LEARNING SEMINAR ABOUT COHERENT SHEAVES ON MODULI SPACE OF LANGLANDS PARAMETERS

GYUJIN OH

We would like to learn about the recent development on the moduli spaces of global/local Langlands parameters and coherent sheaves on them. These are expected to encode global/local Langlands correspondences. Some of the expectations are as follows.

- The formal completion at a point of the moduli of local parameters is identified with the deformation space.
- For each open compact subgroup $G(F)$ for a local field F, there is a coherent sheaf A_K on the moduli of local Langlands parameters, such that $\mathrm{End}\, A_K$ is identified with the Hecke algebra for K.
- The coherent sheaf for the Iwahori K has an interpretation analogous to the Springer theory.
- The restriction of A_K to the formal completion at a point is identified with a "patching module" over a Galois deformation ring, constructed using the patching method in automorphic lifting theorems.
- For each level $K \subset G(\mathbb{A})$, there is a coherent sheaf A_K on the moduli of global Langlands parameters, such that its cohomology computes the cohomology of the Shimura variety/moduli of shtukas corresponding to the level.
- The global and local pictures satisfy the local-global compatibility, in that $A_{\prod K_v} = f^! \left(\boxtimes_v A_{K_v}\right)$, where $f : X \to \prod_v X_v$ is the natural map from the moduli of global Langlands parameters to the product of moduli of local Langlands parameters.

The main reference is [Zh], which discusses both the local and global conjectures. There are more references about the local conjectures, including [He], [DHKM], [BCHN], [FS].

Talk 1: Springer theory and Kazhdan–Lusztig's proof of the Deligne–Langlands conjecture ([KL], [CG]).
State in a way that the results fit into the context, cf. [BCHN, §1.1-1.2]. Speaker: Marco.

Talk 2: Review of local representation theory ([EH, §2-§4]).
Bernstein centers/components, (super)cuspidal support, Bernstein–Zelevinsky theory, modified local Langlands correspondence. Speaker: Leo.

Talk 3: The works of [EH] and [Hel].
Summarize the properties (and maybe the construction) of the local Langlands correspondence in families. We only need to understand to the extent where they can be used in [He, §4]. Speaker: Eric.

Talk 4: The local conjectures, following [He].

Briefly review the geometry of moduli of tame L-parameters ([He, §2]) and state the conjectures, [He, Conjecture 3.2, 3.6]. In particular, explain how [EH] gives an answer in the Iwahori level case for $G = \mathrm{GL}_n$. Speaker: Sam.

Talk 5: Categorical archimedean local Langlands correspondence.
Discuss the result of [BN] in the context of the seminar, and relate it with another more well-known geometric formulation of archimedean local Langlands, most notably [ABV], [So2]. Discuss why this is related to Koszul duality, [BGS]. Relate it with [Zh, §4.6]. Speaker: Gyujin.

Talk 6: Construction of moduli of Langlands parameters.
Follow [Zh, §2-§3] and/or [FS], arguing as formally as possible. Show their geometric properties, e.g. dimension, flatness, local complete intersection. One may try to relate to a more classical approach taken by [DHKM]. Speaker: Gyujin.

Talk 7: The local conjectures revisited, in the context of [Zh, §4.1-§4.5].
Speaker: Gyujin.

Talk 8: Review of local/global Langlands correspondence over function field, following [La] and [GL].
Speaker: Eric.

Talk 9: The work of [LZ], its relation with Kottwitz conjectures, and its generalization in [Zh, §4.7].
Speaker: Linus.

Talk 10: State the number field version of the global conjecture [Zh, §4.7], and relate it with various existing works, including [XZ] and the patching construction (cf. [CEGGPS]).
Speaker: Lue.

Possible extra topics include:

- Relation to geometric Langlands [AGKRRV].
- Relation to Emerton–Gee stack [EG1], [EG2].
- Relation to Fargues–Scholze [FS].
- Proof of Emerton–Helm conjecture using integral Bernstein center [HM].
- Proof of categorical local Langlands for GL_n as in [BCHN, §6].
- Proof of Soergel's conjecture as in [BV], Soergel bimodules [So1].

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