tau^{n_\tau}.$$

Note that as $\sum_\tau n_\tau = 0$, $v_\pi\left(\prod_{\tau\in\Sigma}\tau(\pi)^{n_\tau}\right) = 0$. Thus, there exists $u_{\underline{n}} \in \widehat{E^{\mathrm{nr}}}$ such that
$$\frac{\varphi_E(u_{\underline{n}})}{u_{\underline{n}}} = \left(\prod_{\tau\in\Sigma}\tau(\pi)^{n_\tau}\right)^{-1}.$$
Thus, by taking $u = \prod_\tau t_\tau^{n_\tau} u_{\underline{n}}$, we have $\varphi_E(u) = u$.

Proposition 2.3. *(1) The element u is a unit of $(E\otimes_{E_0} B_{\mathrm{cris}})^{\varphi_E=1}$.*
(2) Every unit is of this form up to $E^\times$.

Proposition 2.4. *(1) Every rank 1 $E\otimes_{\mathbb{Q}_p} B_e$-representation of G_E is of the form $(E\otimes_{\mathbb{Q}_p} B_e)(\delta)$, for some $\delta : G_E \longrightarrow E^\times$, where here it really means twist by character δ (namely, a tensor product).*
(2) If, as G_E-representaitons, $(E\otimes_{\mathbb{Q}_p} B_e)(\delta) \cong E\otimes_{\mathbb{Q}_p} B_e$, then δ is de rham, and the sum of its weights is zero.
(3) If δ is de Rham whose weights sum up to zero, then there is a potentially unramified character $\eta : G_E \longrightarrow E^\times$ such that $(E\otimes_{\mathbb{Q}_p} B_e)(\delta\eta) \cong E\otimes_{\mathbb{Q}_p} B_e$ as G_E-representations.

Remark 2.2. The above result does not hold if $E \neq K$.

Now we want to study extensions of B-pairs (which is also studied by Nakamura). For a B_K^E-pair W, there is a good definition of Galois cohomology $H^i(G_K, W)$ for $i \geq 0$, which is an E-vector space. Although we will not define it here, we note some properties.

(1) $H^1(G_K, W)$ classifies extensions of B by W.
(2) There is an exact sequence
$$W_{\mathrm{dR}}^{G_K} \longrightarrow H^1(G_K, W) \longrightarrow H^1(G_K, W_e) \oplus H^1(G_K, W_{\mathrm{dR}}^+) \longrightarrow H^1(G_K, W_{\mathrm{dR}}).$$
This is a very natural compatibility statement.

We want classes in $H^1(G_K, W)$ to be zero, so we want $W_{\mathrm{dR}}^{G_K} = 0$. This is not as simple as you might think even for characters, where one is easily led to think that $W_{\mathrm{dR}}^{G_K} = 0$ if and only if the character is not de Rham; this is not true as there are many embeddings of K, so that it might happen that the character is de Rham via one embedding but not de Rham via other embedding. This is called **partially de Rham** (coined by Yiwen Ding). This is why we need $\Delta(\mathbb{Q}_p)$-triangulinity, because in this setting W is de Rham if and only if $W_{\mathrm{dR}}^{G_K} \neq \{0\}$.

Now the last ingredient is a triangulability condition.

Proposition 2.5. *Let X_e, Y_e be irreducible $E\otimes_{\mathbb{Q}_p} B_e$-representations of G_K. If $X_e\otimes Y_e$ is split trianguline, then $X_e\otimes Y_e$ is actually a direct sum of rank 1 objects.*

3. **Applications.**

We hope to show that for any p-adic G_K-representations V, V', if $V\otimes V'$ is trianguline, then V and V' are potentially crystalline.

Using Theorem 1.1, Andrea Conti showed a statement of the form
$$\text{“}\mathrm{Sym}^k\, V \text{ trianguline} \Longrightarrow V \text{ potentially trianguline”},$$
and used this to study eigenvarieties.

1. The Deligne-Langlands conjecture for Hecke modules.

Let $F/\mathbb{Q}_p$ be a finite extension, and $G = \mathrm{GL}_n(F)$. Let I be the Iwahori subgroup of G, and let $H = \mathbb{C}[I\backslash G/I]$, the **complex Iwahori-Hecke algebra**. Let $\widehat{G} = \mathrm{GL}_n(\mathbb{C})$. The group W_F/P_F, the quotient of W_F by wild inertia subgroup, is generated by φ, the Frobenius, and v, a lift of monodromy, where $\varphi v \varphi^{-1} = v^q$.

The local Langlands correspondence for tame representations can be thought as a bijection

$$\left\{ \begin{array}{c} \text{certain} \\ \text{representations} \\ W_F/P_F \longrightarrow \widehat{G} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{irreducible} \\ \text{complex smooth} \\ G\text{-representations} \\ \text{with } V^I \neq 0 \end{array} \right\}.$$

Using the explicit nature of W_F/P_F, we can transfer the language into

$$\left\{ \begin{array}{c} \text{certain } (s,t) \in \widehat{G} \\ \text{with } sts^{-1} = t^q \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{irreducible} \\ H\text{-modules} \end{array} \right\}.$$

This is the **Deligne-Langlands conjecture for Hecke modules**. This is proved by Kazhdan-Lusztig in 1985.

Proof. Let $\widehat{G}/\widehat{B} =: \widehat{\mathfrak{B}}$ be the dual flag variety, with an action of $\widehat{G}$ on the left. Let

$$K^{\widehat{G}}(\widehat{\mathfrak{B}}) = K_0(\{\widehat{G}\text{-equivariant coherent } \mathcal{O}_{\widehat{\mathfrak{B}}}\text{-modules}\}),$$

$$R(\widehat{G}) = K_0(\{\widehat{G}\text{-representations}\}).$$

Note that $R(\widehat{G})$ is a ring and $K^{\widehat{G}}(\widehat{\mathfrak{B}})$ is naturally a module over it.

Now the faithful H-action is $R(\widehat{G})_{\mathbb{C}}$-linear, so we get a map

$$\mathscr{A}_{\mathbb{C}} : H \hookrightarrow \mathrm{End}_{R(\widehat{G})_{\mathbb{C}}}(K^{\widehat{G}}(\widehat{\mathfrak{B}})_{\mathbb{C}}),$$

such that $\mathscr{A}_{\mathbb{C}}|_{Z(H)}$ identifies $Z(H)$ with $R(\widehat{G})_{\mathbb{C}}$. Now, given $(s,t) \in \widehat{G}$ with s semisimple and $sts^{-1} = t^q$, we have an action of $H \otimes_{Z(H),s} \mathbb{C}$ on $K^{\widehat{G}}(\widehat{\mathfrak{B}}) \otimes_{R(\widehat{G}),s} \mathbb{C}$. As s is semisimple, $K^{\widehat{G}}(\widehat{\mathfrak{B}}) \otimes_{R(\widehat{G}),s} \mathbb{C}$ is just the Grothendieck group of the category of coherent sheaves on the fixed point part, $K(\widehat{\mathfrak{B}}^s)_{\mathbb{C}}$. This $\widehat{\mathfrak{B}}^s$ is a finite set of points, so $K(\widehat{\mathfrak{B}}^s)_{\mathbb{C}}$ is a finite-dimensional vector space, and we use t to single out the corresponding simple subquotient. □

2. The mod p situation.

We should rather use the pro-p Iwahori subgroup $I^{(1)} \subset I$, and study the mod p pro-p Iwahori Hecke algebra $H^{(1)} = \overline{\mathbb{F}}_p[I^{(1)}\backslash G/I^{(1)}]$. This has been extensively studied by many people (Vigneras, Ollivier, Schneider, Grosse-Klönne, $\cdots$).

Theorem 2.1 (Ollivier). *The center of $H^{(1)}$ contains $Z^0(H^{(1)}) = \overline{\mathbb{F}}_p[w_1, \cdots, w_{n-1}, w_n^{\pm 1}]$, such that $H^{(1)}$ is a module finite over $Z^0(H^{(1)})$.*

Here w_i's correspond to fundamental weights.

Definition 2.1. *A simple $H^{(1)}$-module is called **supersingular** if $Z^0(H^{(1)})$ acts by $w_i = 0$ for all $i < n$.*

Theorem 2.2 (Breuil, Vigneras, Ollivier, Grosse-Klönne)**.** *There is a functorial bijection*

$$\left\{\begin{array}{c}\textit{supseringular}\\ H^{(1)}\textit{-modules of}\\ \textit{dimension } n\end{array}\right\}\xrightarrow{\sim}\left\{\textit{irreducible } W_F \longrightarrow \widehat{G}\right\},$$

where $\widehat{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$.

Remark 2.1.
- Construction is completely algebraic using (φ, Γ)-modules.
- There is a fully faithful functor inducing the bijection.

Question. Is there some geometric mod p Kazhdan-Lusztig-like construction for the inverse direction of Galois to automorphic?

The first step would be to find all supersingular $H^{(1)}$-modules of dimension n in some equivariant cohomology of $\widehat{G}$ or $\widehat{\mathfrak{B}}$ $\cdots$. We first decompose $H^{(1)}$ into parts using $\overline{\mathbb{F}}_p[\mathbb{T}] \subset H^{(1)}$ where $\mathbb{T} = \begin{pmatrix} \mathbb{F}_q^\times & & \\ & \cdots & \\ & & \mathbb{F}_q^\times \end{pmatrix} \cong I/I^{(1)}$. It is a semisimple ring, so

$$H^{(1)} = \prod_{\gamma \in \mathbb{T}^\vee/W_0} H^{(\gamma)},$$

where W_0 is the Weyl group of G and $H^{(\gamma)}$'s are subalgebras of $H^{(1)}$.

3. **The Iwahori case.**

We study the most complicated case, $\gamma = 1$, where then

$$H^{(\gamma)} = \overline{\mathbb{F}}_p[I\backslash G/I] = \bigoplus_{w \in W} \overline{\mathbb{F}}_p T_w,$$

where $W = W_0\Lambda$ is the Iwahori-Weyl group, $\Lambda = \bigoplus_{i=1}^d \mathbb{Z}\eta_i$ and $\eta_i(x) = \begin{pmatrix} 1 & & & & & \\ & \cdots & & & & \\ & & 1 & & & \\ & & & x & & \\ & & & & 1 & \\ & & & & & \cdots \\ & & & & & & 1 \end{pmatrix}$, with x at the i-th position.

There is a nice presentation of W. Namely, if we let $u = \eta_1(1, \cdots, n) \in W$, then $W = \Omega W_{\mathrm{aff}}$ where $\Omega = \langle u \rangle \cong \mathbb{Z}$ and $W_{\mathrm{aff}} = \langle s_0 = us_1u^{-1}, s_1, \cdots, s_{n-1}\rangle$, where s_i's are simple reflections in W_0. Thus, $H^{(1)} = \overline{\mathbb{F}}_p[S_1, \cdots, S_{n-1}, U^{\pm 1}]$ where $S_i = T_{s_i}$, $U = T_u$ with full set of relations

- $S_i^2 = -S_i$,
- $Us_{i+1} = S_iU$,
- $U^2S_1 = S_{n-1}U^2$,
- $S_iS_jS_i = S_jS_iS_j$,
- $S_iS_j = S_jS_i$ for $|i - j| \geq 2$.

We want H-action on $K^{\widehat{G}}(\widehat{\mathfrak{B}})_{\overline{\mathbb{F}}_p}$, where now $\widehat{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ and $\widehat{\mathfrak{B}} = \widehat{G}/\widehat{B}$. There is a construction of Demazure (1976) of H_0-action on $K^{\widehat{G}}(\widehat{\mathfrak{B}})_{\overline{\mathbb{F}}_p}$, which works for any characteristic. The construction as follows: for each i, let $\widehat{P}_i$ be the minimal parabolic subgroup which corresponds to the partition $(1, \cdots, 1, 2, 1, \cdots, 1)$ where 2 is at the i-th position. Then, the natural map $\pi_i : \widehat{\mathfrak{B}} \longrightarrow \widehat{G}/\widehat{P}_i$ is a $\mathbb{P}^1$-bundle. In this way the Grothendieck group of coherent sheaves over $\widehat{G}/\widehat{P}_i$ sits naturally inside the Grothendieck group of coherent sheaves over $\widehat{\mathfrak{B}}$, and the idempotent can be given by $\pi_i^*(\pi_i)_*$; let $-s_i$ act as this.

Theorem 3.1 (Pepin-Schmidt)**.** *The Demazure representation extends uniquely to a faithful H-representation*

$$\mathscr{A} : H \hookrightarrow \mathrm{End}_{R(\widehat{G})_{\overline{\mathbb{F}}_p}}(K^{\widehat{G}}(\widehat{\mathfrak{B}})_{\overline{\mathbb{F}}_p}),$$

such that

$$\mathscr{A}|_{Z(H)} : Z(H) \xrightarrow{\sim} Z^0(H^{(1)}) \cong \overline{\mathbb{F}}_p[w_1, \cdots, w_{n-1}, w_n^{\pm 1}].$$

Now we try to play the same game. Take a semisimple element $s \in \widehat{G}$, and let $H_s := H \otimes_{Z,s} \overline{\mathbb{F}}_p$ act on $K(\widehat{\mathfrak{B}}^s)_{\overline{\mathbb{F}}_p}$.

Theorem 3.2 (Pepin-Schmidt)**.** *If the corresponding central character $\theta_s : Z(H) \longrightarrow \overline{\mathbb{F}}_p$ is supersingular, then $K(\widehat{\mathfrak{B}}^s)_{\overline{\mathbb{F}}_p}$ contains all supersingular H-modules of dimension n corresponding to θ_s; $\mathscr{A}_s$ is injective.*

Example 3.1. If $n = 2$, by dimension count and injectivity, one has $\mathscr{A}_s : H_s \xrightarrow{\sim} \mathrm{End}_{\overline{\mathbb{F}}_p}(K(\widehat{\mathfrak{B}}^s)_{\overline{\mathbb{F}}_p})$.

Notations. Let $F/\mathbb{Q}_p$ be a finite extension, k be the residue field of its ring of integers $\mathcal{O}_F$ and let $\mathbb{F}$ be some finite extension of k. Let $G = \mathrm{GL}_2(F)$, and let $K = \mathrm{GL}_2(\mathcal{O}_F)$. Let I (resp. I_1) be the Iwahori subgroup (resp. pro-p Iwahori subgroup); the normalizer of I_1 is $N(I_1) = I\Pi^{\mathbb{Z}}$, where $\Pi = \left(\begin{smallmatrix} & 1 \\ p & \end{smallmatrix}\right)$. Let $K_1 = \ker(\mathrm{GL}_2(\mathcal{O}_F) \twoheadrightarrow \mathrm{GL}_2(k))$; $N(K_1) = KZ$. Let $F(a,b) = \mathrm{Sym}^{a-b}\,\mathbb{F}^2 \otimes \det{}^b$ be a KZ-representation, where $\left(\begin{smallmatrix} p & \\ & p \end{smallmatrix}\right)$ acts trivially. Let $\chi_{a,b}$ be the character such that $\chi_{a,b}\left(\begin{smallmatrix} [x] & \\ & [y] \end{smallmatrix}\right) = x^a y^b$ for $x, y \in k^\times$, where $[-]$ means Teichmüller lift; in particular, conjugation by Π yields $(\chi_{a,b})^\Pi = \chi_{b,a}$.

1. **mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.**
   - For $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2\,\mathbb{F}$ of form

$$\overline{\rho} \sim \begin{pmatrix} \omega^{a+1} \otimes \mathrm{un}_\lambda & * \\ & \mathrm{un}_{\lambda^{-1}} \end{pmatrix},$$

   for $0 < a < p-3$, the G-representation $\pi(\overline{\rho})$ correponding to $\overline{\rho}$ via the mod p local Langlands correspondence (Colmez, Breuil) can be given by

$$\pi(\overline{\rho}) = \begin{array}{c} \frac{\mathrm{ind}_{KZ}^G(F(p-2,a+1))}{T-\lambda^{-1}} \\ \Big| \\ \frac{\mathrm{ind}_{KZ}^G(F(a,0))}{T-\lambda} \end{array}.$$

   Here, the notation means the unique nonsplit extension with the described subquotients.
   - On the other hand, if $\overline{\rho} = \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{a+1}$, then

$$\pi(\overline{\rho}) = \frac{\mathrm{ind}_{KZ}^G F(a,0)}{T}.$$

The cosocle filtration of $\pi(\overline{\rho})^{K_1}$ is computed by Morra.

- In the first case, it is

$$\begin{tikzcd} F(p-1,a) \arrow[dr, dash] & & F(p-2,a+1) \arrow[dl, dash] \\ & F(a,0) & \end{tikzcd} \quad \text{or} \quad \begin{array}{c} F(p-1,a) \\ \Big| \\ F(a,0) \end{array} \oplus \begin{array}{c} F(a+1,-1) \\ \Big| \\ F(p-2,a+1) \end{array}.$$

- In the second case, it is

$$\begin{array}{c} F(p-2,a+1) \\ \Big| \\ F(a,0) \end{array} \oplus \begin{array}{c} F(a-1,1) \\ \Big| \\ F(p-1,a) \end{array}.$$

We are interested in $\pi(\overline{\rho})^{I_1}$ too; there is an action of Π via conjugation and I_1-invariants of constituents get swapped by Π-conjugation.

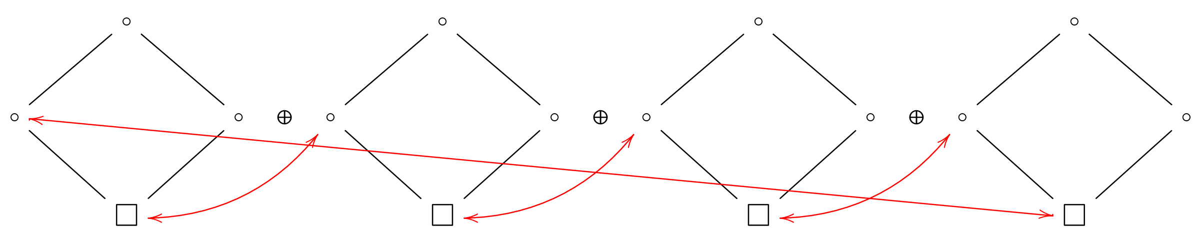
- In the first case, it is

$$\begin{tikzcd}[column sep=small]
F(p-1,a) \arrow[dr, dash] \arrow[dr, dash, dashed, bend right, "\Pi"'] & & F(p-2,a+1) \arrow[dl, dash] \\
& F(a,0) &
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
F(p-1,a) \arrow[d, dash] \arrow[d, dash, dashed, bend right, "\Pi"'] \\
F(a,0)
\end{tikzcd}
\oplus
\begin{tikzcd}
F(a+1,-1) \arrow[d, dash] \arrow[d, dash, dashed, bend left, "\Pi"] \\
F(p-2,a+1)
\end{tikzcd}.$$

- In the second case, it is

$$\begin{tikzcd}
F(p-2,a+1) \arrow[d, dash] & & F(a-1,1) \arrow[d, dash] \\
F(a,0) \arrow[rr, dash, dashed, bend right, "\Pi"'] & \oplus & F(p-1,a)
\end{tikzcd}.$$

For $F = \mathbb{Q}_{p^2}$, for $\overline{\rho} : G_F \longrightarrow \mathrm{GL}_2 \mathbb{F}$ an irreducible and generic Galois representation ($0 < a < p-3$), Buzzard expected what $\pi(\overline{\rho})^{K_1}$ should look like:



Here, all weights are explicitly given, and in particular, squared weights are in $W^{\mathrm{BDJ}}(\overline{\rho})$, the explicit expected set of weights predicted by Buzzard-Diamond-Jarvis. The red arrows are Π-conjugations. In particular, the expectation comes from global consideration.

The pro-p Iwahori Hecke algebra in this case has a presentation

$$H(G, I_1) = \mathbb{F}[N(I_1)/I_1, S],$$

where $S = \sum_{\lambda \in K} \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{F}[K]$. In the diagram, S acts by 0 on π^{I_1}. From this, Breuil-Paskunas were led to the following

Definition 1.1 (Breuil-Paskunas). *A* ***diagram*** *is $\mathscr{D}^{N(k)} \subset \mathscr{D}$ where $\mathscr{D}^{N(k)}$ has an involution by Π and $\mathscr{D}$ has a $\mathrm{GL}_2(k)$-action.*

For example, for a smooth G-representation π over $\mathbb{F}$, to which $\begin{pmatrix} p & \\ & p \end{pmatrix}$ acts trivially, $\pi^{I_1} \subset \pi^{K_1}$ is a diagram. We denote this diagram as $\mathscr{D}(\pi)$.

2. Global construction.

Let $\tilde{F}/\tilde{F}^+$ be a CM extension, and let $v \mid p$ in $\tilde{F}^+$ be unramified and split in $\tilde{F}$. Let $\mathfrak{G}$ be a definite unitary group over $\tilde{F}^+$.

Given an open compact subgroup $K^{v,\infty} \subset \mathfrak{G}(\mathbb{A}_{\tilde{F}^+}^{v,\infty})$ and an $\mathbb{F}[K^{v,\infty}]$-module L, consider the space of algebraic automorphic forms

$$S(K^v, L) = \{f : \mathfrak{G}(\tilde{F}^+ \backslash \mathfrak{G}(\mathbb{A}_{\tilde{F}^+}) \longrightarrow L \mid f(gk) = k^{-1} f(g) \text{ for } g \in \mathfrak{G}(\mathbb{A}_{\tilde{F}^+}), k \in K^v\} = \varinjlim_{K_v} S(K^v K_v, L).$$

Suppoer that we are given a modular Galois representation $\overline{r} : G_{\tilde{F}} \longrightarrow \mathrm{GL}_2(\mathbb{F})$. Denote $\overline{\rho} = \overline{r}|_{G_{\tilde{F}_{\tilde{v}}}}$, for a place $\tilde{v} \mid v$ in $\tilde{F}$. Let

$$\pi(\overline{\rho}) = S(K^v, L)[\mathfrak{m}],$$

which is naturally a G-representation.

Theorem 2.1 (Breuil-Diamond). *If $\overline{\rho}$ is generic, then $\mathscr{D}(\pi(\overline{\rho}))$ determines $\overline{\rho}$.*

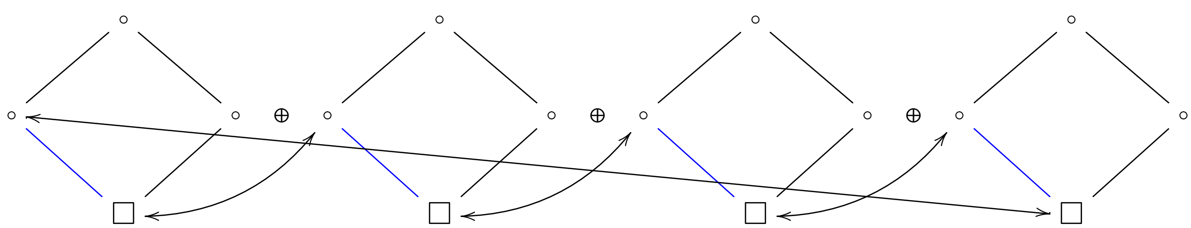
Theorem 2.2 (Hu-Wang, Le-Morra-Schraen, Hu, Dotto-Le). *If $\overline{\rho}$ is generic, then $\overline{\rho}$ determines $\mathscr{D}(\pi(\overline{\rho}))$. In particular, $\pi(\overline{\rho})^{K_1}$ is the maximal KZ-representation such that*

- $\mathrm{soc}_{KZ}\, \pi(\overline{\rho})^{K_1} = \bigoplus_{\sigma \in W^{\mathrm{BDJ}}(\overline{\rho})} \sigma$,
- *$\sigma \in W^{\mathrm{BDJ}}(\overline{\rho})$ appears as a Jordan-Holder factor of $\pi(\overline{\rho})^{K_1}$ with multiplicity 1.*

Remark 2.1. (1) In the aforementioned case of $\mathbb{Q}_{p^2}$, using

$$S' = \sum_{\lambda \in k} \lambda^s \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{F}[K],$$

π^{I_1} with $N(I_1)$-action is determined by the scalar action of $(S'\Pi)^4$, as you can see from the following diagram about the effect of S' (blue arrows) that you get back to the original weight after applying $S'\Pi$ four times:



(2) Breuil-Diamond gave a way to construct a (φ, Γ)-module out of diagram, $\mathscr{D}(\pi) \mapsto M(\mathscr{D}(\pi))$, so that via Fontaine's functor, $M(\mathscr{D}(\pi))$ corresponds to $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}_p}} \overline{\rho}^\vee$.

3. **Use of patching axioms.**

We list **patching axioms** which should be satisfied the patched module M_∞ (cf. Caraiani-Emerton-Gee-Geraghty-Paskunas-Shin). It has an $R_\infty[[K]]$-action, such that

- M_∞ is a finitely generated $S_\infty[[K]]$-module,
- if $\mathfrak{m}$ is the maximal ideal corresponding to $\overline{\rho} : G_F \longrightarrow \mathrm{GL}_2(\mathbb{F})$, then $(M_\infty/\mathfrak{m})^\vee \cong M_\infty[\mathfrak{m}]^\vee \cong \pi(\overline{\rho})$,
- $\mathrm{Supp}_{R_\infty} M_\infty(\sigma(\tau)^\circ) \subset \mathrm{Spec}\, R_\infty(\tau)$, where $\sigma(\tau)^\circ$ is the type corresponding to a tame inertial type, $R_\infty(\tau) = R_\infty \otimes_{R_{\overline{\rho}}^{\square}} R_{\overline{\rho}}^{\tau,\square}$ and $M_\infty(V) := \mathrm{Hom}_{W(\mathbb{F})[K]}(V, M_\infty^\vee)^\vee$,
- $M_\infty(V)$ is maximal Cohen-Macaulay over its support $Z(V) \subset \mathrm{Spec}\, R_\infty$,
- and $\dim_{\overline{\mathbb{Q}}_p} M_\infty(\sigma(\tau)^\circ) \otimes_{R_\infty} \overline{\mathbb{Q}}_p = 1$ for all $\overline{\mathbb{Q}}_p$-points of $\mathrm{Spec}\, R_\infty(\tau)$.

These axioms are properties that are expected to be satisfied by a module constructed out of Taylor-Wiles-Kisin patching (in a modern context). The following two easy algebraic lemmas will be the key of showing niceness of patched modules.

Lemma 3.1. *If M is finitely generated and maximally Cohen-Macaulay over a regular local ring R, then M is free over R.*

Proof. The condition and Auslander-Buchsbaum formula imply that $\mathrm{proj.\,dim}\, M = 0$ and thus the freeness of M over R. $\square$

Lemma 3.2. *If $0 \longrightarrow V \longrightarrow V_1 \oplus V_2 \longrightarrow V_3 \longrightarrow 0$ is an exact sequence of finite $W(\mathbb{F})[K]$-modules, such that $V_1 \longrightarrow V_3$ and $V_2 \longrightarrow V_3$ are surjective, $M_\infty(V_i)$ is cyclic (i.e. free over $Z(V_i)$) for $i = 1, 2, 3$, and $Z(V_1) \cap Z(V_2) = Z(V_3)$, then $M_\infty(V)$ is cyclic.*

Proof. This follows from

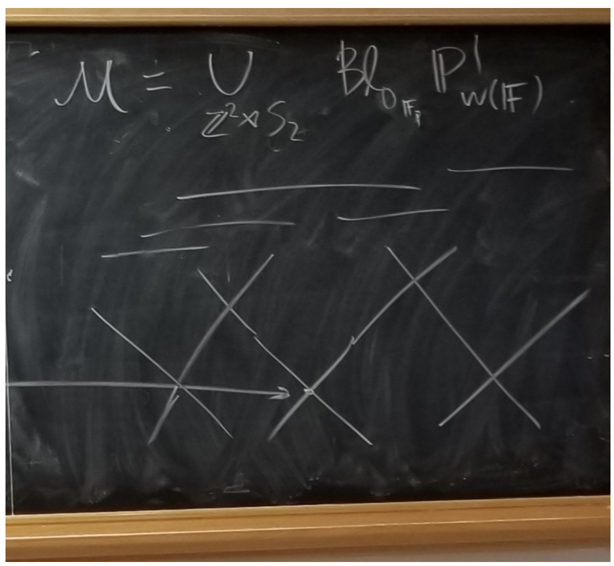
$$0 \longrightarrow R/I \cap J \longrightarrow R/I \oplus R/J \longrightarrow R/I + J \longrightarrow 0.$$

□

The main geometric input for deformation ring is

Theorem 3.1 (Le)**.** *If $\overline{\rho} : G_F \longrightarrow \mathrm{GL}_2(\mathbb{F})$ is generic, then there is $x(\overline{\rho}) \in \mathcal{M}(\mathbb{F})$ such that the tame Barsotti-Tate deformation ring for $\overline{\rho}$ is the local ring of $\mathcal{M}$ at $x(\overline{\rho})$, adjoined with some formal variables, where $\mathcal{M} = \bigcup_{\mathbb{Z}^2 \rtimes S_2} \mathrm{Bl}_{0_{\mathbb{F}_p}} \mathbb{P}^1_{W(\mathbb{F})}$ is the relevant local model.*

The local model is the result of gluing infinite copies of blow-ups of $\mathbb{P}^1_{W(\mathbb{F})}$ at a single point at the special fiber, so that the special fiber is just an infinite chain of $\mathbb{P}^1_{\mathbb{F}}$'s (horizontal lines above the chain of $\mathbb{P}^1_{\mathbb{F}}$'s depict the generic fiber which is just the disjoint union of countably many $\mathbb{P}^1_{W(\mathbb{F})[1/p]}$'s):



In particular, the worst singularity you can get is $W(\mathbb{F})[[x, y]]/(xy - p)$, which is still regular. Thus, we can apply the above lemmas and patching axioms to deduce that $M_\infty(V)$'s are cyclic, and thus multipliciy one (following Emerton-Gee-Savitt).

1. Motivation.

The A_{inf}-cohomology is a p-adic interpolation of existing cohomology theories. Let $K/\mathbb{Q}_p$ be a finite extension, $C = \widehat{\overline{K}} \supset \mathcal{O}_C \twoheadrightarrow k$. Consider a smooth proper formal scheme $\mathfrak{X}/\mathcal{O}_C$, with adic generic fiber X and special fiber $\mathfrak{X}_s$. We have following cohomology theories:

- $R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$, with filtration.
- $R\Gamma_{\text{ét}}(X, \mathbb{Z}_p)$.
- $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_s/W(k))$, with Frobenius.

We first review the crystalline site. Let (A, I, γ) be a PD ring, which means $\gamma_n : I \longrightarrow I$ does the work of "$\gamma_n(x) = \frac{x^n}{n!}$." For an A/I-algebra R, we define

$$\mathrm{CRIS}(R) = \left\{ \begin{array}{c} \begin{tikzcd} A \arrow[rr] \arrow[d] & & B \arrow[d] \\ A/I \arrow[r] & R \arrow[r] & B/J \end{tikzcd} \text{, where } (B, J, \gamma') \text{ is a} \\ \text{(pro-nilpotent) PD thickening} \end{array} \right\}.$$

With indiscrete topology, this defines a crystaline site, and the cohomology of the sheaf $\mathcal{O}_{\mathrm{cris}}$ on the crystalline site given by

$$\mathcal{O}_{\mathrm{cris}} : \quad \begin{tikzcd} & B \arrow[d] \\ R \arrow[r] & B/J \end{tikzcd} \mapsto B,$$

gives the crystalline cohomology, $R\Gamma_{\mathrm{cris}}(R/A)$.

2. Interpolation.

We want to interpolate the crystalline cohomology, using "$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$". Let $A_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat})$, and let $\mu := [\epsilon] - 1 \in A_{\mathrm{inf}}$ where $\epsilon = (1, \zeta_p, \zeta_{p^2}, \cdots) \in \mathcal{O}_{C^\flat}$. Let $p^\flat = [(p, p^{1/p}, \cdots)] \in A_{\mathrm{inf}}$, and $\xi = \frac{\mu}{\varphi^{-1}(\mu)} = \frac{[\epsilon]-1}{[\epsilon]^{1/p}-1} \in A_{\mathrm{inf}}$. Let $\theta : A_{\mathrm{inf}} \longrightarrow \mathcal{O}_C$, whose kernel is a principal ideal generated by ξ and also $p - p^\flat$.

Theorem 2.1 (Bhatt-Morrow-Scholze). *There is a perfect complex $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \in D(A_{\mathrm{inf}})$ such that*

1. $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}} \mathcal{O}_C \cong R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$,
2. $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})[1/\mu] \cong R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes A_{\mathrm{inf}}[1/\mu]$,
3. $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}} A_{\mathrm{cris}} \cong R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})$ *(note this is not the special fiber but just mod p reduction),*
4. $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}} W(k) \cong R\Gamma_{\mathrm{cris}}(\mathfrak{X}_s/W(k))$,
5. *the cohomology $H^*_{A_{\mathrm{inf}}}(\mathfrak{X})$ is valued in Breuil-Kisin-Fargues modules,*
6. *and if $\mathfrak{X}$ is defined over $\mathcal{O}_K$, then (3) would induces a (G_K, φ)-equivariant isomorphism*

$$R\Gamma_{\mathrm{cris}}(\mathfrak{X}_s/W(k)) \otimes B_{\mathrm{cris}} \xrightarrow{\sim} R\Gamma_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes B_{\mathrm{cris}}.$$

Remark 2.1.

- In Theorem 2.1, (3) is "the most difficult part," and it implies (4), (5), (6).
- The proof of Theorem 2.1(3) involves de Rham-Witt complexes. Bhatt-Lurie-Mathew gave an alternative proof of (4) by reinterpreting the de Rham-Witt complex and working backwards.

Our main goal is the following

Proposition 2.1 (Yao). *There is a functorial φ-equivarian map*

$$h : R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}/p}/A_{\mathrm{cris}}) \longrightarrow R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})\widehat{\otimes}^{\mathbb{L}} A_{\mathrm{cris}},$$

such that it is compatible with the de Rham comparison after $\otimes^{\mathbb{L}}_{A_{\mathrm{cris}}} \mathcal{O}_C$.

Remark 2.2.

- There is a similar construction over B^+_{dR}.
- Proposition 2.1 implies Theorem 2.1(4), (5), (6).
- There is a variant for "generalized semistable" $\mathfrak{X}$ using log formal scheme.
- If $\mathfrak{X}$ is smooth over $\mathcal{O}_C$, one can upgrade h to be an isomorphism.

3. A_{inf}-cohomology.

Consider the natural functor

$$\nu : X_{\mathrm{pro\acute{e}t}} \longrightarrow \mathfrak{X}_{\mathrm{\acute{e}t}}.$$

Then, the definition of A_{inf}-cohomology can be given as

$$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) := R\Gamma(\mathfrak{X}_{\mathrm{\acute{e}t}}, L\eta_\mu R\nu_* \mathbb{A}_{\mathrm{inf},X}).$$

We try to justify some notations. The pro-étale site $X_{\mathrm{pro\acute{e}t}}$ has typical objects consisted of "towers of finite étale covers." Given a tower

$$\cdots \longrightarrow \mathcal{U}_i \longrightarrow \mathcal{U}_{i-1} \longrightarrow \cdots \longrightarrow \mathcal{U}_0 \longrightarrow X,$$

where each $\mathcal{U}_i \longrightarrow \mathcal{U}_{i-1}$ is finite étale surjective and $\mathcal{U}_0 \longrightarrow X$ is étale, we say this is **affinoid perfectoid** if $\mathcal{U}_i = \mathrm{Spa}(R_i, R_i^+)$ and $(\varinjlim R_i^+)^\wedge_p[1/p] =: R$ is perfectoid. It is known that these objects form a basis for $X_{\mathrm{pro\acute{e}t}}$.

When we also consider the natural functor

$$\omega : X_{\mathrm{pro\acute{e}t}} \longrightarrow X_{\mathrm{\acute{e}t}},$$

we can define various sheaves,

- $\mathcal{O}_X^+ := \omega^* \mathcal{O}^+_{X_{\mathrm{\acute{e}t}}}$,
- $\widehat{\mathcal{O}}_X^+ := \varprojlim_r \mathcal{O}_X^+/p^r$,
- $\widehat{\mathcal{O}}_{X^\flat}^+ := \varprojlim_\varphi \mathcal{O}_X^+/p$,
- and $\mathbb{A}_{\mathrm{inf},X}$"="$W(\widehat{\mathcal{O}}^+_{X^\flat})$ (using derived completions).

Now we define the decalage operator.

Definition 3.1. *Given a ring A, a non-zerodivisor $f \in A$ and an f-torsionfree cochain complex $M^\bullet$, the complex $\eta_f M^\bullet \subset M^\bullet[1/f]$ is defined by*

$$(\eta_f M^\bullet)^i = \{x \in f^i M^i \mid dx \in f^{i+1} M^{i+1}\}.$$

Then, one can derive this functor to get the ***decalage operator*** *$L\eta_f : D(A) \longrightarrow D(A)$. This preserves algebra objects.*

In particular, $H^i(\eta_f M^\bullet) = H^i(M^\bullet)/H^i(M^\bullet)[f]$.

4. **Construction of h in Proposition 2.1.**
It suffices to construct, for Spf $R \subset \mathfrak{X}_{\text{ét}}$,

$$h_R : R\Gamma_{\text{cris}}((R/p)/A_{\text{cris}}) \longrightarrow A\Omega_R \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}},$$

where $A\Omega_R = R\Gamma(\text{Spf } R, L\eta_\mu R\nu_* \mathbb{A}_{\text{inf}})$. The idea is to work locally with nice quotients of perfectoid rings, **quasiregular semiperfectoid rings**. This is the natural analogue of locally complete intersection condition in the world of perfectoids.

Example 4.1. Instead of giving the definition of quasiregular semiperfectoid rings, we give some examples.

- $\mathcal{O}_C/p = \mathcal{O}_{C^\flat}/p^\flat$.
- $\overline{\mathbb{F}}_p[[X^{1/p^\infty}, Y^{1/p^\infty}]]/(X - Y) = \overline{\mathbb{F}}_p[[X^{1/p^\infty}]]\widehat{\otimes}_{\mathbb{F}_p[X]}\overline{\mathbb{F}}_p[[X^{1/p^\infty}]]$.
- $\mathcal{O}_C\langle X^{\pm 1/p^\infty}\rangle/(X - 1)$.

We work on a slightly larger category, namely **quasisyntomic rings** over $\mathcal{O}_C$. This is the analogue of locally complete intersection without finite-typeness conditions. This makes sense because there is still a cotangent complex so you can define the notion with Tor-amplitude in $[-1, 0]$.

Lemma 4.1 (Bhatt-Morrow-Scholze)**.** *The category of quasiregular semiperfectoid rings over $\mathcal{O}_C$ forms a basis for the category of quasisyntomic rings over $\mathcal{O}_C$.*

Lemma 4.2. *The functor $R \mapsto A\Omega_R \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$ forms a sheaf from the category of quasisynotmic rings over $\mathcal{O}_C$ to $D(A_{\text{cris}})$.*

This means that

$$A\Omega_R \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} = \text{holim}_{R \to S, S \text{ quasiregular semiperfectoid}} A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}.$$

Thus, it suffices to construct h_R to $A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$.

Lemma 4.3. *(1) $A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$ is an algebra and topologically free over A_{cris}.*
(2) The natural map

$$A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} \longrightarrow \mathbb{L}\Omega_{S/\mathcal{O}_C} \longrightarrow S \longrightarrow S/p,$$

is a PD thickening.

This means that this lies in CRIS(S/p). As $A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$ is "affine", we get the natural map

$$R\Gamma_{\text{cris}}(R/A_{\text{cris}}) \longrightarrow A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}.$$

5. **Crystalline comparison.**
From this we can quickly prove the crystalline comparison, Theorem 2.1(4). From

$$h_R : R\Gamma_{\text{cris}}(R/A_{\text{cris}}) \longrightarrow A\Omega_R \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}},$$

we get the candidate for crystalline comparison map,

$$h_R \widehat{\otimes}^{\mathbb{L}} W(k) : R\Gamma_{\text{cris}}(R_s/W(k)) \longrightarrow A\Omega_R \widehat{\otimes}^{\mathbb{L}} W(k).$$

By derived Nakayama's lemma, we only need to check after $\otimes^{\mathbb{L}}_{W(k)} k$. But the diagram

$$\begin{tikzcd} A_{\mathrm{cris}} \arrow[r] \arrow[d] & \mathcal{O}_C \arrow[d] \\ W(k) \arrow[r] & k \end{tikzcd}$$

commutes, and by de Rham comparison, $h_R \otimes^{\mathbb{L}} \mathcal{O}_C$ is an isomorphism. Thus, $h_R \widehat{\otimes}^{\mathbb{L}} k$ is an isomorphism, as desired.

1. **Main result.**

Let $K/\mathbb{Q}_p$ be a finite extension, and let Δ be a finite set (e.g. the set of simple roots; see Gergely Zabradi's talk). Let $G_{K,\Delta} := \prod_{\alpha\in\Delta} G_K$.

Theorem 1.1 (Main theorem, preliminary statement)**.** *The category of continuous representations of finite $\mathbb{Z}_p$-modules is equivalent to the category of "(φ,Γ)-modules over some rings." Also, a similar statement for representations on finite-dimensional $\mathbb{Q}_p$-vector spaces.*

2. **Classical picture.**

Let us recall the classical story when $|\Delta| = 1$. When $K = \mathbb{Q}_p$, everything becomes extremely explicit. Namely we have a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{\mathcal{E}}^{\dagger} \arrow[r, hook] \arrow[d] & \mathcal{O}_{\mathcal{E}} \arrow[d] \\ \widetilde{\mathcal{O}}_{\mathcal{E}}^{\dagger} \arrow[r, hook] & \widetilde{\mathcal{O}}_{\mathcal{E}} \end{tikzcd}$$

where

- $\mathcal{O}_{\mathcal{E}} = \mathbb{Z}_p((\pi))^{\wedge}$, the p-adic completion,
- $\mathcal{O}_{\mathcal{E}}^{\dagger} \subset \mathcal{O}_{\mathcal{E}}$ is a subring consisted of formal Laurent series on π convergent for $1-\varepsilon < |\pi| < 1$ for some $\varepsilon > 0$,
- both $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{O}_{\mathcal{E}}^{\dagger}$ have actions of $\varphi(\pi) = (1+\pi)^p - 1$ and, for all $\gamma \in \mathbb{Z}_p^{\times}$, $\gamma(\pi) = (1+\pi)^{\gamma} - 1$,
- $\widetilde{\mathcal{O}}_{\mathcal{E}} = W(\mathbb{F}_p((t)))^{\mathrm{perf},\wedge} = \left\{\sum_{n=0}^{\infty} p^n[\overline{x}_n] \mid \overline{x}_n \in \mathbb{F}_p((\pi))\right\}$, the p-adic completion of perfect closure, so that $\mathcal{O}_{\mathcal{E}} \longrightarrow \widetilde{\mathcal{O}}_{\mathcal{E}}$ is defined by $1+\pi \longmapsto [1+\pi]$, the Teichmuller representative, and
- $\widetilde{\mathcal{O}}_{\mathcal{E}}^{\dagger} \subset \widetilde{\mathcal{O}}_{\mathcal{E}}$ is the subring consisted of $\sum p^n[\overline{x}_n]$ with $v_\pi(\overline{x}_n) \geq -an - b$ for some $a, b > 0$, which satisfies $\widetilde{\mathcal{O}}_{\mathcal{E}}^{\dagger} \cap \mathcal{O}_{\mathcal{E}} = \mathcal{O}_{\mathcal{E}}^{\dagger}$.

Definition 2.1. *A (φ,Γ)-module over $\mathcal{O}_{\mathcal{E}}^{(\dagger)}$ or $\widetilde{\mathcal{O}}_{\mathcal{E}}^{(\dagger)}$ is a finite module M over that ring plus semilinear actions of φ, Γ which commute, i.e $\varphi^*M \xrightarrow{\sim} M$ and $\gamma^*M \xrightarrow{\sim} M$ for all $\gamma \in \Gamma$.*

Theorem 2.1 (Fontaine, Cherbonnier-Colmez)**.** *If $|\Delta| = 1$ and $K = \mathbb{Q}_p$, then $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ is equivalent to the categories of (φ,Γ)-modules over these rings.*

If $K \neq \mathbb{Q}_p$, then by induction process, one formally gets a similar statement where all rings are replaced with some finite étale extensions and Γ is replaced with Γ_K, the image of $\mathrm{Gal}(K(\mu_{p^\infty})/K)$ in $\mathbb{Z}_p^{\times}$ via cyclotomic character.

Remark 2.1. The trickiest part is $\mathcal{O}_{\mathcal{E}}^{\dagger}$, which needs Lazard's theory of analytic group actions (i.e. the p-adic Lie group Γ acting on $\mathcal{O}_{\mathcal{E}}^{\dagger}$).

3. **Multivariable picture.**

Now we want to mimic the classical story with $|\Delta|$-many variables. We can define an analogous diagram

$$\begin{tikzcd}
\mathcal{O}^{\dagger}_{\mathcal{E},\Delta} \arrow[r, hook] \arrow[d] & \mathcal{O}_{\mathcal{E},\Delta} \arrow[d] \\
\widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E},\Delta} \arrow[r, hook] & \widetilde{\mathcal{O}}_{\mathcal{E},\Delta}
\end{tikzcd}$$

For example, we define $\mathcal{O}_{\mathcal{E},\Delta} = \left(\bigotimes_{\alpha\in\Delta,\mathbb{Z}_p} \mathcal{O}_{\mathcal{E},\alpha}\right)^{\wedge}$, where the completion is taken with respect to **weak topology**.

Example 3.1. If $K = \mathbb{Q}_p$, then $\mathcal{O}_{\mathcal{E},\Delta}/(p^r) = \mathbb{Z}/p^r\mathbb{Z}[[\pi_\alpha : \alpha \in \Delta]][\pi_\alpha^{-1} : \alpha \in \Delta]$. We are thus completing with respect to some valuation that considers every π_α's. For each $\alpha \in \Delta$, there are corresponding φ_α-action and Γ_α-action, only acting on π_α and not other variables.

Then, Theorem 1.1 says that $\mathrm{Rep}_{\mathbb{Z}_p}(G_{K,\Delta})$ is equivalent to the category of multivariable $(\varphi_\Delta, \Gamma_\Delta)$-modules over these four rings.

4. **Components of proofs.**

One proves the case of $\widetilde{\mathcal{O}}_{\mathcal{E},\Delta}$ using perfectoid spaces, and deduce the Theorem for other rings using this case. The latter step can be done in a similar way that is done in the proof of the classical setting of $|\Delta| = 1$.

To prove the case of $\widetilde{\mathcal{O}}_{\mathcal{E},\Delta}$, WLOG we can now assume that we are in the torsion case. When $|\Delta| = 1$, this case is proved first by Fontaine-Wintenberger, who proved that $G_{K(\mu_{p^\infty})} \cong G_{K(\mu_{p^\infty})^\flat}$. Then the rest follows using the so-called nonabelian Artin-Schreier-Witt theory or Lang's thesis; see Katz in SGA 7.

To do this for the case of $|\Delta| > 1$, we first use induction to reduce to the case of $|\Delta| = 2$. Then, we need to relate

$$\pi_{1,\text{ét}}(\operatorname{Spa} F_1) \times \pi_{1,\text{ét}}(\operatorname{Spa} F_2) \cong \pi_1\left(\text{“}\frac{\operatorname{Spa} F_1 \times \operatorname{Spa} F_2}{\varphi_1}\text{”}\right),$$

where F_1, F_2 are equicharacteristic perfectoid fields, and φ_1 is the "partial Frobenius" acting as Frobenius only at the first factor. This is because the tilting can only handle the product of the two partial Frobenii.

Remark 4.1. This reminds us the so-called "Drinfeld's lemma", which is an analogous statement for schemes over $\mathbb{F}_p$.

Remark 4.2. In fact, "$\frac{\operatorname{Spa} F_1 \times \operatorname{Spa} F_2}{\varphi_1}$" is the Fargues-Fontaine curve for F_1 with coefficients in F_2.

To prove this, we first reduce to the case where F_1, F_2 are algebraically closed. Then, the case of $F_1 = \mathbb{C}_p$ reduces to the classification of vector bundles on Fargues-Fontaine curves, which was done by Weinstein (when $F_2 = \mathbb{C}_p$), Fargues, Scholze.

To enlarge F_1, we need an auxiliary argument using "convergence polygons" for p-adic differential equations. Roughly speaking, one proceeds step by step by adding one transcendental variable at a time to F_1, and one can view this as a problem regarding a connection on a piece of a relative curve. Now one uses that the Fargues-Fontaine curves are "proper" to show that the convergence polygons behave in some uniform way. This way one can reduce the problem to the case of abelian covers where one can use Artin-Schreier theory.