REDUCED GENERALIZED GELFAND–GRAEV REPRESENTATIONS, AND RANKIN–SELBERG INTEGRALS WITH NON-UNIQUE LOCAL MODELS

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Eulerianity of some period integral is hard to show because naive multiplicity one fails for "degenerate" Whittaker models. If the degenerate Whittaker model is more degenerate than the "wave-front of the representation", the local model is infinite-dimensional (Moeglin–Waldspurger). However, there has been a Japanese school of mathematicians (Yamashita, Oda, Shintani, Murase, Sugano, $\cdots$) who realized that one can salvage the situation by considering the action of unipotent group as well as its certain extension by a reductive group. The extension lies between the corresponding parabolic and its unipotent radical. This yields new period integrals with Eulerian property (our example being the GGP period integral for $\mathrm{U}(2,1)\times\mathrm{U}(1,1)$) as well as an explanation of certain period integrals that have Eulerianity but were not known to have local models with multiplicity one property (our example being the Kohnen–Skoruppa integral for the spin L-function of Sp_4).

1. Generalities on the "RGGGR"s

Here RGGGR stands for *Reduced Generalized Gelfand–Graev Representations*, a terrible name due to Yamashita. Recall that Gelfand–Graev representations are of form $\mathrm{Ind}_N^G \chi$ in the context of finite groups of Lie type which is why the name appears in our discussion on salvaging degenerate Whittaker models.

The usual degenerate Whittaker models for a local representation π is $\mathrm{Hom}_N(\pi|_N, \psi)$ for a unipotent subgroup $N \subset G$ and a character ψ of N. The problem is that if N is smaller than what appears on the wave-front set of π (cf. [GGS]), this space is infinite-dimensional. On the other hand, let $P = L \ltimes N$ be the Levi decomposition of a parabolic subgroup. Let $S \subset L$ be the centralizer of $[\psi] \in \widehat{N}$; then, ψ extends to a representation of $R = S \ltimes N$, which we also denote by ψ. One can do exactly the same for a more general unitary representation of N. Finally, let $\rho \in \widehat{S}$.

Slogan. The RGGGR $\pi_{\rho,\psi} = \mathrm{Ind}_R^G(\rho \otimes \psi)$ has multiplicity one property, namely $\dim_G(\pi, \pi_{\rho,\psi})^\circ \leq 1$.

The superscript $\circ$ indicates some extra care should be imposed when we are dealing with real group representations. Recall that, even in the classical theory of Jacquet, the space of Whittaker functionals is of dimension larger than one; but *all but one dimension are spanned by Whittaker functionals of exponential growth.* So over the reals one can expect that one has to impose an appropriate moderate growth condition. We exhibit how to use these models arise in some period integrals via the theory of *Fourier expansions along closed subgroups* (cf. [Od]), which, formally speaking, is just the decomposition of an R-representation into η-isotypic parts with η running over $\eta \in \widehat{R}$.

2. The GGP integral for $\mathrm{U}(2,1)\times\mathrm{U}(1,1)$

The standard L-function of $\mathrm{U}(2,1)$ has an integral representation of "GGP form" by Gelbart–Piatetski-Shapiro [GPS], which we briefly review. Consider an imaginary quadratic field $K/\mathbb{Q}$ and a Hermitian vector space V/K with the form given by the matrix $\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}\right)$ for a certain basis. There is a natural embedding $\mathrm{U}(1,1)\times\mathrm{U}(1)\to\mathrm{U}(2,1)$, $((\begin{smallmatrix}*&*\\ *&*\end{smallmatrix}),(\,!\,))\mapsto\left(\begin{smallmatrix}*&0&*\\0&!&0\\ *&0&*\end{smallmatrix}\right)$. Let π be an automorphic cuspidal representation of $\mathrm{U}(2,1)(\mathbb{A})$. Let $B\subset\mathrm{U}(1,1)$ be the upper triangular Borel, and let E be an Eisenstein series on $\mathrm{U}(1,1)(\mathbb{A})$ with respect to B. For $f\in\pi$, consider the integral

$$I=\int_{[\mathrm{U}(1,1)]}f(h)E(h)dh.$$

Since $E(h)=\sum_{\gamma\in B(\mathbb{Q})\backslash\mathrm{U}(1,1)(\mathbb{Q})}F(\gamma h)$ for some section F, I unfolds into

$$I=\int_{B(\mathbb{Q})\backslash\mathrm{U}(1,1)(\mathbb{A})}f(h)F(h)dh.$$

Here, $[G]=G(\mathbb{Q})\backslash G(\mathbb{A})$ as usual. Let $N\subset B$ be the unipotent radical. Then, as F is $N(\mathbb{A})$-invariant,

$$I=\int_{B(\mathbb{Q})N(\mathbb{A})\backslash\mathrm{U}(1,1)(\mathbb{A})}F(h)\int_{N(\mathbb{Q})\backslash N(\mathbb{A})}f(nh)dndh.$$

Let $f_{00}(g):=\int_{N(\mathbb{Q})\backslash N(\mathbb{A})}f(ng)dn$.

This turns out to be the sum of Whittaker–Fourier coefficients. Namely, let $\widetilde{B}\subset\mathrm{U}(2,1)$ be the upper-triangular Borel, and $\widetilde{N}$ be its unipotent radical. Then, $\widetilde{N}$ is a "Heisenberg group" with center N. In particular, $[\widetilde{N}/N]=N(\mathbb{A})\widetilde{N}(\mathbb{Q})\backslash\widetilde{N}(\mathbb{A})$ is an abelian group which is compact, so one can Fourier-expand it; the Fourier coefficients corresponding to nontrivial characters of $[\widetilde{N}/N]$ are precisely Whittaker–Fourier coefficients. On the other hand, the integral against the trivial character is

$$\int_{[\widetilde{N}/N]}f_{00}(ng)dn=\int_{N(\mathbb{A})\widetilde{N}(\mathbb{Q})\backslash\widetilde{N}(\mathbb{A})}\int_{N(\mathbb{Q})\backslash N(\mathbb{A})}f(n'ng)dn'dn=\int_{[\widetilde{N}]}f(ng)dn=0,$$

because f is cuspidal. Now since $f_{00}(g)=\sum_{b\in N(\mathbb{A})\backslash B(\mathbb{Q})N(\mathbb{A})}W_\psi(bg)$ for any choice of nontrivial character ψ,

$$I=\int_{N(\mathbb{A})}W_\psi(h)F(h)dh,$$

which is Eulerian.

Now what happens if we replace E by a cuspform? Consider

$$I'=\int_{\mathrm{GU}(1,1)(\mathbb{Q})\times\mathrm{GU}(1)(\mathbb{Q})\backslash\mathrm{GU}(1,1)(\mathbb{A})\times\mathrm{GU}(1)(\mathbb{A})}f(h)f'(h)\mu(h)dh,$$

where f' is a cuspform on $\mathrm{GU}(1,1)(\mathbb{A})$ and μ is a character of $\mathrm{GU}(1)(\mathbb{A})$, such that the central $\mathbb{G}_m$ acts trivially for all f,f',μ. Here we switched to GU from U to avoid nuisances regarding center. We will put G in front of letters to denote the analogue in GU.

We consider the Fourier expansion along $G\widetilde{N}$. Note that the unitary representations of $\widetilde{N}(F)$, for F a local field, are either Weil representations (characterized by their central characters) or characters. Let

$$f^{\psi}(g) = \int_{[GN]} f(zg)\overline{\psi(z)}dz,$$

where ψ is a character of N which is extended to GN. These are "Fourier–Jacobi coefficients" of f, by the following reason. As $[N]$ is compact abelian,

$$f(g) = f_{00}(g) + \sum_{\psi \neq 1} f^{\psi}(g).$$

Recall that $f_{00}(g)$ is a sum of Whittaker coefficients; so, if f were a Hermitian modular form (i.e. if π_∞ is a holomorphic discrete series), $f_{00}(g) = 0$, and the rest of the sum would be precisely the usual Fourier–Jacobi expansion. For π_∞ belonging to a generic discrete series, however, one has to consider both Fourier–Jacobi coefficients and Whittaker coefficients.

Since $f_{00}(g)$ is left-$GN(\mathbb{A})$-invariant, f' being cuspidal makes the integral against f_{00} vanish. On the other hand, $f^{\psi}(g)$ transforms under the left $\widetilde{N}(\mathbb{A})$-action as the Weil representation with central character ψ transforms. Now we use the theory of RGGGR: if we let R to be the centralizer of N in $\widetilde{B}$, then $\mathrm{Ind}_R^G(\chi \otimes \omega_\psi)$ is of multiplicity one ([GR] for the nonarchimedean case, [Is] for the archimedean case), where ω_ψ is the Weil representation with central character ψ and χ is the character of the reductive part, which is $S = \mathrm{U}(1)$. So f^{ψ} decomposes further into RGGGR-coefficients

$$f^{\psi}(g) = \sum_{\chi} f^{\psi,\chi}(g),$$

where now $f^{\psi,\chi}$ *has Eulerianity*. As $\mathrm{U}(1,1)$-automorphic forms are basically elliptic modular forms, we thus have

$$\begin{aligned}
I' &= \int_{[\mathrm{GU}(1,1)\times \mathrm{GU}(1)]} \sum_{\psi\neq 1,\chi} f^{\psi,\chi}(h) f'(h)\mu(h)dh \\
&= \int_{[\mathrm{GU}(1,1)]} \sum_{\psi\neq 1} f^{\psi,\mu^{-1}}(h) f'(h)dh \\
&= \int_{[\mathrm{GU}(1,1)]} \left(\int_{GN(\mathbb{Q})\backslash \mathrm{GU}(1,1)(\mathbb{Q})} f^{\psi,\mu^{-1}}(xh)dx \right) f'(h)dh \\
&= \int_{N(\mathbb{Q})\backslash \mathrm{U}(1,1)(\mathbb{A})} f^{\psi,\mu^{-1}}(h) f'(h)dh \\
&= \int_{N(\mathbb{A})\backslash \mathrm{U}(1,1)(\mathbb{A})} f^{\psi,\mu^{-1}}(h) \left(\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f'(nh)\psi(n)dn \right) dh \\
&= \int_{N(\mathbb{A})\backslash \mathrm{U}(1,1)(\mathbb{A})} f^{\psi,\mu^{-1}}(h) W_{f'}^{\psi}(h)dh,
\end{aligned}$$

where $W_{f'}^{\psi}(h)$ is the Whittaker function for f' with respect to ψ. Now the last integral is manifestly Eulerian (most notably using the multiplicity one property of RGGGR)!

3. The Kohnen–Skoruppa integral for Sp_4

Warning. We will have to be sloppy at several places due to some extra complications that need to be resolved. We will indicate where we are being sloppy. The section will exhibit some formal manipulations which should be regarded as a suggestion.

The Kohnen–Skoruppa integral, an integral representation for the spin L-function of Sp_4, is recast in modern language by [PS]. We freely use the notations of *loc. cit.*, although we would have to conjugate the embedding $\mathrm{GL}_{2,L}^* \hookrightarrow \mathrm{GSp}_4$ slightly. For simplicity, we use the split case, $H := \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2 \hookrightarrow G := \mathrm{GSp}_4$, given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) \mapsto \begin{pmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{pmatrix}.$$

The integral we consider is

$$J(\phi, s) = \int_{Z(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} E(g, s, \Phi)\phi(g)dg,$$

where ϕ is a cusp form of a cuspidal automorphic representation π on GSp_4, $E(g, s, \Phi)$ is the Klingen Eisenstein series, and Z is the center. This unfolds into

$$J(\phi, s) = \int_{B'(\mathbb{Q})Z(\mathbb{A})\backslash \mathrm{GL}_{2,L}^*(\mathbb{A})} f^{\Phi}(g, s)\phi(g)dg,$$

where $B'(\mathbb{Q}) = Q(\mathbb{Q}) \cap H(\mathbb{Q})$ and $Q(\mathbb{Q}) \subset \mathrm{GSp}_4(\mathbb{Q})$ is the Klingen parabolic, $\begin{pmatrix} * & & * & * \\ * & * & * & * \\ * & & * & * \\ & & & * \end{pmatrix}$. Let $B' = LN$ be its Levi decomposition. Then, $L = \mathrm{GL}_2(\mathbb{Q}) \times T$, where $T = \left(\begin{smallmatrix} * & \\ & 1 \end{smallmatrix} \right) \in \mathrm{GL}_2(\mathbb{Q})$, and N is abelian, just consisted of $\begin{pmatrix} 1 & & & \\ & 1 & & * \\ & & 1 & \\ & & & 1 \end{pmatrix}$. Since $[N]$ is a part of the region of integral, from the Fourier expansion $\phi(g) = \sum_{\psi \in \widehat{[N]}} \phi_\psi(g)$, only the coefficient for the trivial character $\mathbf{1}$ survives, and we get

$$J(\phi, s) = \int_{L(\mathbb{Q})N(\mathbb{A})Z(\mathbb{A})\backslash H(\mathbb{A})} f^{\Phi}(g, s)\phi_{\mathbf{1}}(g)dg.$$

Now consider U, the unipotent radical of Q, the Klingen parabolic. It is also a Heisenberg group, with center N. The irreducible representations of U with N acting trivially are precisely characters, so

$$\phi_{\mathbf{1}}(g) = \sum_{\chi \in \widehat{[U/N]}} \phi_\chi(g) = \sum_{\gamma \in T} \phi_{\chi_0}(\gamma g),$$

where subscript means the corresponding character for the Fourier coefficient and χ_0 is any choice of nontrivial character. We thus get

$$J(\phi, s) = \int_{\mathrm{GL}_2(\mathbb{Q})N(\mathbb{A})Z(\mathbb{A})\backslash H(\mathbb{A})} f^{\Phi}(g, s)\phi_{\chi_0}(g)dg,$$

where $\mathrm{GL}_2(\mathbb{Q})$ in the region is the "first factor." Now $\phi_{\chi_0}(g)$ is also an element of the degenerate Whittaker model with respect to Klingen parabolic, which is typically infinite-dimensional. We can try to use RGGGR in this case, noting that $S = \mathrm{GL}_2(\mathbb{Q})$ is the reductive part of the centralizer of N:

$$\phi_{\chi_0}(g) = \text{“} \sum_{\sigma \in [\widehat{\mathrm{GL}_{2,\mathbb{Q}}}]} \phi_{\sigma \otimes \chi_0} \text{”}. \tag{1}$$

One big problem is that the unitary dual of $\mathrm{GL}_2(F)$ is much more complicated, involving continuous spectrum. Let us pretend that such sum makes sense as we would want. Then, as the region of integral involves $[\mathrm{GL}_{2,\mathbb{Q}}]$, among $\{\phi_{\sigma\otimes\chi_0}\}_{\sigma \in [\widehat{\mathrm{GL}_{2,\mathbb{Q}}}]}$, only the one corresponding to the trivial representation of GL_2, $\phi_{\mathbf{1}\otimes\chi_0}$, survives:

$$J(\phi, s) = \text{“} \int_{R(\mathbb{A})\backslash H(\mathbb{A})} f^{\Phi}(g, s)\phi_{\mathbf{1}\otimes\chi_0}(g)dg \text{”}.$$

Now we would win the Eulerianity of the integral if we have multiplicity one property of the RGGGR's associated with $\phi_{\mathbf{1}\otimes\chi_0}(g)$. For non-archimedean local models, this is exactly achieved in Section 3 of [MS]. For archimedean local models, this is achieved in [Hi], with the usual caveat that the multiplicty one holds when one imposes extra *moderate growth condition.* If one settles the question on how to make the sum (1) rigorous, then, as in Jacquet's classical theory, one should be able to show that, for cusp forms, only archimedean local RGGGR functionals of moderate growth can appear.

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