A QUOI SERVENT LES MOTIFS? (WHAT'S THE POINT OF MOTIVES?)

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- All started with standard conjectures.
- First approach: instead of building cycles, try to build vector bundles and take their Chern classes.
- Two obstructions to Hodge conjecture:
  (1) Atiyah-Hirzebruch showed Hodge conjecture is false integrally.
  (2) (From Hodge theory) Natural living place of codimension d cycle is **NOT** $H_{\mathbb{Z}}^{d,d}$, but its **extension by the intermediate Jacobian** $J^d(X) := H^{2d-1}(X, \mathbb{C})/F^d + H^{2d-1}(X, \mathbb{Z})$. To make sure, we review why. Recall that $H^{2d-1}(X, \mathbb{R}) \to H^{2k-1}(X, \mathbb{C})/F^k H^{2k-1}(X)$ is an isomorphism, where the filtration F^i is "holomorphic part has degree $\geq i$". As $\mathrm{rk}_{\mathbb{Z}} H^{2k-1}(X, \mathbb{Z}) = \dim_{\mathbb{R}} H^{2k-1}(X, \mathbb{R})$, this defines a full lattice L_k inside the complex vector space $V_k = H^{2k-1}(X, \mathbb{C})/F^k H^{2k-1}(X)$. Then $J^k(X) = V_k/L_k$. By Poincare duality we can instead see this as $J^k(X) = F^{n-k+1}H^{2n-2k+1}(X)^\vee / H_{2n-2k+1}(X, \mathbb{Z})_{torsfree}$. Now given a homologically trivial codimension k-cycle Z, there is a chain C_Z of dimension $2n-2k+1$ such that $\partial C_Z = Z$. Integrating over C_Z gives a functional $A^{2n-2k+1}(X)^\vee$. Now does not descend to an element of $H^{2n-2k+1}(X, \mathbb{C})^\vee$, but for an exact form $\omega = d\psi$, $\int_{C_Z} \omega = \int_Z \psi$ is zero unless ψ is a $(n-k, n-k)$-form as Z is a collection of complex manifolds with boundary of complex dimension $n-k$. Thus it really descends to an element of $F^{n-k+1}H^{2n-2k+1}(X, \mathbb{C})^\vee$ as $F^{n-k+1}H^{2n-2k+1}(X, \mathbb{C}) = \frac{F^{n-k+1}A^{2n-2k+1}(X) \cap \ker d}{dF^{n-k+1}A^{2n-2k}(X)}$. Ambiguity of choice of cycle from class gives an ambiguity of $H_{2n-2k+1}(X, \mathbb{Z})_{torsfree}$, so we get the desired **Abel-Jacobi map**. This factors through the Chow group of homologously trivial cycles (i.e. rationally trivial cycles are sent to zero).
- Over a field k, we want $\mathcal{M}(k)$, the category of pure k-motives, to be a $\mathbb{Q}$-linear semisimple graded (by weights) abelian category with finite dimensional Hom groups. Also, we want $\mathcal{MM}(k)$, the category of mixed k-motives, to be a $\mathbb{Q}$-linear abelian category where each mixed motive M admits an increasing filtration W_*, **weight filtration**, so that $\mathrm{gr}_n^W(M)$ is pure of weight n.
- An algebraic variety X over k must have **motivic cohomology group** $H^i_{\mathcal{M}}(X)$ as an object of $\mathcal{MM}(k)$. If X is nonsingular and projective over k, it should be pure of weight i.
- For each of the usual cohomology theories H, there must be a **realization functor** and **comparison isomorphism**.
- Realization functors must be compatible with tensor products. This opens up a possibility of using Tannakian formalism. In particular if everything is true one can find an affine $\mathbb{Q}$-group G, **motivic Galois group** such that the category of pure k-motives are isomorphic to the category of linear representations of G. Note however that inner forms define the same representation category, so there are different possible choices of G. Thus the motivic Galois group is about the choice of G as well as the equivalence $\mathcal{M}(k) \cong \mathrm{Rep}(G)$, which is the same as picking a **fiber functor** ω of $\mathcal{M}(k)$, an exact functor from $\mathcal{M}(k)$ to the category of $\mathbb{Q}$-vector spaces, compatible with $\otimes$. Then ω gives G via $G = \mathrm{Aut}^{\otimes}(\omega)$. So $\omega \circ H_{\mathcal{M}}$ is a

choice of cohomology theory. Caveat: ℓ-adic étale cohomology is not a $\mathbb{Q}$-valued cohomology theory..

- For any full subcategory $\mathcal{M}$ of $\mathcal{MM}(k)$ closed under $\otimes$, dual and subquotients, the functor $M \mapsto \mathrm{gr}^W(M)$ from $\mathcal{M}$ to the full subcategory of $\mathcal{MM}(k)$ generated by direct sums of pure motives gives rise to the **motivic pro-unipotent group scheme** U whose representation category (valued in the category of (direct sums of) pure motives over k) is equivalent to $\mathcal{M}$.
- The **abelianisation of a motivic Galois group** $G_{\mathcal{M}}^{\mathrm{ab}}$ is conjectured to be independent of choice of realization. Also, for any $k_1 \subset k_2$ two algebraically closed fields, the $G_{\mathcal{M}}^{\mathrm{ab}}$'s are conjectured to stay the same. Also the corresponding subcategory is conjectured to be generated by $H^1_{\mathcal{M}}$ of CM abelian varieties.
- A **1-motive** K^* is a complex of reduced group schemes in degrees $[-1, 0]$ with, over the algebraic closure, K^{-1} is a free finite type $\mathbb{Z}$-module and K^0 is an extension of abelian variety by a torus. We can produce 1-motive as follows. Let $\overline{X}$ be a projective smooth connected curve over $k = \overline{k}$, and S, T disjoint finite subsets of $\overline{X}$. Let $J_T(\overline{X})$ be the generalized Jacobian classifying invertible sheaves of degree zero over $\overline{X}$ with a trivialization over T. It is an extension of an abelian variety $\mathrm{Pic}^0(\overline{X})$ by the torus $\ker(\mathbb{Z}^T \xrightarrow{\text{add}} \mathbb{Z})$. Each $s \in S$ induces an invertible sheaf $\mathcal{O}(s)$, trivialized over T and of degree 1, so we have a map $\ker(\mathbb{Z}^S \to \mathbb{Z}) \to J_T(\overline{X})$. This is a 1-motive.

Archimedean contemplation.

- Let S be a $\mathbb{C}$-scheme and M be a motive "parametrized by S". Its archimedean realization should give you a variation of Hodge structures over $S(\mathbb{C})$, polarized if M is. There you get a map $\varphi : S(\mathbb{C}) \to \mathscr{C}$ where $\mathscr{C}$ is the moduli space of some type of polarized Hodge structures (of Hodge numbers those of M_{Hodge}). The VHS gives a vector bundle $M_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces with integrable connection together with a continuously varying Hodge filtration F^*. The holomorphy and Griffiths transversality conditions can be seen in this context as follows. Let $\varphi : TS \otimes_{\mathbb{R}} \mathbb{C} \to TS$ be the $\mathbb{C}$-linear extension of the identity map, with the complex structure on TS coming from that of S. Let $T'' \subset TS \otimes_{\mathbb{R}} \mathbb{C}$ be $\ker \varphi$. Define the Hodge filtration of $TS \otimes_{\mathbb{R}} \mathbb{C}$ as $F^{-1} = TS \otimes_{\mathbb{R}} \mathbb{C}$, $F^0 = T''$, $F^1 = 0$. For t and m C^∞-sections of $TS \otimes_{\mathbb{R}} \mathbb{C}$ and $M_{\mathbb{C}}$, respectively, $\nabla_t m$ expresses holomorphy and transversality: holomorphy is $\nabla_t m \in F^i$ for $t \in T''$, $m \in F^i$, and transverality is $\nabla_t m \in F^{i-1}$ for $m \in F^i$.

  In terms of the map $\varphi : S(\mathbb{C}) \to \mathscr{C}$, there exists a complex structure on $\mathscr{C}$ and holomorphic distribution $\tau \subset TS$ such that holomorphy + Griffiths transversality is holomorphy of φ + tangency of φ wrt τ. The distribution τ is in general not integrable and is quite mysterious. When $\tau = TS$ though, $\mathscr{C}$ is an arithmetic quotient of a Hermitian symmetric domain. So $\mathbb{C}$-points of Shimura varieties is a moduli of Hodge structures, which suggests that a Shimura variety should be a moduli of motives! More specifically, given a Shimura variety $\mathrm{Sh}_K(G, X)$ and a field $F \supset E(G, X)$, an F-point of $\mathrm{Sh}_K(G, X)$ should correspond to an exact $\otimes$-functor $x : \mathrm{Rep}(G) \to \mathcal{M}(F)$ and its integral structure, which means there is an isomorphism of $\otimes$-functors $x(V)_{\mathbb{A}^f} \xrightarrow{\sim} V \otimes \mathbb{A}^f$ compatible with composition of an element of K. If we embed $F \hookrightarrow \mathbb{C}$ we should get a realization a VHS, independent of the embedding. De Rham realization, see weights, ...

It warns that, over a "trait" (S, η, s) (S Spec of dvr), an abvar over η with semistable reduction admits a rigid analytic description as a cokernel of a defining morphism of 1-motive. In paralle, in Hodge theory, if H is a PVHS with type $(1,0),(0,1)$, over a punctured disc D^*, the monodromy provides a weight filtration which makes H a VMHS around 0. In general this kind of nontrivial transversality behavior can't be expected to be seen in the theory of motives (?). In general these kinds of "asymptotic behavior" only exists over a tangent space (e.g. Schmid's asymptotic nilpotent orbit). Saito's theory of mixed Hodge modules is in part inspired by the ℓ-adic theory and the motivic point of view. Conversely, it suggests that if we consider motives over a base S, it might be better to consider realizations as perverse sheaves. It is only setting that one can expect weight filtration.

Absolute cohomology.

- One can think everything in the derived category. Note: not every triangulated category with a t-structure is the derived category of its heart. Anyways let's assume the existence of a triangulated category $\mathcal{DM}(k)$ with a t-structure whose heart is $\mathcal{MM}(k)$ (as remarked $\mathcal{DM}(k)$ may not be $D^b(\mathcal{MM}(k))$), equipped with realization functors, and also the operation of Tate twist which is compatible with Tate twists of each realization. Finally suppose $\mathcal{M}(k)$ is Tannakian and the ℓ-adic cohomology is a fiber functor. It implies that if $\varphi : M_1 \to M_2$ is a morphism of motives whose ℓ-adic realization is an isomorphism, then φ is an isomorphism too. Denote $\mathrm{Ext}^i(M_1, M_2) = \mathrm{Hom}_{\mathcal{DM}(k)}(M_1, M_2[i])$. This is the same Ext for $i = 0, 1$, and may be different for $i > 1$ precisely because $\mathcal{DM}(k)$ may be different from $D^b(\mathcal{MM}(k))$.
- We can now define the **absolute cohomology**. Let X/k be an algebraic variety. Then $H^i_{abs}(X, \mathbb{Q}(j)) := \mathrm{Hom}_{\mathcal{DM}(k)}(1, R\Gamma_{\mathcal{M}}(X)(j)[i])$. This is also called motivic cohomology group? Oh yeah, because here $H^i_{\mathcal{M}}$ is motive-valued and H^i_{abs} is really vector space valued. It is surely different from $H^i_{\mathcal{M}}(X)$. In particular there is a spectral sequence
$$E_2^{p,q} = \mathrm{Ext}^p(1, H^q_{\mathcal{M}}(X)) \Rightarrow H^{p+q}_{abs}(X),$$
or
$$E_2^{p,q} = H^p_{abs}(H^q_{\mathcal{M}}(X)) \Rightarrow H^{p+q}_{abs}(X).$$
The ℓ-adic analogue is
$$E_2^{p,q} = H^p(\mathrm{Gal}(\overline{k}/k), H^q_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)) \Rightarrow H^i_{\text{ét}}(X, \mathbb{Q}_\ell).$$
The ℓ-adic realization functor is a fiber functor, so the Hard Lefschetz for ℓ-adic étale cohomology will give you $c_1(\mathcal{L})^i : H^{N-i}_{\mathcal{M}}(X) \to H^{N+i}_{\mathcal{M}}(X)(i)$ an isomorphism, given that there is "motivic $c_1(\mathcal{L})$." This will give you $R\Gamma_{\mathcal{M}}(X) \cong \oplus H^i_{\mathcal{M}}(X)[-i]$ noncanonically. This is essentially a rational statement, so it is not expected to hold integrally.
- Given a codimension d algebraic cycle Z of k-smooth variety X, $[Z]$ should lie in $H^{2d}_{abs}(X, \mathbb{Q}(d))$, and this only depends on the linear equivalence class. If further X is projective, using the noncanonical decomposition, one could split the absolute cohomology to graded pieces of the motivic spectral sequence, $H^n_{abs}(H^{2d-n}_{\mathcal{M}}(X)(d))$.
  - Abel-Jacobi map is really an example of "going one step further through the filtration". Given any cohomology theory, the analogue of relative-to-absolute motivic spectral sequence shows that at least there is a map from Chow cycles to cohomology group $H^{2d}(X, \mathbb{Q}(d))$, which is the last graded piece of the filtration (quotient of the absolute cohomology). This is the **cycle class map**. If its image is zero (**homologically trivial**), then its image in the absolute cohomology lies in the next step of the filtration and one can go to the next graded piece, $\mathrm{Ext}^1(\mathbb{Z}, H^{2d-1}(X)(d))$.
- If X is smooth over a field, the K-groups can be decomposed by eigenspaces of the Adams operations, $K_n(X)_{\mathbb{Q}} = \bigoplus K_n(X)^{(j)}$. Here the Adams operation Ψ_k acts on $K_n(X)^{(j)}$ as multiplication by k^j. For example $K_0(X)^{(j)}$ is the Chow cycles of codimension j ($\mathbb{Q}$-coefficients). There should also be **Chern classes** $\mathrm{ch}^j : K_n(X)_{\mathbb{Q}} \to H^{2j-n}_{abs}(X, \mathbb{Q}(j))$ factoring through $K_n(X)^{(j)}$. The best hope is ch^j gives an isomorphism $K_n(X)^{(j)} \cong H^{2j-n}_{abs}(X, \mathbb{Q}(j))$. Bloch has suggested interpretations of $K_n(X)^{(j)}$ as "higher Chow groups".

- Beilinson suggested that this "best hope conjecture" will follow if we assume that (a) the functor $X \mapsto K_*(X)$ factors through $R\Gamma_{\mathcal{M}}$, and (b) $\mathcal{DM}(k) = D^b(\mathcal{MM}(k))$. These will give striking vanishings like $\mathrm{Hom}^i(1, M) = 0$ for M weight w and $i > -w$ and $\mathrm{Hom}^i(1, M(w + b)) = 0$ for M weight w, $b > 0$ and $i > w + b$.

Seems like absolute cohomology theories are:

- ℓ-adic: absolute étale cohomology (no base-change to algebraic closure of the base field)
- p-adic: syntomic cohomology
- Archimedean: "absolute Hodge cohomology" or whatnot.