HEIGHTS OF ALGEBRAIC CYCLES, BY SHOU-WU ZHANG

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1. Introduction

1.1. **Weil Heights.** The first idea of heights is Weil height, appeared in the proof of Mordell-Weil theorem. To start with, there is a natural way of defining heights on $\mathbb{P}^n(\mathbb{Q})$, via the map

$$h : \mathbb{P}^n(\mathbb{Q}) \to \mathbb{R}, x = [x_0, \dots, x_n] \mapsto \log \max_i(|x_i|), \tag{1}$$

where $x_i \in \mathbb{Z}$ with $\gcd(x_0, \dots, x_n) = 1$.

Given a projective variety X and an embedding $\phi : X \hookrightarrow \mathbb{P}^n$, we can define a height $h_\phi : X(\mathbb{Q}) \to \mathbb{R}$ by $x \mapsto h(\phi(x))$. To extend this to $X(\overline{\mathbb{Q}})$, one can try to embed $X(K)$ for a general number field $K/\mathbb{Q}$ into $\mathbb{Q}$-points of some bigger projective variety, for example $X(K) = \mathrm{Res}_{K/\mathbb{Q}} X(\mathbb{Q})$, and use the height on the bigger projective variety $\mathrm{Res}_{K/\mathbb{Q}} X$.

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1.1.1. *Height Machine.* Let K be a number field and X a projective variety over K. Given a line bundle L over X, one can define a height $h_L : X(\bar{K}) \to \mathbb{R}$ up to bounded functions (denoted $O(1)$ below) such that

(1) $h_{L_1+L_2} = h_{L_1} + h_{L_2} \mod O(1)$, and
(2) if we take $L = \phi^*\mathscr{O}_{\mathbb{P}^n}(1)$, then $h_L = \text{Weil height} \mod O(1)$.

1.1.2. *Local Weil Heights.* Let $D \hookrightarrow \mathbb{P}^n$ be a divisor. For simplicity we first take D to be the divisor defined by $x_0 = 0$. Then the Weil height h as defined in Equation (1), when restricted to $\mathbb{P}^n - D = \mathbb{A}^n$, has the following local decomposition:

$$h(1, x_1, \cdots, x_n) = \sum_{p \le \infty} h_p(1, x_1, \cdots, x_n), \tag{2}$$

where

$$h_p(1, x_1, \cdots, x_n) = \log \max(1, |x_1|_p, \cdots, |x_n|_p). \tag{3}$$

Now assume D is given by $F = 0$, where F is a homogeneous polynomial of degree d. Then we can similarly define local heights $\mathbb{P}^n - D \to \mathbb{R}$ by

$$h_{p,D}(x_0, \cdots, x_n) = \log \frac{\max(|x_0|_p, \cdots, |x_n|_p)}{|F(x_0, \cdots, x_n)|_p^{1/d}}, \tag{4}$$

and still have

$$\sum_p h_{p,D} = h. \tag{5}$$

Note that the right-hand-side does not depend on the divisor D.

1.2. **Néron-Tate heights.** We want to eliminate the ambiguity of a bounded function in height machine.

We first introduce Tate's idea. Let A be an abelian variety and $\phi : A \hookrightarrow \mathbb{P}^n$ a *symmetric embedding*. That is, we assume that there is an involution σ on $\mathbb{P}^n$ such that σ act as $[-1]$ on A. This is always possible by enlarging n if necessary: given $A \hookrightarrow \mathbb{P}^n$, we get an embedding of A into $\mathbb{P}^n \times \mathbb{P}^n$, the first component being the original embedding and the second precomposed with $[-1]$, then the composition $A \hookrightarrow \mathbb{P}^n \times \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ gives a symmetric embedding. Then this gives a symmetric ample line bundle $L = \phi^*\mathcal{O}(1)$ such that $[-1]^*L = L$. Then we $[m]^*L = L^{\otimes m^2} = m^2 L$, which gives $h_L(mx) = m^2 h_L(x) + O(1)$ for any m. In particular $h_L(2^n x) = 4^n h_L(x) + O(1)$ so $\lim_{n\to\infty} \frac{h_L(2^n x)}{4^n}$ exists. The *canonical height* $\widehat{h}$ is defined by this limit. That is,

$$\widehat{h}(x) := \lim_{n\to\infty} \frac{h_L(2^n x)}{4^n}. \tag{6}$$

The canonical height $\widehat{h}$ is a semipositive quadratic form on $A(\overline{K})$, and $\widehat{h}(x) = 0$ if and only if x is torsion. This idea can be applied to any variety with an endomorphism satisfying certain properties (Northcott).

Néron's idea is to find a canonical local height for $\widehat{h}$, using Néron models and Poincaré line bundles. But we will not go to details for now.

1.3. **Arakelov intersection theory.** Note that if we apply Néron-Tate idea to function field, then height is naturally the degree of the line bundle extended to the "family" (a variety over a function field is naturally a family of varieties). Arakelov theory is the extension of this idea, applied to curves over a number field K. Suppose we have a curve C over K equipped with a line bundle L. Take a model $\mathfrak{X} \to \operatorname{Spec} \mathcal{O}_K$ with an extended line bundle $\mathscr{L}$. But $\operatorname{Spec} \mathcal{O}_K$ is not compact, so we compactify it by adding all infinite places.

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, Rightarrow] \arrow[d] & \mathfrak{X} \coprod (\coprod_v \mathfrak{X}_v(\mathbb{C})) \arrow[d] \\
\operatorname{Spec} \mathcal{O}_K \arrow[r, Rightarrow] & \operatorname{Spec} \mathcal{O}_K \coprod \{v \mid \infty\}
\end{tikzcd}$$

where each $\mathfrak{X}_v(\mathbb{C})$ is a Riemann surface. Also we get a line bundle $\mathscr{L}_v$ on each $\mathfrak{X}_v(\mathbb{C})$, and we can define a metric $|| \; ||_v$ on $\mathscr{L}_v$ by requiring $||a\ell||_v = |a|_v ||\ell||_v$ for any $a \in \mathbb{C}$ and local generator ℓ (not unique, see below).

Why do we want a norm at each Archimedean place? Note that the model already gives us a $\mathbb{Z}_p$-structure on each $\mathbb{Q}_p$-vector space, that is, a p-adic norm for each finite prime p. (The set $\mathbb{Z}_p$ is just the unit ball in $\mathbb{Q}_p$.)

But there are many ways of defining a metric. However we have an invariant, **curvature**. Given a line bundle L with a metric on a compact complex manifold X, on a trivializable neighborhood U, with trivialization given by a choice of a local section ℓ, you can define a curvature of $\|\ell\| : U \to \mathbb{R}_{>0}$ by $\frac{\partial\overline{\partial}}{\pi i} \log \|\ell\|$. Note that if the metric differs by a harmonic function, the curvature is unchanged. So we get a well-defined $(1,1)$-form $c_1(L, \|\cdot\|)$ on X. This form also recovers the metric up to scaling, because a harmonic function on a compact complex manifold is ought to be a constant.

Example 1.1. The Weil height on $\mathbb{P}^n_{\mathbb{Q}}$ is defined by $\mathcal{O}(1)$ on $\mathbb{P}^n_{\mathbb{Z}}$ with a metric

$$\|\ell_i\|(z_0, \cdots, z_n) = \frac{|z_i|}{\max(|z_0|, \cdots, |z_n|)},$$

where $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) = \sum_{i=0}^n \mathbb{Z}\ell_i$. This ℓ^∞-norm is however not smooth, so what we get is a $(1,1)$-current. If one instead uses an ℓ^p-norm, that is to define

$$\|\ell_i\|(z_0, \cdots, z_n) = \frac{|z_i|}{\sqrt[p]{\sum |z_i|^p}},$$

we would get an honest form.When $p = 2$, this is known as the Fubini-Study metric.

Remark 1.1. The form is denoted as c_1 as this form in the second de Rham cohomology is equal to the first Chern class of the line bundle. Conversely, any $(1,1)$-form representing the Chern class will give you a metric on the line bundle.

If X is a complex curve, then a $(1,1)$-form is a measure on $X(\mathbb{C})$, and we have

$$\deg L = \int_{X(\mathbb{C})} c_1(L).$$

Conversely if we have a form that integrates into an integer, we can form a line bundle as well as a metric (note that $H^2(X(\mathbb{C})) = \mathbb{Z}$).

Arakelov's compactification is a model over $\mathcal{O}_K$ plus a (smooth positive) probability measure $d\mu_v$ on $X_v(\mathbb{C})$ for each archimedean prime v, such that all metrized line bundles $(L, \|\cdot\|_v)$ on $X_v(\mathbb{C})$ have curvature proportional to $d\mu_v$, that is,

$$c_1(L, \|\cdot\|_v) = \deg L \cdot d\mu_v.$$

This property is called **admissibility**. We call the compatified

$$\overline{\mathfrak{X}} = (\mathfrak{X}, \{d\mu_v\}_{v|\infty})$$

an **arithmetic surface**. For a section ℓ of an admissible line bundle $L = (\mathscr{L}, (\|\cdot\|_v))$, we define $\widehat{\mathrm{div}}(\ell) = \mathrm{div}(\ell)_f + \mathrm{div}(\ell)_\infty$ where

$$\mathrm{div}(\ell)_\infty = \sum_{v|\infty} c_v[X_v],$$

and

$$c_v = \int_{X_v(\mathbb{C})} -\log \|\ell\|_v.$$

This can be seen as an element of

$$\mathrm{Div}(\overline{\mathfrak{X}}) := \mathrm{Div}(\mathfrak{X}) + \sum_{v|\infty} \mathbb{R}[\mathfrak{X}_v].$$

1.3.1. *Intersection Numbers.* We now want to define an intersection number on $\mathrm{Div}(\overline{\mathfrak{X}})$. We want to define local intersection numbers so that for properly intersecting divisors,

$$D_1 \cdot D_2 = \sum_v (D_1 \cdot D_2)_v = \sum_v (D_{1,v} \cdot D_{2,v}).$$

At finite primes, classical intersection theory works. At archimedean places, when one divisor is vertical, say if $D_1 = \mathfrak{X}_v$, then we define $(D_1 \cdot D_2) = \epsilon_v \deg D_2$ where $\epsilon_v = 1$ if v is real and $\epsilon_v = 2$ if v is complex. The harder question is, for two horizontal divisors D_1, D_2, how do we define an intersection number at an archimedean place v? Note that $D_{1,v}$ and $D_{2,v}$ are divisors (finite sum of points) on $X_v(\mathbb{C})$, and we would like to define their intersection number.

Definition 1.1. *Given two points on $X_v(\mathbb{C})$, with a probability measure $d\mu_v$, the intersection number is given by the Green's function $g_v : X_v(\mathbb{C}) \times X_v(\mathbb{C}) \backslash \Delta \to \mathbb{R}$ (Δ is the diagonal) such that*

$$\frac{\partial_x \overline{\partial}_x}{\pi i} g_v(x, y) = \delta_y(x) - d\mu(x)$$

as distributions. That is, if $x \neq y$, we have

$$\frac{\partial_x \overline{\partial}_x}{\pi i} g_v(x, y) = -d\mu(x)$$

and near y we have

$$g_v(x, y) = -\log z(x) + O(1),$$

where z is a local coordinate at y.

What is the relation between Neron-Tate theory and Arakelov theory? Given a curve C/K, the Jacobian of C, $\mathrm{Jac}(C)$, is isomorphic to the group of degree zero divisors $\mathrm{Div}^0(C)$ modulo rational equivalence (denoted as $\sim$ below). We consider the Neron-Tate height on $\mathrm{Jac}(C)$ and Arakelov height on $\mathrm{Div}^0(C)/\sim$.

Note that if D is a divisor of degree zero on C, when extending the line bundle $\mathscr{L} = \mathcal{O}(D)$, admissibility says that the metric has curvature zero (as degree is zero). So this comes from a local system by the Riemann-Hilbert correspondence: in more down-to-earth terms in this setting, systems of germs of parallel vector fields (with respect to the chosen metric) form a local system.

For a nonarchimedean prime, we take the Zariski closure $\overline{D}$ of D to extend to a special fiber, but the Zariski closure is not necessarily degree zero. On the other hand we can adjust $\overline{D}$ by a vertical divisor $\sum c_i F_i$, where the F_i's are components of the special fiber, so that $(\overline{D} + \sum c_i F_i) \cdot F_j = 0$ for each j. The extension $\overline{D}' := \overline{D} + \sum c_i F_i$ is called the flat extension. Then the relation between the

Neron-Tate height and the Arakelov height is that $h_{NT}([D]) = -\overline{D}' \cdot \overline{D}'$. (Hodge index theorem, by Hrijac-Faltings.)

Zhang's treatment is to give probability measures also on the dual graph of the reduction graph at nonarchimedean places. One can also define admissibility in this setting. This gives a uniform treatment of both archimedean and non-archimedean places and had applications in proving Bogomolov conjecture and Gross-Zagier formula.

1.4. **Goal.** The purpose of the course is to develop admissible pairing for higher dimensional varieties, in a fashion of Grothendieck's standard conjectures, with the hope of applications to Gan-Gross-Prasad conjectures (higher-dimensional analogues of Gross-Zagier). Let X be an n-dimensional variety. We would like to define intersection number $Y \cdot Z$ for Y, Z subvarieties of dimension p, q with $p + q = n - 1$. Note that in this way we get the correct dimension counting in the corresponding arithmetic variety $((p + 1) + (q + 1) = (n + 1))$. We will start with archimedean local pairing, then non-archimedean local pairing and finally develop a global pairing.

2. Summary of Notations

We will develop a theory that will parallel the classical Archimedean theory. Thus, we will try to keep the notations similar for everything. This will be helpful for bookkeeping purposes. We tried to follow Zhang's preliminary notes on the material.

Role[1]	Notation[1]	Cases[2]	Definition
$N^i(\mathfrak{X})$	$N^i(X)$	$\mathbb{R}$	Pairs (Trivial cycle, numerically trivial[4] Green current)
	$N^i(\mathfrak{X})$	$\mathbb{Q}_p$	Vertical numerically trivial[5] cycles
$Z^i(\mathfrak{X})$	$\widetilde{Z}^i(X)$	$\mathbb{R}$	Pairs (Cycle, Green current)
	$Z^i(\mathfrak{X})$	$\mathbb{Q}_p$	Cycles of $\mathfrak{X}$
$\widehat{Z}^i(\mathfrak{X})$	$\widehat{Z}^i(\mathfrak{X})$	Local	$Z^i(\mathfrak{X})/N^i(\mathfrak{X})$
	$\widehat{\mathrm{CH}}^i(\mathfrak{X})$	$\mathbb{Q}$	Arithmetic Chow group of $\mathfrak{X}$
	$\mathrm{TC}^i(\mathfrak{X})$	$\mathbb{F}_p(t)$	Tate cycles, $H^{2i}(\mathfrak{X}_{\overline{k}}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}/k)}$
$Z_0^i(\mathfrak{X})$	$\widetilde{Z}_0^i(X)$	$\mathbb{R}$	Pairs (Cohomologically trivial cycle, Green current)
	$Z_0^i(\mathfrak{X})$	$\mathbb{Q}_p$	Cycles of $\mathfrak{X}$ restricting to cohomologically trivial cycles of X
$\widehat{Z}_0^i(\mathfrak{X})$	$\widehat{Z}_0^i(\mathfrak{X})$	Local	$Z_0^i(\mathfrak{X})/N^i(\mathfrak{X})$
	$\widehat{\mathrm{CH}}_0^i(\mathfrak{X})$	$\mathbb{Q}$	$\ker(\widehat{\mathrm{CH}}^i(\mathfrak{X}) \to C^i(X))$
$Z_1^i(\mathfrak{X})$	$\widetilde{Z}_1^i(X)$	$\mathbb{R}$	Pairs (Trivial cycle, Green current)
	$Z_1^i(\mathfrak{X})$	$\mathbb{Q}_p$	Cycles supported in the special fiber ("vertical")
$\widehat{Z}_1^i(\mathfrak{X})$	$\widehat{Z}_1^i(\mathfrak{X})$	Local	$Z_1^i(\mathfrak{X})/N^i(\mathfrak{X})$
	$\widehat{\mathrm{CH}}_1^i(\mathfrak{X})$	$\mathbb{Q}$	$\ker(\widehat{\mathrm{CH}}^i(\mathfrak{X}) \to \mathrm{CH}^i(X))$ ("vertical")
	$\mathrm{TC}_1^i(\mathfrak{X})$	$\mathbb{F}_p(t)$	$\ker(\mathrm{TC}^i(\mathfrak{X}) \to \mathrm{TC}^i(X))$
$Z_2^i(\mathfrak{X})$	$\widetilde{Z}_2^i(X)$	$\mathbb{R}$	Harmonic $(i-1, i-1)$ forms, $\ker \partial\overline{\partial} \cap A^{i-1,i-1}$
	$Z_2^i(\mathfrak{X})$	$\mathbb{Q}_p$	"Movable vertical cycles", $\langle \overline{Z} \cap [\mathfrak{X}_k] \rangle_{Z \in Z^i(X)}$
$\widehat{Z}_2^i(\mathfrak{X})$	$\widehat{Z}_2^i(\mathfrak{X})$	Local	$(Z_2^i(\mathfrak{X}) + N^i(\mathfrak{X}))/N^i(\mathfrak{X})$
	$\widehat{\mathrm{CH}}_2^i(\mathfrak{X})$	$\mathbb{Q}$	"Movable vertical cycles", $\widehat{\mathrm{CH}}^{i-1}(\mathfrak{X}) \cdot \widehat{\mathrm{CH}}^1(\operatorname{Spec} \mathcal{O}_K)$
	$\mathrm{TC}_2^i(\mathfrak{X})$	$\mathbb{F}_p(t)$	$\ker \omega_1$, where $\omega_1 = \omega\vert_{\mathrm{TC}_1^i(\mathfrak{X})}$
$B^i(\mathfrak{X})$		$\mathbb{R}$	$\widehat{Z}_1^i(X)/\widehat{Z}_2^i(X) \cong \partial\overline{\partial}(A^{i-1,i-1})$
		$\mathbb{Q}_p$	$\widehat{Z}_1^i(\mathfrak{X})/\widehat{Z}_2^i(\mathfrak{X})$
		$\mathbb{Q}$	$\widehat{\mathrm{CH}}_1^i(\mathfrak{X})/\widehat{\mathrm{CH}}_2^i(\mathfrak{X})$
		$\mathbb{F}_p(t)$	$\mathrm{TC}_1^i(\mathfrak{X})/\mathrm{TC}_2^i(\mathfrak{X})$

<table>
<tr><td colspan="2" rowspan="4">$C^i(X)$</td><td>$\mathbb{R}$</td><td>Cohomology classes of algebraic cycles</td></tr>
<tr><td>$\mathbb{Q}_p$</td><td>Numerically equivalent classes of X, $\cong \widehat{Z}_2^{n+1-i}(\mathfrak{X})^\vee$</td></tr>
<tr><td>$\mathbb{Q}$</td><td>Numerically equivalent classes of X, $\cong \widehat{\mathrm{CH}}_2^{n+1-i}(\mathfrak{X})^\vee$</td></tr>
<tr><td>$\mathbb{F}_p(t)$</td><td>$\mathrm{im}(C^i(\mathfrak{X}) \to H^{2i}(X_{\overline{\eta}}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{K}/K)})$ [6]</td></tr>
<tr><td rowspan="4">$\widehat{C}^i(\mathfrak{X})$</td><td rowspan="3">$\widehat{C}^i(\mathfrak{X})$</td><td>$\mathbb{R}$</td><td>$\partial\overline{\partial}$-closed forms, image of the curvature map $\omega : \widehat{Z}^i(X) \to A^{i,i}$</td></tr>
<tr><td>$\mathbb{Q}_p$</td><td>$\widehat{Z}_1^{n+1-i}(\mathfrak{X})^\vee$</td></tr>
<tr><td>$\mathbb{Q}$</td><td>$\widehat{\mathrm{CH}}_1^{n+1-i}(\mathfrak{X})^\vee$</td></tr>
<tr><td>$C^i(\mathfrak{X})$</td><td>$\mathbb{F}_p(t)$</td><td>$\mathrm{im}(\mathrm{TC}^i(\mathfrak{X}) \to H^0(B_{\overline{k}}, R^{2i}\pi_*\mathbb{Q}_\ell(i))) \cong \mathrm{TC}_1^{n+1-i}(\mathfrak{X})^\vee$ [3]</td></tr>
<tr><td rowspan="4">$\widehat{C}_1^i(\mathfrak{X})$</td><td rowspan="3">$\widehat{C}_1^i(\mathfrak{X})$</td><td>$\mathbb{R}$</td><td>$\partial\overline{\partial}$-exact forms</td></tr>
<tr><td>$\mathbb{Q}_p$</td><td>$B^{n+1-i}(\mathfrak{X})^\vee \cong \ker(\widehat{C}^i(\mathfrak{X}) \to C^i(X))$</td></tr>
<tr><td>$\mathbb{Q}$</td><td>$B^{n+1-i}(\mathfrak{X})^\vee$</td></tr>
<tr><td>$C_1^i(\mathfrak{X})$</td><td>$\mathbb{F}_p(t)$</td><td>$\ker(C^i(\mathfrak{X}) \to C^i(X))$</td></tr>
<tr><td colspan="2" rowspan="4">ω
(Curvature)</td><td>$\mathbb{R}$</td><td>$\omega : \widehat{Z}^i(X) \twoheadrightarrow \widehat{C}^i(X) \subset A^{i,i}$, $\omega(Z, g) := \delta_Z - \frac{\partial\overline{\partial}}{\pi i} g$</td></tr>
<tr><td>$\mathbb{Q}_p$</td><td>$\omega : \widehat{Z}^i(\mathfrak{X}) \to \widehat{C}^i(\mathfrak{X}) = \widehat{Z}_1^{n+1-i}(\mathfrak{X})^\vee$, intersection pairing on $\mathfrak{X}$</td></tr>
<tr><td>$\mathbb{Q}$</td><td>$\omega : \widehat{\mathrm{CH}}^i(\mathfrak{X}) \to \widehat{C}^i(\mathfrak{X}) = \widehat{\mathrm{CH}}_1^{n+1-i}(\mathfrak{X})^\vee$, intersection pairing on $\widehat{\mathrm{CH}}^*(\mathfrak{X})$</td></tr>
<tr><td>$\mathbb{F}_p(t)$</td><td>$\omega : \mathrm{TC}^i(\mathfrak{X}) \to C^i(\mathfrak{X})$, obvious map by definition</td></tr>
</table>

[1] The notations in the "Role" column mainly follows that of the nonarchimedean case. Except the archimedean case, $\mathfrak{X}$ is a good model of X, the object over which we want to define a height pairing, and theories are developed over $\mathfrak{X}$. On the other hand in the Archimedean case $X = \mathfrak{X}$ (but something involving $\mathfrak{X}$ should contain additional information about Green currents).

[2] Each symbol represents the following: $\mathbb{R}$, archimedean local field; $\mathbb{Q}_p$, non-archimedean local field; $\mathbb{Q}$, number field; $\mathbb{F}_p(t)$, global function field.

[3] B is the base curve where $K = k(B)$ is the base function field, and $\pi : \mathfrak{X} \to B$ is a good model of X.

[4] W.r.t. pairing defined as integration of wedge product.

[5] W.r.t. intersection pairing on $\mathfrak{X}$.

[6] $\overline{\eta} \in B(\overline{K})$ is a geometric generic point.

Part 1. **Archimedean Local Pairing**

Let X be a projective complex variety (or more generally Kähler manifold). Given a Kähler form ω on X, we would like to define an archimedean local pairing, in a sense that for $Y, Z \subset X$ subvarieties that do not intersect with each other and $\dim Y + \dim Z = \dim X - 1$, we would like to define $\langle Y, Z\rangle \in \mathbb{R}$. This was foreseen in the last lecture where we defined the intersection number of two distinct points x, y of a curve X by using Green's function $g(x, y)$. Note that, as y approaches to x, $g(x, y) = -\log|z|(y) + O(1)$, where z denotes a local coordinate around x. One can also define Archimedean local height in a more general case where one of the subvarieties is a point, say $\dim Z = 0$. For a fixed divisor $Y \subset X$, the Archimedean height pairing $g(\cdot, Y) : X\backslash|Y| \to \mathbb{R}$ has at most log singularities. This has a similar estimate as before; if Y is locally defined by $f = 0$, then $g(z, Y) = -\log|f|(z) + O(1)$, as $z \to Y$. Equivalently, this defines a hermitian metric $\|\cdot\|$ on $\mathcal{O}(Y)$ such that $\exp g(z, Y) = \|1\|_z$, the norm of the constant section 1 evaluated at z. As in the case of curves, Green's function $g(z, Y)$ can be also thought as a function $g(z)$ on $X\backslash|Y|$ such that as distributions

$$\frac{\partial\overline{\partial}}{\pi i}g = \delta_{|Y|} - \omega \tag{7}$$

for a smooth (1,1)-form ω on X. More precisely, Equation 7 means that

$$\int_X g\frac{\partial\bar{\partial}}{\pi i}f = \int_{|Y|} f - \int_X \omega f \tag{8}$$

In generalizing this, we need some Hodge theory.

3. Complex Hodge Theory

3.1. **Hodge decomposition.** Let (X, ω) be a Kähler manifold, where $\omega = \sum g_{ij}dz_i \wedge d\bar{z}_j$ is a closed, positive definite (i.e. $[g_{ij}]$ is positive definite) $(1,1)$-form. Let $\mathscr{A}^{p,q}(X)$ be the sheaf of (p, q)-forms on X. Locally on an open set $U \hookrightarrow X$ with local coordinates $z_1, \dots, z_n$, $\alpha \in \mathscr{A}^{p,q}(U)$ can be written as $\alpha = \sum g_{IJ}dz_I d\bar{z}_J$, where the sum is over $(I, J) \subset \{1, \dots, n\}$, $|I| = p$, $|J| = q$, and $dz_I = dz_{n_1}\cdots dz_{n_p}$ if $I = \{n_1, \dots, n_p\}$ and $n_1 < \cdots < n_p$. We have a double complex of sheaves on X,

$$\begin{tikzcd} \mathscr{A}^{p,q} \arrow[r, "\partial"] \arrow[d, "\overline{\partial}"'] & \mathscr{A}^{p+1,q} \arrow[d, "\overline{\partial}"] \\ \mathscr{A}^{p,q+1} \arrow[r, "\partial"'] & \mathscr{A}^{p+1,q+1} \end{tikzcd}$$

We can also define the total complex, $\mathscr{A}^r = \oplus_{p+q=r}\mathscr{A}^{p,q}$, with differential $d : \mathscr{A}^r \to \mathscr{A}^{r+1}$ given by $d = \partial + \overline{\partial}$. The classical de Rham theory says that over a contractible open subset $U \subset X$ (for example, when $U \cong B^n$, an open unit ball in $\mathbb{C}^n$), $\mathbb{C} \to \mathscr{A}^\bullet(U)$ is exact. Thus, one can compute the singular cohomology $H^i(X, \mathbb{C})$ as the cohomology of the complex $\Gamma(X, \mathscr{A}^\bullet) =: A^\bullet$, that is

$$H^i(X, \mathbb{C}) = \frac{\ker(d|_{A^i})}{d(A^{i-1})}.$$

Do not forget that we were given with a Kähler form; in particular, this gives a volume form ω^n on X, and a pre-Hilbert space structure on $A^{p,q}$ by

$$\langle\alpha, \beta\rangle = \int_X \langle\alpha(x), \beta(x)\rangle\omega^n,$$

for $\alpha, \beta \in A^{p,q}$. As this may or may not be complete, we define $A^{p,q}_{L^2}$ to be the Hilbert space completion of $A^{p,q}$ with respect to this structure. Then $\partial : A^{p,q}_{L^2} \to A^{p+1,q}_{L^2}$ has a dual $\partial^* : A^{p+1,q}_{L^2} \to A^{p,q}_{L^2}$.

But as we are really working with a manifold, we can in fact define ∂^* on the level of $A^{p,q}$'s, by using the Hodge star operator $\star : A^{p,q} \to A^{n-q,n-p}$. The inner product then can be rewritten as (up to a scalar factor)

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge \overline{\star\beta}.$$

As $\star^2 = 1$ if $p+q$ is even and $\star^2 = -1$ if $p+q$ is odd, using the expected adjoint property, we can define ∂^* using the following computation,

$$\langle \partial\alpha, \beta \rangle = \int_X \partial\alpha \wedge \overline{\star\beta} = -\int_X \alpha \wedge (\partial\overline{\star\beta}).$$

Namely, we would like

$$\overline{\star\partial^*\beta} = \partial\overline{\star\beta}$$

so one can define $\partial^*\beta = \pm \star \overline{\partial\overline{\star\beta}}$ where the sign is determined by the degree of β. Using this, we define the Laplacian as

$$\Delta_\partial := \partial\partial^* + \partial^*\partial.$$

The first main point of Hodge theory can be then summarized as

$$A^{p,q} = (\ker \Delta_\partial) \oplus \operatorname{im} \partial \oplus \operatorname{im} \partial^*.$$

Thus formally one has a corollary

$$\frac{\ker(\partial|_{A^{p,q}})}{\partial A^{p-1,q}} \cong \ker(\Delta_\partial|_{A^{p,q}}).$$

We can similarly define $\Delta_d = dd^* + d^*d, \Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. As ω is a closed form, it is easy to see that $\Delta_\partial = \Delta_{\overline{\partial}}$, and $\Delta_d = \Delta_\partial + \Delta_{\overline{\partial}}$. In other words $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$. Thus, as far as the kernel of some Laplacian is concerned, one could take any kind of Laplacian; thus we will drop the subscript when we are talking about the kernel of a Laplacian. Some formal consequences are the following.

- $H^i(X, \mathbb{C}) = \ker(\Delta|_{A^i})$.
- If one defines $H^{p,q} := \ker(\Delta|_{A^{p,q}})$, we have the Hodge decomposition

$$H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}.$$

- $H^{p,q} = H^q(X, \Omega^p)$, where Ω^p is the sheaf of **holomorphic** p-forms on X. This is because $\Omega^p \to (A^{p,\bullet}, \overline{\partial})$ gives a resolution of Ω^p.

3.2. **Lefschetz operators.** Another big content of the classical Hodge theory is the theory of Lefschetz operators. The Lefschetz operator $\mathbb{L} : A^{p,q} \to A^{p+1,q+1}$ is just the wedge of ω. The amazing properties of this operator on the other hand are somehow very formal consequences of the setting we have set so far. To illustrate the point we assume a toy model like the following:

- V is an n-dimensional $\mathbb{C}$-vector space, and $\overline{V}$ be the same space but $\mathbb{C}$ acts via a complex conjugation.
- For each $p, q \in \{1, \ldots, n\}$, let $V^{p,q} := \wedge^p V \otimes \wedge^q \overline{V}$.
- Let $\mathbb{L} : V^{*,\bullet} \to V^{*+1,\bullet+1}$ be a collection of $\mathbb{C}$-linear operators.

Then, one can always find the "lowering operator" $\Lambda : V^{*,\bullet} \to V^{*-1,\bullet-1}$ (as opposed to $\mathbb{L}$ being "raising") such that $\mathbb{L}$ and Λ together form an $\mathfrak{sl}_2$-action, i.e. $[\mathbb{L}, \Lambda]$ on $V^{p,q}$ acts by the scalar $p+q-n$. This has the following formal consequences.

Corollary 3.1 (Hodge index theorems)**.** *Let $p+q \leq n$. Then,*

(1) (Lefschetz type) $\mathbb{L}^{n-(p+q)} : V^{p,q} \to V^{n-q,n-p}$ *is an isomorphism,*
(2) (Hodge type) *For $\alpha \in V^{p,q}$, $\mathbb{L}^{n+1-(p+q)}\alpha = 0$, and $i^{(p+q)}\alpha \wedge \mathbb{L}^{n-(p+q)}\overline{\alpha} > 0$.*

This corollary, as well as the existence of Λ operator, holds for $H^*(X, \mathbb{C})$, which is Hodge-Lefschetz Theorem.

3.3. **Archimedean Hodge-Lefschetz Theorem.** In order to generalize to non-archimedean cases, we would like to do Hodge theory algebraically, and the operators that are algebraic in nature are d and $\mathbb{L}$, so we would like to ask: can we recover the notion of harmonic forms from only these operators? We have $\mathrm{im}(\Delta) = \mathrm{im}(\partial) \oplus \mathrm{im}(\partial^*) = \mathrm{im}(\overline{\partial}) \oplus \mathrm{im}(\overline{\partial}^*)$. But one also has $\{\partial, \partial^*\}$ and $\{\overline{\partial}, \overline{\partial}^*\}$ anti-commuting with each other, so

$$\mathrm{im}(\Delta) = \mathrm{im}(\partial\overline{\partial}) \oplus \mathrm{im}(\partial\overline{\partial}^*) \oplus \mathrm{im}(\partial^*\overline{\partial}) \oplus \mathrm{im}(\partial^*\overline{\partial}^*).$$

Thus we have two formulas for $H^{p,q}$,

$$H^{p,q} = \frac{\ker\partial \cap \ker\overline{\partial}}{\mathrm{im}(\partial\overline{\partial})} = \frac{\ker(\partial\overline{\partial})}{\mathrm{im}(\partial) + \mathrm{im}(\overline{\partial})}.$$

As we have a Hodge index theorem for the $\mathfrak{sl}_2$-action formed by $\mathbb{L}$ and Λ, the Hodge index theorem will follow for any $\mathfrak{sl}_2$-submodule. This is true for $H^{*,*}, \mathrm{im}(\partial\overline{\partial}^*)$ and $\mathrm{im}(\partial^*\overline{\partial})$. This is basically because ω is closed, so $0 = [\mathbb{L}, \partial] = [\mathbb{L}, \overline{\partial}] = [\Lambda, \partial^*] = [\Lambda, \overline{\partial}^*]$. On the other hand $[\mathbb{L}, \partial^*] = i\overline{\partial}$ and so on. Thus, $\mathrm{im}(\partial\overline{\partial})$ is a $\mathbb{L}$-module (but not a Λ-module), and $\mathrm{im}(\partial^*\overline{\partial}^*)$ is a Λ-module (but not a $\mathbb{L}$-module). The Hodge-Lefschetz Theorem goes as follows

Theorem 3.1 Hodge-Lefschetz Theorem**.** *For $p+q \leq n+1$, $\mathbb{L}^{n+1-p-q}$ gives an isomorphism*

$$\mathrm{im}(\partial\overline{\partial})^{p,q} \xrightarrow{\sim} \mathrm{im}(\partial\overline{\partial})^{n+1-q,n+1-p}$$

and $\Lambda^{n+1-p-q}$ gives an isomorphism

$$\mathrm{im}(\partial^*\overline{\partial}^*)^{n-q,n-p} \xrightarrow{\sim} \mathrm{im}(\partial^*\overline{\partial}^*)^{p-1,q-1}.$$

Moreover, if $\alpha \in \mathrm{im}(\partial\overline{\partial})^{p,q}$ (resp. $\alpha \in \mathrm{im}(\partial^*\overline{\partial}^*)^{n-q,n-p}$), $\alpha \neq 0$ and $\mathbb{L}^{n+2-p-q}\alpha = 0$ (resp. $\Lambda^{n+2-p-q}\alpha = 0$), then

$$i^{p-q}(-1)^{(p+q-2)(p+q-3)/2} \int \beta \mathbb{L}^{n+1-p-q} \frac{\partial\overline{\partial}}{\pi i} \overline{\beta} > 0,$$

$$(\textit{resp. } i^{p-q}(-1)^{(p+q-2)(p+q-3)/2} \int \beta \Lambda^{n+1-p-q} \frac{\partial^*\overline{\partial}^*}{\pi i} \overline{\beta} > 0,)$$

for any $\beta \in A^{p-1,q-1}$ (resp. $\beta \in A^{n+1-q,n+1-p}$) with $\alpha = \frac{\partial\overline{\partial}}{\pi i}\beta$ (resp. $\alpha = \frac{\partial^*\overline{\partial}^*}{\pi i}\beta$).

Because of the discrepancy of Hodge index theorems, this gives a canonical splitting $\ker(d) = \mathrm{im}(\partial\overline{\partial}) \oplus H^{*,*}$ for the short exact sequence of $\mathbb{L}$-modules. This illustrates that harmonic forms can be defined if one has standard conjectures.

Corollary 3.2. *The decomposition*

$$\ker(d) = \mathrm{im}(\partial\overline{\partial}) \oplus H^{*,*} \tag{9}$$

gives a canonical spliting to the following exact sequence of $\mathbb{L}$ modules:

$$0 \to \mathrm{im}(\partial\overline{\partial}) \to \ker d \to H^{*,*} \to 0. \tag{10}$$

4. Cycles and the Curvature Map

4.1. **Green currents.** The space of currents $D_p(X)$ is defined as $\mathrm{Hom}_{\mathrm{cont}}(A^p(X), \mathbb{C})$. Some elementary observations:

- A form $\alpha \in A^p$ can be thought as an element of $D_{n-p}(X)$ via $(\alpha, \beta) = \int \alpha \wedge \beta$. Thus, $D^p(X) := D_{n-p}(X)$ is also another useful indexing (superscript usually means dimension while subscript usually means codimension).
- A codimension p subvariety $Y \hookrightarrow X$ defines a current in $D_p(X)$ via $\beta \mapsto \int_Y \beta$.
- Combining these, given a codimension p variety $Y \hookrightarrow X$ and a smooth (or more generally L^1) $(p-q)$-form η on Y, $q \leq p$, one define a current in $D_p(X)$ by $\beta \mapsto \int_Y \eta(\beta|_Y)$.

The following is a general existence theorem of Green currents.

Theorem 4.1 [GS90]**.** *Let $Y \hookrightarrow X$ be a codimension p subvariety. Let $\alpha \in A^{p,p}(X)$ be a closed form representing the homological cycle $[Y]$ (i.e. the cohomology class of α is the Poincaré dual of $[Y]$). Then, there is a current $g \in D^{p-1,p-1}(X)$ such that*

$$\frac{\partial\overline{\partial}}{\pi i} g = \delta_Y - \alpha. \tag{11}$$

Moreover, such g is unique up to addition of an element of $\mathrm{im}\,\partial \oplus \mathrm{im}\,\overline{\partial}$.

Remark 4.1. (1) Suppose that g' is a Green current for another α' representing $[Y]$. Then,

$$\frac{\partial\overline{\partial}}{\pi i}(g - g') = \alpha - \alpha',$$

which is an exact real form. This can then be solved by harmonic analysis.

(2) We can choose g to be smooth on $X \backslash |Y|$ and to have at worst logarithmtic singularity on X. This roughly means that, around every point $y_0 \in Y$ with local coordinate x and Y cut by $\{f_1 = \cdots = f_m = 0\}$,

$$g(x) = \alpha \log \rho(x) + O(1),$$

where $\rho(x) = \sum |f_i|^2$. More precisely, there is a dominant morphism $\pi : \widetilde{X} \to X$ such that $E = \pi^{-1}(Y)$ is a normal crossings divisor, $\widetilde{X} - E \xrightarrow{\sim} X - Y$ and there is a current $\widetilde{g}$ on $\widetilde{X}$ such that g is the direct image of $\widetilde{g}$, and locally around a point of E where E is locally cut by $z_1 \cdots z_k = 0$, $\widetilde{g} = \alpha \log \sum_{i=1}^k |z_i|^2 + O(1)$ for a smooth $(p-1, p-1)$-form α. For example, around $Y = (0, \cdots, 0) \in \mathbb{C}^n = X$,

$$\log\left(\sum_{i=1}^n |z_i|^2\right)\left(\frac{\partial\overline{\partial}}{\pi i}\log \sum |z_i|^2\right)^{n-1},$$

would be something of log singularity. For more details, consult to [GS90].

(3) The existence part of the above Theorem 4.1 can be equivalently phrased as follows. Let $Y \hookrightarrow X$ be a codimension p subvariety. Then, there is a current $g \in D^{p-1,p-1}(X)$, smooth on $X \backslash |Y|$ with at worst logarithmtic singularity on X, such that

$$\omega(Y, g) := \delta_Y - \frac{\partial\overline{\partial}}{\pi i} g \tag{12}$$

is smooth on X. (Then $\omega(Y, g) \in A^{p,p}(X)$ is a closed form representing the homological cycle $[Y]$.)

4.2. **Cycles.** Now one can consider the following set of pairs,

$$\widetilde{Z}^i(X) = \left\{ (Y,g) \mid Y \hookrightarrow X \text{ codim } i \text{ cycle}, \ g \text{ current for } Y \text{ s.t. } \frac{\partial\overline{\partial}}{\pi i} g - \delta_Y \text{ is smooth} \right\}. \tag{13}$$

Note that if g_1, g_2 are Green's currents of cycles Z_1, Z_2 respectively, then by Theorem 4.1 Equation 11, $g_1 + g_2$ is a Green's current of the cycle $Z_1 + Z_2$. So $\widetilde{Z}^i(X)$ is in fact a group under addition. Now let $Z^i(X)$ be the group of codimension i cycles on X and $C^i(X) \subset H^{i,i}(X,\mathbb{C})$ the group of cohomological classes of cycles in $Z_i(X)$. We have the following surjections

$$\widetilde{Z}^i(X) \twoheadrightarrow Z^i(X) \twoheadrightarrow C^i(X), \quad (Y,g) \mapsto Y \mapsto \text{Poincaré dual of } [Y]. \tag{14}$$

We define $\widetilde{Z}^i_0(X)$ to be the kernel of the composite of the above two projections, and $\widetilde{Z}^i_1(X)$ to be the kernel of the first projection. That is,

$$\widetilde{Z}^i_0(X) = \ker\left(\widetilde{Z}^i(X) \to C^i(X)\right) = \left\{ (Y,g) \in \widetilde{Z}^i(X) \mid Y \text{ is cohomologically trivial.} \right\} \tag{15}$$

and

$$\begin{aligned} \widetilde{Z}^i_1(X) &= \ker\left(\widetilde{Z}^i(X) \to Z^i(X)\right) = \left\{ (Y,g) \in \widetilde{Z}^i(X) \mid Y \text{ is the empty cycle.} \right\} \\ &= \left\{ (\text{empty cycle}, g) \mid \frac{\partial\overline{\partial}}{\pi\sqrt{-1}} g \text{ is smooth and exact} \right\}, \end{aligned} \tag{16}$$

where the last equality is by applying Theorem 4.1 to $Y =$ the empty cycle. But since $\mathrm{im}(d) \cap A^{i,i} = \mathrm{im}(\partial\overline{\partial}) \cap A^{i,i}$, we have that mapping (empty cycle, g) to g gives the isomporphism

$$\widetilde{Z}^i_1(X) \cong A^{i-1,i-1}. \tag{17}$$

Now we identify $\widetilde{Z}^i_1(X)$ with $A^{i-1,i-1}$ and consider the pairing:

$$\widetilde{Z}^i_1(X) \times \widetilde{Z}^{n+1-i}(X) \to \mathbb{R}, \quad (\phi, (Y,g)) \mapsto \int_X \phi \wedge \omega(Y,g), \tag{18}$$

and let $N^i(X) \subset \widetilde{Z}^i_1(X)$ be its left kernel. Observe that

$$\mathrm{im}(\partial + \overline{\partial}) \cap A^{i-1,i-1} \subset N^i(X) \subset \ker(\partial\overline{\partial}) \cap A^{i-1,i-1} \tag{19}$$

so we have the surjection

$$\frac{\ker(\partial\overline{\partial}) \cap A^{i-1,i-1}}{\mathrm{im}(\partial + \overline{\partial}) \cap A^{i-1,i-1}} \twoheadrightarrow \frac{\ker(\partial\overline{\partial}) \cap A^{i-1,i-1}}{N^i(X)}, \tag{20}$$

which corresponds to

$$H^{i-1,i-1}(X) = H^{n+1-i,n+1-i}(X)^\vee \twoheadrightarrow C^{n+1-i}(X)^\vee. \tag{21}$$

Remark 4.2.
(1) We have the isomorphism
$$\frac{\ker(\partial\overline{\partial}) \cap A^{i-1,i-1}}{N^i(X)} \cong C^{n+1-i}(X)^\vee$$
because the Pairing 18 induces a perfect pairing
$$\frac{\ker(\partial\overline{\partial}) \cap A^{i-1,i-1}}{N^i(X)} \times C^{n+1-i}(X) \to \mathbb{R}.$$
(2) By Grothendieck's standard conjecture $C^{n+1-i}(X)^\vee$ would be isomorphic to $C^{i-1}(X)$ but we do not assume it here.

We now define the following groups of numerical equivalence classes.

$$\widehat{Z}^i(X) = \widetilde{Z}^i(X)/N^i(X), \tag{22}$$

$$\widehat{Z}^i_0(X) = \widetilde{Z}^i_0(X)/N^i(X), \tag{23}$$

$$\widehat{Z}^i_1(X) = \widetilde{Z}^i_1(X)/N^i(X), \tag{24}$$

$$\widehat{Z}^i_2(X) = \frac{\ker(\partial\overline{\partial}) \cap A^{i-1,i-1}}{N^i(X)} \cong C^{n+1-i}(X)^\vee. \tag{25}$$

Finally we define

$$B^i(X) = \widehat{Z}^i_1(X)/\widehat{Z}^i_2(X) \cong \partial\overline{\partial}(A^{i-1,i-1}). \tag{26}$$

Then we have short exact sequences

$$0 \to \widehat{Z}^i_1(X) \to \widehat{Z}^i(X) \to Z^i(X) \to 0, \tag{27}$$

and

$$0 \to \widehat{Z}^i_2(X) \to \widehat{Z}^i_1(X) \to B^i(X) \to 0. \tag{28}$$

4.3. **the curvature map.** Consider the map

$$\omega : \widehat{Z}^i(X) \to A^{i,i}, \qquad [Y, g] \mapsto \omega(Y, g), \tag{29}$$

where $\omega(Y, g)$ is as defined in Equation 12. The map is well-defined because of the second containment in (19). We call this map the **curvature map**. Let $\widehat{C}^i(X)$ be the image of the curvature map, that is,

$$\widehat{C}^i(X) := \omega(\widehat{Z}^i(X)). \tag{30}$$

Then taking cohomological class gives us a surjection

$$\widehat{C}^i(X) \twoheadrightarrow C^i(X). \tag{31}$$

(It can be seen from Theorem 4.1 that this map is indeed surjective.) Let

$$\widehat{C}^i_1(X) := \ker(\widehat{C}^i(X) \twoheadrightarrow C^i(X)). \tag{32}$$

Then we have a short exact sequence

$$0 \to \widehat{C}^i_1(X) \to \widehat{C}^i(X) \to C^i(X) \to 0, \tag{33}$$

which can be considered as the "smooth" dual of the exact sequence

$$0 \to \widehat{Z}^{n+1-i}_2(X) \to \widehat{Z}^{n+1-i}_1(X) \to B^{n+1-i}(X) \to 0. \tag{34}$$

We also have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \widehat{Z}^i_1(X) \arrow[r] \arrow[d, "\omega_1"] & \widehat{Z}^i(X) \arrow[r] \arrow[d, "\omega"] & Z^i(X) \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & \widehat{C}^i_1(X) \arrow[r] & \widehat{C}^i(X) \arrow[r] & C^i(X) \arrow[r] & 0 \end{tikzcd}, \tag{35}$$

where ω_1 is the restriction of ω and c is mapping to cohomological class. Consider the induced long exact sequence

$$0 \to \ker(\omega_1) \to \ker(\omega) \to \ker(c) \to \operatorname{coker}(\omega_1) \to 0 (= \operatorname{coker}(\omega)). \tag{36}$$

But note also that by definition $\widehat{C}_1(X)(\subset A^{i,i})$ is in fact contained in $\mathrm{im}(d)\cap A^{i,i}$, which is the same as $\mathrm{im}(\partial\overline{\partial})\cap A^{i,i}$, so ω_1 is in fact surjective and we have $\mathrm{coker}(\omega_1)=0$, and the above Sequence 36 is in fact a short exact sequnce. From the above discussions we also have

$$\ker\omega_1 = \widehat{Z}^i_2(X) \cong C^{n+1-i}(X)^\vee, \tag{37}$$

and the isomorphism is canonical. Finally if we define

$$\widehat{Z}^i(X)^0 := \ker(\omega), \tag{38}$$

and

$$Z^i(X)^0 := \ker(c), \tag{39}$$

Sequence 36 becomes the following short exact sequence

$$0 \to C^{n+1-i}(X)^\vee \to \widehat{Z}^i(X)^0 \to Z^i(X)^0 \to 0. \tag{40}$$

5. Admissible Cycles and Archimedean Local Pairing

Roughly speaking, for $Y, Z \hookrightarrow X$ subvarieties with $\dim Z + \dim Y = n-1$, we would like to define

$$\langle Y, Z\rangle := \int_Z g_Y, \tag{41}$$

but g_Y is not well-defined and the above integral depends on, for example, $\omega(Y, g_Y)$. We will make the above rigorous by choosing g_Y to be **admissible**. By the Hodge-Lefchetz Theorem, the exact sequence 28 and 33 each has a canonical splitting:

$$\widehat{Z}^i_1(X) = \widehat{Z}^i_2(X) \oplus B^i_{\mathbb{L}}(X), \tag{42}$$

where

$$B^i_{\mathbb{L}}(X) = \mathrm{im}(\partial^*\overline{\partial}^* : A^{i,i} \to \widehat{Z}^i_1(X)), \tag{43}$$

and

$$\widehat{C}^i(X) = \widehat{C}^i_1(X) \oplus C^i_{\mathbb{L}}(X), \tag{44}$$

where

$$C^i_{\mathbb{L}}(X) = H^{i,i}(X) \cap \widehat{C}^i(X) \tag{45}$$

is the space of (i,i)-harmonic forms in the image of the curvature map.

Let $Z^*_{\mathbb{L}}(X)$ denote the cycles with hamonic curvatures, which is also the orthogonal complement of $B^*_{\mathbb{L}}(X)$, then we have the following exact sequence:

$$0 \to C^{n+1-i}(X)^\vee \to Z^*_{\mathbb{L}}(X) \to Z^*(X) \to 0, \tag{46}$$

which is split by the lifting $Y \mapsto (Y, g_Y)$, where g_Y is chosen so that

$$\int_X g_Y h = 0, \quad \forall h \in C^{n+1-i}_{\mathbb{L}}(X). \tag{47}$$

Such a Green's current is called **admissible**, and we can now define the pairing using Equation 41 by choosing an admissible g_Y.

Part 2. **Non-archimedean Local Pairing**

6. Cycles and Curvature Maps

Useful references are [Fa92], [BS58], [SGA6] and [Fu98].

Let R be a dvr, $K = \mathrm{Frac}(R)$, and k be the residue field of R. Let $\mathfrak{X}$ be a regular flat projective scheme over R, an integral model of $X = \mathfrak{X}_K$. Then we have maps

$$Z^{*-1}(X) \to Z^*(\mathfrak{X}) \to Z^*(X) \to C^*(X),$$

(all with $\mathbb{Q}$-coefficients) where Z^* is the usual $\mathbb{Q}$-vector space formally generated by codimension i cycles and C^* is the space of numerical equivalence classes, and the map $Z^{*-1}(X) \to Z^*(\mathfrak{X})$ is defined by $Z \mapsto \overline{Z} \cap [\mathfrak{X}_k]$.

Definition 6.1 ($Z_0^*(\mathfrak{X}), Z_1^*(\mathfrak{X}), Z_2^*(\mathfrak{X})$)**.** *Define a filtration on* $Z^*(\mathfrak{X})$ *as*

$$Z_0^*(\mathfrak{X}) = \ker(Z^*(\mathfrak{X}) \to C^*(X)),$$

$$Z_1^*(\mathfrak{X}) = \ker(Z^*(\mathfrak{X}) \to Z^*(X)) \text{ (“vertical cycles”)},$$

$$Z_2^*(\mathfrak{X}) = \mathrm{im}(Z^{*-1}(X) \to Z^*(\mathfrak{X})) \text{ (“movable cycles”)}.$$

Here “vertical” means cycles are supported in the special fiber, and “movable” means you can “move out to the generic fiber.” For example, if $X = \mathbb{P}^1$ and $\mathfrak{X}$ is a semistable model, then Z_2 is spanned by connected components (as opposed to Z_1 being spanned by irreducible components).

Let $n = \dim \mathfrak{X} - 1$. Then we can define $Z^p(\mathfrak{X}) \times Z_1^{n+1-p}(\mathfrak{X}) \to \mathbb{Q}$ by just intersecting.

Definition 6.2 ($N^*(\mathfrak{X}), \widehat{Z}^*(\mathfrak{X})$)**.** *Denote the null space of the intersection pairing as* $N^*(\mathfrak{X}) \subset Z_1^*(\mathfrak{X})$ *(vertical, numerically trivial cycles). Define* $\widehat{Z}_i^*(\mathfrak{X}) = (Z_i^*(\mathfrak{X}) + N^*(\mathfrak{X}))/N^*(\mathfrak{X})$.

Remark 6.1. Note that $Z^{*-1}(X) \to \widehat{Z}_2^*(\mathfrak{X})$ induces an isomorphism $C^{*-1}(X) \xrightarrow{\sim} \widehat{Z}_2^*(\mathfrak{X})$. This is because the specialization map preserves intersection numbers, e.g. [Fu98, Corollary 20.3].

The intersection pairing descends to

$$\widehat{Z}^p(\mathfrak{X}) \times \widehat{Z}_1^{n+1-p}(\mathfrak{X}) \to \mathbb{Q}.$$

Definition 6.3 (Curvature map)**.** *The map*

$$\widehat{Z}^p(\mathfrak{X}) \xrightarrow{\omega} \widehat{Z}_1^{n+1-p}(\mathfrak{X})^\vee =: \widehat{C}^p(\mathfrak{X}),$$

coming from the intersection pairing, is called the ***curvature map****.*

Remark 6.2.
1. The curvature map ω is surjective because we modded out by N^*.
2. As movable cycles can be moved out of the special fiber, $\omega(\widehat{Z}_2^*(\mathfrak{X})) = 0$. This strengthens our analogy with Kählerian setting, where $\widehat{Z}_2(\mathfrak{X})$ was the space of **harmonic forms** (i.e. curvature-zero forms).

Definition 6.4 ($B^*(\mathfrak{X}), \widehat{C}_1^*(\mathfrak{X})$)**.** *We define* $B^*(\mathfrak{X}) := \widehat{Z}_1^*(\mathfrak{X})/\widehat{Z}_2^*(\mathfrak{X})$ *and* $\widehat{C}_1^*(\mathfrak{X}) = B^{n+1-*}(\mathfrak{X})^\vee$.

From the above definitions, many direct but entangled consequences can be observed.

Remark 6.3.
1. Obviously we have an exact sequence

$$0 \to \widehat{Z}_2^*(\mathfrak{X}) \to \widehat{Z}_1^*(\mathfrak{X}) \to B^*(\mathfrak{X}) \to 0.$$

Taking the linear dual, we have

$$0 \to \widehat{C}_1^{n+1-*}(\mathfrak{X}) \to \widehat{C}^{n+1-*}(\mathfrak{X}) \to \widehat{Z}_2^*(\mathfrak{X})^\vee \to 0.$$

Note however that we have $\widehat{Z}_2^*(\mathfrak{X}) \cong C^{*-1}(X)$, and the intersection pairing restricts to a **perfect pairing** $C^{*-1}(X) \times C^{n+1-*}(X) \to \mathbb{Q}$. Thus $\widehat{Z}_2^*(\mathfrak{X})^\vee \cong C^{n+1-*}(X)$ and we have a short exact sequence

$$0 \to \widehat{C}_1^*(\mathfrak{X}) \to \widehat{C}^*(\mathfrak{X}) \to C^*(X) \to 0.$$

(2) The curvature map $\omega : \widehat{Z}^*(\mathfrak{X}) \twoheadrightarrow \widehat{C}^*(\mathfrak{X})$ restricts to $\omega_1 : \widehat{Z}_1^*(\mathfrak{X}) \to \widehat{C}_1{}^*(\mathfrak{X})$, because harmonic forms get killed by ω. In fact we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \widehat{Z}_1^*(\mathfrak{X}) \arrow[r] \arrow[d, "\omega_1"] & \widehat{Z}^*(\mathfrak{X}) \arrow[r] \arrow[d, "\omega"] & Z^*(X) \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & \widehat{C}_1^*(\mathfrak{X}) \arrow[r] & \widehat{C}^*(\mathfrak{X}) \arrow[r] & C^*(X) \arrow[r] & 0 \end{tikzcd}$$

(we will call this "the usual commutative diagram"), where vertical arrows are induced from intersection pairings.

7. Standard Conjectures and Pairing

We can now form the relevant Standard Conjectures. Let us assume that we have an ample line bundle $\mathscr{L}$ on $\mathfrak{X}$ which induces a Lefschetz operator $\mathbb{L}$ induced by cupping with $c_1(\mathscr{L})$.

Conjecture 7.1 (Standard Conjectures for $B^*(\mathfrak{X})$, Zhang)**.** *For $i \leq (n+1)/2$,*

- *(Lefschetz type Standard Conjecture) $\mathbb{L}^{n+1-2i} : B^i(\mathfrak{X}) \to B^{n+1-i}(\mathfrak{X})$ is an isomorphism,*
- *(Hodge Standard Conjecture) $(-1)^i\alpha\mathbb{L}^{n+1-2i}\alpha > 0$, if $\alpha \in B^i(\mathfrak{X})$, $\alpha \neq 0$, $\mathbb{L}^{n+2-2i}\alpha = 0$.*

Note that the usual Grothendieck Standard Conjecture (Lefschetz type) for X insists that $\mathbb{L}^{n-2i} : C^i(X) \to C^{n-i}(X)$ is an isomorphism, so there is a difference of index by 1. We will usually refer to this Standard Conjecture as the **Grothendieck Standard Conjecture for** $C^*(X)$.

We record some consequences of the above Standard Conjecture.

Corollary 7.1 (Enhanced Beilinson-Bloch-type conjecture)**.** *If the (Lefschetz type) Standard Conjecture for $B^*(\mathfrak{X})$ (Conjecture 7.1) holds, then $\widehat{C}_1^*(\mathfrak{X}) = \omega(\widehat{Z}_0^*(\mathfrak{X}))$.*

Proof. The Standard Conjecture implies that $B^i(\mathfrak{X}) \times B^{n+1-i}(\mathfrak{X}) \to \mathbb{Q}$ is a perfect pairing, so that $\ker \omega_1 = \widehat{Z}_2^*(\mathfrak{X})$ and $\operatorname{coker} \omega_1 = 0$. From the snake lemma applied to the usual commutative diagram, the statement follows. □

Corollary 7.2. *Assume that the (Lefschetz type) Standard Conjecture for $B^*(\mathfrak{X})$ (Conjecture 7.1) and the (Lefschetz type) Grothendieck Standard Conjecture for $C^*(X)$ hold with respect to the same Lefschetz operator $\mathbb{L}$.*

1. *The sequence $0 \to \widehat{C}_1^*(\mathfrak{X}) \to \widehat{C}^*(\mathfrak{X}) \to C^*(X) \to 0$ splits uniquely as $\mathbb{L}$-modules. We call the above lifting of $C^*(\mathfrak{X})$, denoted as $C_{\mathbb{L}}^*(\mathfrak{X})$, as the space of* ***harmonic forms*** *(with respect to $\mathbb{L}$).*
2. *(Beilinson-Bloch-type) Every cycle $\alpha \in Z^*(X)$ has an extension $\widehat{\alpha} \in \widehat{Z}^*(\mathfrak{X})$ with harmonic curvature, which is unique up to $\widehat{Z}_2^*(\mathfrak{X})$. We call such $\widehat{\alpha}$ an* ***admissible cycle*** *(with respect to $\mathbb{L}$). We denote the space of admissible cycles as $Z_{\mathbb{L}}^*(\mathfrak{X})$.*
3. *The sequence $0 \to \widehat{Z}_2^*(\mathfrak{X}) \to \widehat{Z}_1^*(\mathfrak{X}) \to B^*(\mathfrak{X}) \to 0$ splits uniquely as $\mathbb{L}$-modules. We denote the lifting of $B^*(\mathfrak{X})$ in $\widehat{Z}_1^*(\mathfrak{X})$ as $B_{\mathbb{L}}^*(\mathfrak{X})$.*
4. *$\widehat{Z}^*(\mathfrak{X}) = B_{\mathbb{L}}^*(\mathfrak{X}) \oplus Z_{\mathbb{L}}^*(\mathfrak{X})$, and $0 \to \widehat{Z}_2^*(\mathfrak{X})(\cong C^{*-1}(X)) \to Z_{\mathbb{L}}^*(\mathfrak{X}) \to Z^*(X) \to 0$ is exact.*

Proof. All follow easily from the index discrepancy between the Standard Conjectures for $B^*(\mathfrak{X})$ and $C^*(X)$. □

Remark 7.1. If $\mathfrak{X}$ has a regular special fiber, then $\overline{\alpha}$ already satisfies the necessary conditions for $\widehat{\alpha}$, so the Beilinson-Bloch type conjectures are obviously true.

For a cycle $\alpha \in Z^*(X)$, we can even find a **unique** extension $\widehat{\alpha} \in \widehat{Z}^*(\mathfrak{X})$ with harmonic curvature by asserting an additional condition: that $\widehat{\alpha} - \overline{\alpha} \in \widehat{Z}_1^*(\mathfrak{X})$ is annihilated by all harmonic forms in $C_{\mathbb{L}}^{n-*}(\mathfrak{X})$, where $\overline{\alpha}$ is the Zariski closure of α. We call such $\widehat{\alpha}$ a **normalized** (or **rigidified**) **admissible cycle**. Here one can think of $\widehat{\alpha} - \overline{\alpha}$ as a Green function, and the normalizing condition is requiring $\int g = 0$.

From this we can finally define a non-archimedean local height pairing.

Definition 7.1 (Non-archimedean local height pairing)**.** *Assuming the Standard Conjectures for $B^*(\mathfrak{X})$ and $C^*(X)$, given $Y \in Z^p(X), Z \in Z^{n+1-p}(X)$ with $|Y| \cap |Z| = \emptyset$, we define $(Y, Z) := \widehat{Y} \cdot \widehat{Z} = \deg(\widehat{Y}|_{\widehat{Z}}) = \deg(\widehat{Z}|_{\widehat{Y}})$.*

Remark 7.2. One can also think of the local intersection pairing as $(Y, Z) = \deg(\widehat{\Delta}|_{\overline{Y \times Z}})$, where $\widehat{\Delta}$ is the normalized admissible cycle extension of the diagonal $\Delta \subset X \times X$ with respect to $(\mathscr{L}, \mathscr{L})$ (let's assume $\mathfrak{X} \times_R \mathfrak{X}$ is regular; otherwise we apply alterations beforehand) and $\overline{Y \times Z}$ is the Zariski closure of $Y \times_K Z$ in $\mathfrak{X} \times_R \mathfrak{X}$. Or, we can take $g = \widehat{\Delta} - \overline{\Delta}$ to be "the" Green form, and then we have $(Y, Z) = \overline{Y} \cdot \overline{Z} + \deg(g|_{\overline{Y \times Z}})$.

What can be shown?

Theorem 7.1. *The Standard Conjectures hold for divisors (codimension 1).*

Proof. The Lefschetz type Standard Conjecture holds for $C^*(X)$; over a characteristic 0 field this is Lefschetz (1,1) theorem. Over a characteristic $p > 0$ field, the Standard Conjectures for surfaces are shown by Hartshorne. The general case for divisors then follows by a simple induction, by pulling back to $\operatorname{div}(s)$ for a global section s of high power of $\mathscr{L}$ (this does not lose information because Picard group of X is the same as the formal Picard group of the formal completion of X along $\operatorname{div}(s)$ if $\dim X \geq 3$; see Grothendieck).

Now we prove the Hodge standard conjecture for $B^*(\mathfrak{X})$. Let $F_1, \cdots, F_r$ be irreducible components of $\mathfrak{X}_k$. Then $B^1(\mathfrak{X}) = \sum \mathbb{Q}F_i / \mathbb{Q}[\mathfrak{X}_k]$. Now let $\alpha = \sum m_i F_i$. We need to show that $\mathbb{L}^{n+2-2}\alpha = 0$ and $\alpha \neq 0$ implies $\alpha \mathbb{L}^{n+1-2}\alpha < 0$. But note that

$$\alpha \mathbb{L}^{n-1}\alpha = \sum_{i,j} m_i m_j F_i \mathbb{L}^{n-1} F_j = \frac{1}{2}\left(\sum_{i,j}(m_i^2 + m_j^2) F_i \mathbb{L}^{n-1} F_j - \sum_{i,j}(m_i - m_j)^2 F_i \mathbb{L}^{n-1} F_j\right).$$

The first summand can be rewritten as $\sum_i m_i^2 F_i \mathbb{L}^{n-1}[\mathfrak{X}_k]$, which is zero. Thus, $\alpha\mathbb{L}^{n-1}\alpha \leq 0$, and it is zero if and only if $m_1 = m_2 = \cdots = m_r$ ($\mathfrak{X}_k$ is connected, so one can go from one irreducible component to another by passing through intersecting irreducible components). The condition $\mathbb{L}^n \alpha = 0$ implies that m_i's should all be zero if $\alpha \mathbb{L}^{n-1}\alpha = 0$.

Now we prove that the Lefschetz type Standard Conjecture for $B^*(\mathfrak{X})$ follows from the Hodge Standard Conjecture for $B^*(\mathfrak{X})$. Note that the Hodge Standard Conjecture implies that $\mathbb{L}^{n-1} : B^1(\mathfrak{X}) \to B^n(\mathfrak{X})$ is injective. To show that this is surjective, it is sufficient to show that $\dim B^1(\mathfrak{X}) \geq \dim B^n(\mathfrak{X})$. Thus it is sufficient to show that $\omega_1 : B^1(\mathfrak{X}) \to B^n(\mathfrak{X})^\vee = C^1(\mathfrak{X})$ is surjective (i.e. Beilinson-Bloch). Now the snake lemma applied to the usual commutative diagram gives an exact sequence

$$0 \to \ker \omega_1 \to \ker \omega \to \ker c \to \operatorname{coker} \omega_1 \to 0,$$

so it is sufficient to show that $\ker \omega \to \ker c$ is surjective. This (numerically trivial line bundles have flat metric) is proved in [YZ17, Appendix A.4]. A general strategy is as follows.

- By push-pull, if there is a regular flat $\mathfrak{Y}/R$ such that there is a generically finite $f : \mathfrak{X} \to \mathfrak{Y}$ then Beilinson-Bloch for $\mathfrak{Y}$ is equivalent to Beilinson-Bloch for $\mathfrak{X}$.

- A multiple of numerically trivial line bundle is algebraically trivial, so by the above point we can assume that we started with an algebraically trivial line bundle.
- An algebraically trivial line bundle comes as a pullback of an algebraically trivial line bundle on the Albanese variety $\mathrm{Alb}(X)$ via the Abel-Jacobi map $X \to \mathrm{Alb}(X)$.
- Beilinson-Bloch for line bundles is known to be true for abelian schemes (e.g. [Zh95]).

$\square$

A sample consequence is the following. Choose an ample line bundle $\mathscr{L}$ on $\mathfrak{X}$. Consider

$$\mathrm{NS}(X)^0 = \{M \in \mathrm{NS}(X)_{\mathbb{Q}} \mid c_1(M)c_1(\mathscr{L})^{n-1} = 0\}.$$

Then, for any $M \in \mathrm{NS}(X)^0$, there is an extension $\widehat{M} \in \mathrm{Pic}(\mathfrak{X})_{\mathbb{Q}}$ such that the functional $\widehat{Z}^1_1(\mathfrak{X}) \to \mathbb{Q}$, sending $Z \mapsto Z \cdot c_1(\widehat{M})c_1(\mathscr{L})^{n-1}$, is zero, and this $\widehat{M}$ is unique up to an addition of a multiple of $[\mathfrak{X}_k]$.

Part 3. Global Pairing

8. Arithmetic Intersection Theory

Consider an arithmetic scheme $\pi : \mathfrak{X} \to \operatorname{Spec} \mathcal{O}_K$ for a number field K, which means that $\mathfrak{X}$ is a regular projective $\operatorname{Spec} \mathcal{O}_K$-model of $X = \mathfrak{X}_K$ a projective smooth variety, equipped with an ample line bundle $\mathscr{L}$ and a metric $\|\cdot\|$ on $\mathfrak{X}_\infty = \coprod_{v|\infty} \mathfrak{X}_v$, such that $c_1(\mathscr{L}, \|\cdot\|_v)$ gives a Kähler form on $\mathfrak{X}_v$. Here, "ample" means it is "relatively ample" (i.e. $c_1(\mathscr{L}, \|\cdot\|_v) > 0$), as well as horizontally ample, which means for any $[L : K] < \infty$ and $x \in \mathfrak{X}(\mathcal{O}_L)$, $x^*\mathscr{L}$, a metrized line bundle on $\mathcal{O}_L$, satisfies

$$\deg x^*\mathscr{L} = c_1(\mathscr{L}) x(\operatorname{Spec} \mathcal{O}_L) = [L : K] h_{\mathscr{L}}(x) > 0.$$

Under this setting, we review the aritmetic intersection theory by Gillet-Soulé [GS90]. Let

$$\widetilde{Z}^i(\mathfrak{X}) = \left\{ (Z, g) \mid Z \in Z^i(\mathfrak{X}), g \in \widetilde{D}^{i-1,i-1}(\mathfrak{X}_\infty), \frac{\partial\overline{\partial}}{\pi i} g = \delta_{Z_\infty(\mathbb{C})} - \omega \text{ for a smooth form } \omega \right\},$$

("cohomologously approximating δ with a smooth form"), the space of **arithmetic cycles**, where $\widetilde{D} = D/(\operatorname{im} \overline{\partial} + \operatorname{im} \partial)$.

Remark 8.1. Unless otherwise noted, every space is defined with real coefficients (not with $\mathbb{Q}$ or $\mathbb{Z}$-coefficients).

Among this, we can define **principal cycles**, those in the image of the map

$$\bigoplus_{Y \hookrightarrow \mathfrak{X} \text{ codim } i \text{ integral subvar}} K(Y)^\times \to \widetilde{Z}^i(\mathfrak{X}),$$

defined by

$$\sum (Y_i, f_i) \mapsto \sum (\operatorname{div}(f_i), -\log |f_i| \delta_{Y_i}).$$

The quotient is denoted as $\widehat{\mathrm{CH}}^*(\mathfrak{X})$ and is called the **arithmetic Chow group**. These enjoy nice functoriality property as expected. In the arithmetic intersection theory the role of $\widehat{Z}^*$ used before will be played by the arithmetic Chow groups.

Example 8.1. Consider the case of $\mathfrak{X} = \operatorname{Spec} \mathcal{O}_K$. Then, we have an exact sequence

$$0 \to \oplus_{\text{pair of conjugate } \mathbb{R} \text{ places}} \mathbb{R} \to \widetilde{Z}^1(\mathfrak{X}) \to Z^1(\mathfrak{X}) \to 0.$$

This makes sense in particular for cycles with $\mathbb{Z}$-coefficients. Modding out by principal divisors, we get

$$0 \to \frac{\mathbb{R}^{r_1+r_2}}{\log(\mathcal{O}_K^\times)} \to \widehat{\mathrm{CH}}^1(\mathfrak{X})_{\mathbb{Z}} \to \mathrm{Cl}(K) \to 0.$$

Thus, the arithmetic Chow group with $\mathbb{Z}$-coefficients is a combination of regulator and algebraic parts. Of course, $\widehat{\mathrm{CH}}^1(\operatorname{Spec} \mathcal{O}_K)_{\mathbb{R}} \cong \mathbb{R}$.

One can define the degree map $\deg : \widehat{\mathrm{CH}}^1(\operatorname{Spec} \mathcal{O}_K) \to \mathbb{R}$ so that, for example for $\mathcal{O}_K = \mathbb{Z}$, $(\log p) \deg[\infty] = \deg[p]$ for a rational finite prime p, so that $\widehat{\mathrm{CH}}^1(\operatorname{Spec} \mathbb{Z}) \cong \mathbb{R}$. More generally $\deg(Z, g) = \log \#Z + \frac{1}{2} \int_{(\operatorname{Spec} \mathcal{O}_K)_\infty} g$ for $(Z, g) \in \widehat{\mathrm{CH}}^1(\operatorname{Spec} \mathcal{O}_K)$, where $\#(\sum n_i[\mathfrak{p}_i]) = \sum n_i \#(\mathcal{O}_K/\mathfrak{p}_i)$. From this, we can define the degree map for a more general arithmetic scheme $\mathfrak{X}$ by taking the pushforward, $\deg : \widehat{\mathrm{CH}}^{n+1}(\mathfrak{X}) \to \widehat{\mathrm{CH}}^1(\operatorname{Spec} \mathcal{O}_K) \xrightarrow{\sim} \mathbb{R}$. In fact, this is an isomorphism (cf. [GS94, Section 2]).

One could define a natural intersection pairing $\widehat{\mathrm{CH}}^p(\mathfrak{X}) \otimes \widehat{\mathrm{CH}}^q(\mathfrak{X}) \to \widehat{\mathrm{CH}}^{p+q}(\mathfrak{X})$. There is really one definition one could imagine for the intersection pairing (product on each entry), but a nontrivial task is to show that it is a well-defined thing (cf. [So92, III.2]).

One has an identification $\widehat{\mathrm{CH}}^1(\mathfrak{X}) \cong \widehat{\mathrm{Pic}}(\mathfrak{X})$, the space of hermitian line bundles. Let $\overline{\mathscr{L}}$ be an **arithmetically ample** hermitian line bundle, in a sense of [Zh92]. This defines a Lefschetz operator $\mathbb{L}$ on $\widehat{\mathrm{CH}}^*$'s.

Conjecture 8.1 (Gillet-Soulé Standard Conjectures; cf. [GS94])**.** *For $i \le (n+1)/2$, the following hold.*

- *(Lefschetz type Standard Conjecture)* $\mathbb{L}^{n+1-2i} : \widehat{\mathrm{CH}}^i(\mathfrak{X}) \xrightarrow{\sim} \widehat{\mathrm{CH}}^{n+1-2i}(\mathfrak{X})$.
- *(Hodge Standard Conjecture) On* $\widehat{\mathrm{CH}}^i(\mathfrak{X}) \cap \ker \mathbb{L}^{n+2-2i}$*, the pairing* $(\alpha, \beta) \mapsto (-1)^i \alpha \mathbb{L}^{n+1-2i} \beta$ *is positive definite.*

We study a similar refinement of the Standard Conjectures by introducing a filtration in a similar manner as before.

Definition 8.1 $(\widehat{\mathrm{CH}}_1^*(\mathfrak{X}), \widehat{\mathrm{CH}}_2^*(\mathfrak{X}))$**.** *We define* $\widehat{\mathrm{CH}}_1^*(\mathfrak{X}) = \ker(\widehat{\mathrm{CH}}^*(\mathfrak{X}) \to \mathrm{CH}^*(X))$ *("vertical cycles") and* $\widehat{\mathrm{CH}}_2^*(\mathfrak{X}) = \widehat{\mathrm{CH}}^{*-1}(\mathfrak{X}) \pi^* \widehat{\mathrm{CH}}^1(\mathrm{Spec}\, \mathcal{O}_K)$ *("vertical movable cycles").*

Remark 8.2. As before, we have an isomorphism $\widehat{\mathrm{CH}}_2^*(\mathfrak{X}) \cong C^{*-1}(X)$.

Definition 8.2 $(B^*(\mathfrak{X}), \widehat{C}^*(\mathfrak{X}), \widehat{C}_1^*(\mathfrak{X}))$**.** *We define as follows.*

$$B^*(\mathfrak{X}) = \widehat{\mathrm{CH}}_1^*(\mathfrak{X}) / \widehat{\mathrm{CH}}_2^*(\mathfrak{X}),$$
$$\widehat{C}^*(\mathfrak{X}) = \widehat{\mathrm{CH}}_1^{n+1-*}(\mathfrak{X})^\vee,$$
$$\widehat{C}_1^*(\mathfrak{X}) = B^{n+1-*}(\mathfrak{X})^\vee.$$

Thus, the dual of the exact sequence

$$0 \to \widehat{\mathrm{CH}}_2^*(\mathfrak{X}) \to \widehat{\mathrm{CH}}_1^*(\mathfrak{X}) \to B^*(\mathfrak{X}) \to 0,$$

is

$$0 \to \widehat{C}_1^{n+1-*}(\mathfrak{X}) \to \widehat{C}^{n+1-*}(\mathfrak{X}) \to C^{*-1}(X)^\vee \cong C^{n+1-*}(X) \to 0.$$

Also, the intersection pairing $\widehat{\mathrm{CH}}^*(\mathfrak{X}) \times \widehat{\mathrm{CH}}^{n+1-*}(\mathfrak{X}) \to \mathbb{R}$ gives the **curvature map**

$$\omega : \widehat{\mathrm{CH}}^*(\mathfrak{X}) \to \widehat{C}^*(\mathfrak{X}),$$

which induces a natural transformation between exact sequences,

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathrm{CH}}_1^*(\mathfrak{X}) \arrow[r] \arrow[d, "\omega_1"] & \widehat{\mathrm{CH}}^*(\mathfrak{X}) \arrow[r] \arrow[d, "\omega"] & \mathrm{CH}^*(X) \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & \widehat{C}_1^*(\mathfrak{X}) \arrow[r] & \widehat{C}^*(\mathfrak{X}) \arrow[r] & C^*(X) \arrow[r] & 0,
\end{tikzcd}$$

where all vertical arrows are induced from intersection pairings.

9. Global and Local Standard Conjectures

We would like to define the notion of harmonic forms, admissible cycles, and eventually global height pairing, assuming some variants of Standard Conjectures. One could imagine that one would obviously need the Standard Conjecture for $B^*(\mathfrak{X})$. But in fact the global version of Standard Conjectures for $B^*(\mathfrak{X})$ is a consequence of the local Standard Conjectures for $B^*(\mathfrak{X})$.

Proposition 9.1. *Assume the Local Standard Conjectures for $B^*(\mathfrak{X})$ (Conjecture 7.1). Then, for $i < (n+1)/2$, the following ("Global Standard Conjectures for $B^*(\mathfrak{X})$") hold.*

- *(Lefschetz type Standard Conjecture)* $\mathbb{L}^{n+1-2i} : B^i(\mathfrak{X}) \to B^{n+1-i}(\mathfrak{X})$ *is an isomorphism.*

- *(Hodge Standard Conjecture) On $B^i(\mathfrak{X}) \cap \ker \mathbb{L}^{n+2-2i}$, the bilinear pairing $(x, y) \mapsto (-1)^i x \mathbb{L}^{n+1-2i} y$ is positive definite.*

Proof. One only needs to realize that the natural map $\bigoplus_v B^*(\mathfrak{X}_v) \to B^*(\mathfrak{X})$ is a surjective map (because $B^*(\mathfrak{X})$ only deals with homologically trivial cycles, so only finitely many places are involved). Moreover, by using the pullback of the same ample line bundle for every place, the intersection pairings as well as the Lefschetz operators are compatible through this map. Thus, the Hodge Standard Conjecture as well as the surjectivity part of the Lefschetz type Standard Conjecture follow. The injectivity is then a consequence of the Hodge Standard Conjecture. $\square$

We can deduce similar conclusion, given the Global Standard Conjectures for $B^*(\mathfrak{X})$, by following the exact same argument through the snake lemma applied to the "usual commutative diagram".

Proposition 9.2 (Beilinson-Bloch type)**.** *Assume the Local Standard Conjectures for $B^*(\mathfrak{X})$ (Conjecture 7.1). Then, the intersection pairing $B^*(\mathfrak{X}) \otimes B^{n+1-*}(\mathfrak{X}) \to \mathbb{R}$ is perfect. Equivalently, $0 \to C^{*-1}(X) \to \widehat{\mathrm{CH}}^*_1(\mathfrak{X}) \to \widehat{C}^*_1(\mathfrak{X}) \to 0$ is exact. Equivalently, $0 \to C^{*-1}(X) \to \ker \omega \to \ker c \to 0$ is exact.*

We denote $\ker \omega, \ker c$ as $\widehat{\mathrm{CH}}^*(\mathfrak{X})^0, \mathrm{CH}^*(X)^0$, respectively.

Proposition 9.3 (Harmonic Forms Are Well-Defined)**.** *Assume the Local Standard Conjectures for $B^*(\mathfrak{X})$ (Conjecture 7.1) and the usual Grothendieck Standard Conjecture for $C^*(X)$. Then, as $\mathbb{L}$-modules, the following exact sequences are uniquely and canonically split.*

$$0 \to \widehat{\mathrm{CH}}^*_2(\mathfrak{X}) \cong C^{*-1}(X) \to \widehat{\mathrm{CH}}^*_1(\mathfrak{X}) \to B^*(\mathfrak{X}) \to 0,$$

$$0 \to \widehat{C}^*_1(\mathfrak{X}) \to \widehat{C}^*(\mathfrak{X}) \to C^*(X) \to 0.$$

Denote the canonical liftings of $B^*(\mathfrak{X})$ and $C^*(X)$ as $B^*_{\mathbb{L}}(\mathfrak{X})$ and $C^*_{\mathbb{L}}(\mathfrak{X})$, respectively. We call $C^*_{\mathbb{L}}(\mathfrak{X})$ the space of **harmonic forms**. Also, we call $\omega^{-1}(C^*_{\mathbb{L}}(\mathfrak{X})) =: \mathrm{CH}^*_{\mathbb{L}}(\mathfrak{X})$ the space of **admissible cycles**. We then have an orthogonal decomposition $\widehat{\mathrm{CH}}^*(\mathfrak{X}) = B^*_{\mathbb{L}}(\mathfrak{X}) \oplus \mathrm{CH}^*_{\mathbb{L}}(\mathfrak{X})$ and an exact sequence $0 \to \widehat{\mathrm{CH}}^*_2(\mathfrak{X}) \cong C^{*-1}(X) \to \mathrm{CH}^*_{\mathbb{L}}(\mathfrak{X}) \to \mathrm{CH}^*(X) \to 0$.

Now the remaining task is to define the **normalized admissible cycles** as before. More precisely, we want the middle row of the following commutative diagram with exact rows and columns (consequence of Proposition 9.2) to be split,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widehat{\mathrm{CH}}^*_2(\mathfrak{X}) \arrow[r] \arrow[d] & \widehat{\mathrm{CH}}^*(\mathfrak{X})^0 \arrow[r] \arrow[d] & \mathrm{CH}^*(X)^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{\mathrm{CH}}^*_1(\mathfrak{X}) \arrow[r] \arrow[d, "\omega_1"] & \widehat{\mathrm{CH}}^*(\mathfrak{X}) \arrow[r] \arrow[d, "\omega"] & \mathrm{CH}^*(X) \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & \widehat{C}^*_1(\mathfrak{X}) \arrow[r] \arrow[d] & \widehat{C}^*(\mathfrak{X}) \arrow[r] \arrow[d] & C^*(X) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

("The usual commutative diagram")

What Proposition 9.3 does is that we can reduce the above diagram to the following easier diagram with again exact rows and columns (easy to see by diagram chase).

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widehat{\mathrm{CH}}_2^*(\mathfrak{X}) \arrow[r] \arrow[d, equal] & \widehat{\mathrm{CH}}^*(\mathfrak{X})^0 \arrow[r] \arrow[d] & \mathrm{CH}^*(X)^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{\mathrm{CH}}_2^*(\mathfrak{X}) \arrow[r] & \mathrm{CH}^*_{\mathbb{L}}(\mathfrak{X}) \arrow[r] \arrow[d, "\omega"] & \mathrm{CH}^*(X) \arrow[r] \arrow[d, "c"] & 0 \\
 & & C^*_{\mathbb{L}}(\mathfrak{X}) \arrow[r, equal] \arrow[d] & C^*(X) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

("The reduced usual commutative diagram")

To proceed further, we need some more assumptions coming from global versions of Standard Conjectures.

Proposition 9.4 (Standard Conjectures for $\mathrm{CH}^*_{\mathbb{L}}(\mathfrak{X})$)**.** *Assume the Local Standard Conjectures for $B^*(\mathfrak{X})$ (Conjecture 7.1) and the usual Grothendieck Standard Conjecture for $C^*(X)$. Then, the Gillet-Soulé Standard Conjecture (Conjecture 8.1) is equivalent to the following* ***Standard Conjectures for*** $\mathrm{CH}^*_{\mathbb{L}}(\mathfrak{X})$*: for $i \leq (n+1)/2$,*

- *(Lefschetz-type Standard Conjecture) $\mathbb{L}^{n+1-2i} : \mathrm{CH}^i_{\mathbb{L}}(\mathfrak{X}) \to \mathrm{CH}^{n+1-i}_{\mathbb{L}}(\mathfrak{X})$ is an isomorphism,*
- *(Hodge Standard Conjecture) if $x \in \mathrm{CH}^i_{\mathbb{L}}(\mathfrak{X}) \cap \ker \mathbb{L}^{n+2-2i}$, $(-1)^i x \mathbb{L}^{n+1-2i} x > 0$ unless $x = 0$.*

This is just an immediate consequence of the canonical decomposition $\widehat{\mathrm{CH}}^*(\mathfrak{X}) = B^*_{\mathbb{L}}(\mathfrak{X}) \oplus \mathrm{CH}^*_{\mathbb{L}}(\mathfrak{X})$, Proposition 9.1.

Conjecture 9.1 (Standard Conjectures for $\mathrm{CH}^*(X)^0$, Beilinson; cf. [Be87, §5])**.** *For $i \leq (n+1)/2$,*

- *(Lefschetz-type Standard Conjecture) $\mathbb{L}^{n+1-2i} : \mathrm{CH}^i(X)^0 \to \mathrm{CH}^{n+1-i}(X)^0$ is an isomorphism,*
- *(Hodge Standard Conjecture) for $\alpha \in \mathrm{CH}^i(X)^0 \cap \ker \mathbb{L}^{n+2-2i}$, $(-1)^i \alpha \mathbb{L}^{n+1-2i} \alpha > 0$ unless $\alpha \neq 0$.*

The Lefschetz-type Standard Conjecture for $\mathrm{CH}^*(X)^0$ implies that $0 \to \widehat{\mathrm{CH}}_2^*(\mathfrak{X}) \cong C^{*-1}(X) \to \widehat{\mathrm{CH}}^*(\mathfrak{X})^0 \to \mathrm{CH}^*(X)^0 \to 0$ is uniquely and canonically split as $\mathbb{L}$-modules. Let $\mathrm{CH}^*_{\mathbb{L}}(\mathfrak{X})^0$ denote the canonical lifting of $\mathrm{CH}^*(X)^0$ inside $\widehat{\mathrm{CH}}^*(\mathfrak{X})^0$.

The Lefschetz-type Standard Conjecture for $\mathrm{CH}^*_{\mathbb{L}}(\mathfrak{X})$ gives the $\mathfrak{sl}_2$-action on $\mathrm{CH}^*_{\mathbb{L}}(\mathfrak{X})$; namely, we can define Λ, the adjoint of $\mathbb{L}$, which is the unique operator of degree -1 making $[\mathbb{L}, \Lambda] = n+1-2i$. Define $E^*_{\mathbb{L}}(\mathfrak{X}) = \mathfrak{sl}_2 \cdot \widehat{\mathrm{CH}}_2^*(\mathfrak{X}) \subset \mathrm{CH}^*_{\mathbb{L}}(\mathfrak{X})$.

Remark 9.1. It may seem that $E^*_{\mathbb{L}}(\mathfrak{X})$ is something only related to special fibers. However, it is not, as Λ "spreads things out", i.e. sends something supported in a special fiber to something larger. This is analogous to the fact in the Kähler geometry case that Λ sends $\partial\overline{\partial}$-closed forms to nonclosed forms.

Now we can state the full decomposition which enables us to define normalized admissible cycles (or **harmonic cycles**).

Theorem 9.1 (Harmonic Cycles Are Well-Defined)**.** *Assume the following conjectures.*

- *Grothendieck Standard Conjectures for $C^*(X)$,*
- *Local Standard Conjectures for $B^*(\mathfrak{X})$ (Conjecture 7.1),*
- *Gillet-Soulé Standard Conjectures (Conjecture 8.1),*
- *Standard Conjectures for $\mathrm{CH}^*(X)^0$ (Conjecture 9.1).*

Then, we have the following consequences.

(1) *We have a direct sum of $\mathbb{R}[\Lambda, \mathbb{L}]$-modules*

$$\widehat{\mathrm{CH}}^*(\mathfrak{X}) = B^*_{\mathbb{L}}(\mathfrak{X}) \oplus \mathrm{CH}^*_{\mathbb{L}}(\mathfrak{X})^0 \oplus E^*_{\mathbb{L}}(\mathfrak{X}).$$

(2) *$E^*_{\mathbb{L}}(\mathfrak{X})$ is the unique $\mathbb{R}[\Lambda, \mathbb{L}]$-submodule of $\widehat{\mathrm{CH}}^*(\mathfrak{X})$ which fits into the exact sequence of $\mathbb{L}$-modules*

$$0 \to \widehat{\mathrm{CH}}^*_2(\mathfrak{X}) \to E^*_{\mathbb{L}}(\mathfrak{X}) \xrightarrow{\omega} C^i_{\mathbb{L}}(\mathfrak{X}) \to 0.$$

(3) *The above exact sequence is split as $\mathbb{L}$-modules.*

Proof. From the reduced usual commutative diagram, (2) is equivalent to (1). Also, (2) will imply (3) because $\widehat{\mathrm{CH}}^*_2(\mathfrak{X}) \cong C^{*-1}(X)$ while $C^*_{\mathbb{L}}(\mathfrak{X}) \cong C^*(X)$, thereby having an index discrepancy for Leschetz operators. As the three components of (1) are orthogonal to each other, we only need to show that there is nothing missing.

This is achieved by dimension count, which can be done as follows. Let $C^*(X)_0 := \ker \Lambda$ (Λ of $C^*(X)$, note that the exact sequence of (2) is not an exact sequence of $\mathfrak{sl}_2$-modules!). Then by the formalism of Hodge theory, $C^*(X)$ is $\mathbb{L}$-spanned from $C^*(X)_0$. We define $\alpha : C^{i-1}(X) \to E^i_{\mathbb{L}}(\mathfrak{X})$ to be the natural embedding. As this is an $\mathbb{L}$-module homomorphism, $E^*_{\mathbb{L}}(\mathfrak{X})$ must be $\mathbb{L}$-spanned from $\Lambda^j \alpha(C^{i-1}(X)_0)$, $j \geq 0$. Then, everything will follow if we prove that $\Lambda\alpha(C^i(X)_0) \subset E^i_{\mathbb{L}}(\mathfrak{X})$ are all primitive (i.e. in $\ker \Lambda$), because if so we know that the primitive elements of $E^i_{\mathbb{L}}(\mathfrak{X})$ are precisely $(\alpha(C^{i-1}(X)_0) \cap \ker \Lambda) \oplus \Lambda\alpha(C^i(X)_0)$. We know $\mathbb{L}^{n+1-2i} C^i(X)_0 = 0$ (primitivity), and $\mathbb{L}$'s are all the same for all modules involved, so in particular $\mathbb{L}^{n+1-2i}\alpha(C^i(X)_0) = 0$. As the Lefschetz isomorphism for $\alpha(C^i(X)_0)$ is $\mathbb{L}^{n-1-2i}$ (index difference is 2), this means that although $\alpha(C^i(X)_0)$ is not necessarily primitive, at least $\Lambda\alpha(C^i(X)_0)$ is always primitive. This is precisely what we wanted. □

The above proof showed more, that the canonical $\mathbb{L}$-module lift of $C^i_{\mathbb{L}}(\mathfrak{X})$ inside $\mathrm{CH}^*_{\mathbb{L}}(\mathfrak{X})^0$ is in fact $\mathbb{R}[\mathbb{L}]\Lambda\alpha\widehat{\mathrm{CH}}^*_2(\mathfrak{X})_0$, which we denote by $F^*_{\mathbb{L}}(\mathfrak{X})$. This is the space of **harmonic cycles**.

Definition 9.1 (Global height pairing)**.** *Assuming all the Standard Conjectures assumed in Theorem 9.1, for $Y \in Z^p(X), Z \in Z^{n+1-p}(X)$ with $|Y| \cap |Z| = \emptyset$, we define $(Y, Z) := \widehat{Y} \cdot \widehat{Z}$, where $\widehat{Y}, \widehat{Z}$ are Y, Z seen as elements of $C^*(X) \cong C^*_{\mathbb{L}}(\mathfrak{X}) \cong F^*_{\mathbb{L}}(\mathfrak{X})$, and the dot product is the intersection pairing of arithmetic Chow cycles.*

As before, we would like to know if these things can be turned into reality for divisors. This is not quite the case, but the two global Standard Conjectures reduce to the same problem.

Proposition 9.5. *For divisors, the Hodge Standard Conjecture à la Gillet-Soulé holds, and the two global Lefschetz type Standard Conjectures (Gillet-Soulé and Beilinson) are equivalent to that the canonical map $\mathrm{CH}^n(X)^0 \to \mathrm{Alb}(X)(K)$ is injective, where $\mathrm{Alb}(X)$ is the Albanese variety of X.*

We denote the kernel of the map $\mathrm{CH}^n(X)^0 \to \mathrm{Alb}(X)(K)$ as $\mathrm{CH}^n(X)^{00}$.

Proof. For the Hodge Standard Conjecture, we note that the orthogonal complement of the span of $\overline{\mathscr{L}}$, the once-and-all chosen arithmetically ample hermitian line bundle, inside $\mathrm{CH}^1_{\mathbb{L}}(\mathfrak{X})$ is identified with $\mathrm{Pic}^0(X)$, and the bilinear pairing restricts to the Néron-Tate height pairing with respect to the polarization $c_1(\overline{\mathscr{L}})$, which is nondegenerate. Thus we only need to show that $\langle \mathscr{L}, \mathscr{L}\rangle = -\deg(\mathscr{L}\mathbb{L}^{n-1}\mathscr{L}) \neq 0$, and it is a standard calculation that it is equal to $c_1(\overline{\mathscr{L}})^{n+1} \neq 0$.

That the Lefschetz type Standard Conjecture of Beilinson is equivalent to $\mathrm{CH}^n(X)^{00} = 0$ is easy. The Hodge Standard Conjecture of Gillet-Soulé implies that $\mathrm{CH}^n_{\mathbb{L}}(\mathfrak{X}) \to \mathrm{CH}^1_{\mathbb{L}}(\mathfrak{X})^\vee$ is surjective. As the intersection pairings on $C^*(X)$ and $\widehat{\mathrm{CH}}^*_2(\mathfrak{X}) \cong C^{*-1}(X)$ are nondegenerate, the reduced usual commutative diagram shows that the kernel of the pairing on $\mathrm{CH}^n_{\mathbb{L}}(\mathfrak{X})$ comes precisely from that of $\mathrm{CH}^n(X)^0$, which is $\mathrm{CH}^n(X)^{00}$. $\square$

Example 9.1. Let $X = C_1 \times C_2$, a product of two curves. For $p_1, p_2 \in C_1(K), q_1, q_2 \in C_2(K)$, the two global Standard Conjectures for divisors say that $(p_1, q_1) - (p_1, q_2) - (p_2, q_1) + (p_2, q_2)$ is a divisor of some rational function. Is it believable to you? Exercise: persuade yourself that this is true.

Part 4. **Homological Pairing**

10. Tate Cycles and Homological Pairing

In a function field setting, we have no archimedean factor involved, and we can try some tools that are only available in this setting, e.g. Deligne's theory of weights. Let K be a global function field $k(B)$ of a smooth proper curve B/k with k a finite field of characteristic p. Let X be a smooth proper variety over K with a regular flat projective model $\pi : \mathfrak{X} \to B$ (use alteration if not). We once and for all choose an ample line bundle $\mathscr{L}$ on $\mathfrak{X}$.

Remark 10.1. In this theory we will use $\mathbb{Q}_\ell$-coefficients (and ℓ-adic étale cohomology), not $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$-coefficients, for $\ell \neq p$ an auxiliary prime, unless otherwise noted.

A moral reason why we can use Tate cycles is because of the **Tate conjecture**, which asserts that

$$c_\ell : \mathrm{CH}^i(\mathfrak{X}) \otimes \mathbb{Q}_\ell \to H^{2i}(\mathfrak{X}_{\overline{k}}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}/k)},$$

is an isomorphism. The space on the RHS of the Tate conjecture is called the space of (generalized) **Tate cycles**, denoted as $\mathrm{TC}^i(\mathfrak{X})$. This will play the role of $\widehat{Z}^*(\mathfrak{X})$ and $\widehat{\mathrm{CH}}^*(\mathfrak{X})$ in the function field setting. We define all other players appearing in the "usual commutative diagram" from $\mathrm{TC}^i(\mathfrak{X})$.

Definition 10.1 ($\mathrm{TC}^*_1(\mathfrak{X}), \mathrm{TC}^*(X), C^*(\mathfrak{X}), C^*_1(\mathfrak{X}), C^*(X)$)**.** *We define the space of* ***generalized Tate cycles for*** X*,* $\mathrm{TC}^*(X)$*, as*

$$\mathrm{TC}^i(X) := \mathrm{im}(\mathrm{TC}^i(\mathfrak{X}) \to \lim_{U} \mathrm{TC}^i(\mathfrak{X}_U)),$$

where the limit runs over all Zariski open subsets of B. Define $\mathrm{TC}^*_1(\mathfrak{X}) := \ker(\mathrm{TC}^*(\mathfrak{X}) \to \mathrm{TC}^*(X))$.

The spaces of cohomologically equivalent classes are defined as follows; below, η is the generic point of B, and $\overline{\eta}$ is a geometric generic point.

$$C^i(\mathfrak{X}) := \mathrm{im}(\mathrm{TC}^i(\mathfrak{X}) \to H^0(B_{\overline{k}}, R^{2i}\pi_*\mathbb{Q}_\ell(i))),$$

$$C^i(X) = \mathrm{im}(C^i(\mathfrak{X}) \to H^0(\eta, R^{2i}\pi_*\mathbb{Q}_\ell(i)) = H^{2i}(X_{\overline{\eta}}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{K}/K)}),$$

$$C^*_1(X) = \ker(C^*(\mathfrak{X}) \to C^*(X)).$$

The **curvature map** $\omega : \mathrm{TC}^i(\mathfrak{X}) \to C^i(\mathfrak{X})$ is immediately from the definitions surjective. We present some identifications which will suggest why we have defined things in this way.

Proposition 10.1.
1. *The cup product of cohomology classes induces a nondegenerate pairing* $\mathrm{TC}^i(\mathfrak{X}) \otimes \mathrm{TC}^{n+1-i}(\mathfrak{X}) \to \mathbb{Q}_\ell$.
2. *For an affine open subscheme $U \subset B$, if we denote $Z = B - U$ the corresponding closed subscheme, there is an exact sequence*
$$0 \to H^{2i}_{\mathfrak{X}_{Z_{\overline{k}}}}(\mathfrak{X}_{\overline{k}}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}/k)} \to \mathrm{TC}^i(\mathfrak{X}) \to \mathrm{TC}^i(\mathfrak{X}_U) \to 0,$$
where the subscript for H means local cohomology.
3. $\mathrm{TC}^i_1(\mathfrak{X})$ *is the span of* $\mathrm{im}(H^{2i}_{\mathfrak{X}_{Z_{\overline{k}}}}(\mathfrak{X}_{\overline{k}}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}/k)} \to \mathrm{TC}^i(\mathfrak{X}))$ *inside* $\mathrm{TC}^i(\mathfrak{X})$*, running over affine open subschemes $U \subset B$ ($Z = B - U$).*
4. *For an affine open subscheme $U \subset B$, there is a surjective natural map* $\mathrm{TC}^i(\mathfrak{X}_U) \to C^i(X)$.
5. $C^i(\mathfrak{X}) \cong \mathrm{TC}^{n+1-i}_1(\mathfrak{X})^\vee$*, and under this identification, ω is the natural map induced from the pairing on* $\mathrm{TC}^*(\mathfrak{X})$.

(6) *We have the "usual commutative diagram"*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{TC}_1^*(\mathfrak{X}) \arrow[r] \arrow[d, "\omega_1"] & \mathrm{TC}^*(\mathfrak{X}) \arrow[r] \arrow[d, "\omega"] & \mathrm{TC}^*(X) \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & C_1^*(\mathfrak{X}) \arrow[r] & C^*(\mathfrak{X}) \arrow[r] & C^*(X) \arrow[r] & 0 \end{tikzcd}$$

Remark 10.2. Note that the local cohomologies appearing in the above Proposition should be concentrated at points of bad reduction. Indeed we can take a section to a local system away from those points.

Proof. That the cup product induces a perfect pairing is the well-known Poincaré duality for étale cohomology (over an algebraically closed field). For (2), we note there is a local cohomology exact sequence

$$\cdots \to H^{2i}_{\mathfrak{X}_{Z_{\overline{k}}}}(\mathfrak{X}_{\overline{k}}, \mathbb{Q}_\ell(i)) \to H^{2i}(\mathfrak{X}_{\overline{k}}, \mathbb{Q}_\ell(i)) \to H^{2i}(\mathfrak{X}_{U_{\overline{k}}}, \mathbb{Q}_\ell(i)) \to H^{2i+1}_{\mathfrak{X}_{Z_{\overline{k}}}}(\mathfrak{X}_{\overline{k}}, \mathbb{Q}_\ell(i)) \to \cdots .$$

Now $H^{2i}(\mathfrak{X}_{\overline{k}}, \mathbb{Q}_\ell(i))$ and $H^{2i}(\mathfrak{X}_{U_{\overline{k}}}, \mathbb{Q}_\ell(i))$ is pure of weight 0 whereas $H^{2i+1}_{\mathfrak{X}_{Z_{\overline{k}}}}(\mathfrak{X}_{\overline{k}}, \mathbb{Q}_\ell(i)) \to$ is of weight > 0. Thus (2) follows after taking $\mathrm{Gal}(\overline{k}/k)$ invariants. (3) is then an immediate corollary of (2).

For (4), we note that the Leray spectral sequence yields an exact sequence

$$0 \to H^1(U_{\overline{k}}, R^{2i-1}\pi_*\mathbb{Q}_\ell(i)) \to H^{2i}(\mathfrak{X}_{U_{\overline{k}}}, \mathbb{Q}_\ell(i)) \to H^0(U_{\overline{k}}, R^{2i}\pi_*\mathbb{Q}_\ell(i)) \to 0.$$

The third term is identified with $H^{2i}(X_{\overline{K}}, \mathbb{Q}_\ell(i))^{\pi_{1,\text{ét}}(U_{\overline{k}}, \overline{\eta})}$. Taking $\mathrm{Gal}(\overline{k}/k)$-invariants, the surjective map in the exact sequence precisely becomes $\mathrm{TC}^i(\mathfrak{X}_U) \to C^i(X)$ (the basepoint of étale π_1 was the geometric generic point!).

For (5), a general fact about local cohomology is that, for any $s \in B(\overline{k})$, $H^{2i}_{\mathfrak{X}_s}(\mathfrak{X}_{\overline{k}}, \mathbb{Q}_\ell(i)) \times H^{2(n+1-i)}(\mathfrak{X}_s, \mathbb{Q}_\ell(n+1-i)) \to \mathbb{Q}_\ell$ is perfect. Thus, by (3), the kernel of the map $\mathrm{TC}^i(\mathfrak{X}) \to \mathrm{TC}_1^{n+1-i}(\mathfrak{X})^\vee$ is the same as the kernel of the map $\mathrm{TC}^i(\mathfrak{X}) \to \bigoplus_{s \in B(\overline{k})} H^{2i}(\mathfrak{X}_s, \mathbb{Q}_\ell(i))$. The maps $\mathrm{TC}^i(\mathfrak{X}) \to H^{2i}(\mathfrak{X}_s, \mathbb{Q}_\ell(i))$ all factor through $\mathrm{TC}^i(\mathfrak{X}) \to H^0(B_{\overline{k}}, R^{2i}\pi_*\mathbb{Q}_\ell(i)) \to H^{2i}(\mathfrak{X}_s, \mathbb{Q}_\ell(i))$, so the kernel of the map $\mathrm{TC}^i(\mathfrak{X}) \to \mathrm{TC}_1^{n+1-i}(\mathfrak{X})^\vee$ is then identified with the kernel of $\mathrm{TC}^i(\mathfrak{X}) \to H^0(B_{\overline{k}}, R^{2i}\pi_*\mathbb{Q}_\ell(i))$. As $\mathrm{TC}^i(\mathfrak{X}) \to \mathrm{TC}_1^{n+1-i}(\mathfrak{X})^\vee$ is surjective, the result follows. That there is the usual commutative diagram is immediate. ☐

A surprising result in a function field setting is that the analogue for the Beilinson-Bloch conjecture always holds.

Theorem 10.1 (Beilinson-Bloch type)**.** *The usual comutative diagram has* ***surjective veritcal arrows****. Thus, we have an exact sequence*

$$0 \to \ker \omega_1 \to \ker \omega \to \ker c \to 0.$$

We denote these kernels as $\mathrm{TC}_2^*(\mathfrak{X}), \mathrm{TC}^*(\mathfrak{X})^0, \mathrm{TC}^*(X)^0$*, respectively.*

Proof. What we have to prove is that ω_1 is surjective. The same argument as the proof of Proposition 10.1(3), we see that $C_1^i(\mathfrak{X})$ is spanned by the image of Tate cycles inside $H^0_s(B_{\overline{k}}, R^{2i}\pi_*\mathbb{Q}_\ell(i))$, for $s \in B(\overline{k})$. Note that the formal completion of $B_{\overline{k}}$ along s, which we denote as $\widehat{s}$, is strictly henselian, so there is no higher cohomology of the space. Thus the surjectivity of ω_1 will follow from the middle exactness of

$$H^{2i}_{\mathfrak{X}_s}(\mathfrak{X}_{\overline{k}}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}/k)} \to H^{2i}(\mathfrak{X}_s, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}/k)} \to H^{2i}(\mathfrak{X}_{\overline{K}}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}/k)},$$

for all $s \in B(\overline{k})$. The local cohomology long exact sequence coming from $\mathfrak{X}_s \subset \mathfrak{X}_{\widehat{s}}$ gives an exact sequence

$$\cdots \to H^{2i}_{\mathfrak{X}_s}(\mathfrak{X}_{\widehat{s}}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}/k)} \to H^{2i}(\mathfrak{X}_s, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}/k)} \to H^{2i}(\mathfrak{X}_{\eta_{\widehat{s}}}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}/k)} \to \cdots ,$$

where $\eta_{\widehat{s}}$ is the generic point of $\widehat{s}$. As we have observed that $\widehat{s}$ has no higher cohomology, the Theorem will follow from the injectivity of $H^{2i}(\mathfrak{X}_{\eta_{\widehat{s}}}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}/k)} \to H^{2i}(\mathfrak{X}_{\overline{K}}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}/k)}$. The kernel of this map is $H^1(\eta_{\widehat{s}}, H^{2i-1}(\mathfrak{X}_{\overline{K}}, \mathbb{Q}_\ell(i)))^{\mathrm{Gal}(\overline{k}/k)}$, which picks out only weight zero part of $H^1(\eta_{\widehat{s}}, H^{2i-1}(\mathfrak{X}_{\overline{K}}, \mathbb{Q}_\ell(i)))$, but this is of weight ≥ 1. $\square$

Another consequence is the following.

Proposition 10.2. *As before, define $B^*(\mathfrak{X}) = \mathrm{TC}_1^*(\mathfrak{X})/\mathrm{TC}_2^*(\mathfrak{X})$. Then, $B^*(\mathfrak{X}) = C_1^*(\mathfrak{X}) \cong C_1^{n+1-*}(\mathfrak{X})^\vee$. Thus, $B^*(\mathfrak{X}) \otimes B^{n+1-*}(\mathfrak{X}) \to \mathbb{Q}_\ell$ is perfect.*

Proof. From the usual commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{TC}_2^*(\mathfrak{X}) \arrow[r] \arrow[d] & \mathrm{TC}^*(\mathfrak{X})^0 \arrow[r] \arrow[d] & \mathrm{TC}^*(X)^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{TC}_1^*(\mathfrak{X}) \arrow[r] \arrow[d, "\omega_1"] & \mathrm{TC}^*(\mathfrak{X}) \arrow[r] \arrow[d, "\omega"] & \mathrm{TC}^*(X) \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & C_1^*(\mathfrak{X}) \arrow[r] \arrow[d] & C^*(\mathfrak{X}) \arrow[r] \arrow[d] & C^*(X) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

we know that $C_1^*(\mathfrak{X}) \cong B^*(\mathfrak{X})$. Now we also know $\mathrm{TC}_1^*(\mathfrak{X}) = C^{n+1-*}(\mathfrak{X})^\vee$, and the natural map $\mathrm{TC}_1^*(\mathfrak{X}) \to C^*(\mathfrak{X})$ is self-dual under this duality. The self-duality of $B^*(\mathfrak{X})$ then follows. $\square$

Remark 10.3. Another consequence is that, by taking the dual of the exact sequence

$$0 \to \mathrm{TC}_2^*(\mathfrak{X}) \to \mathrm{TC}_1^*(\mathfrak{X}) \to C^*(\mathfrak{X}) \to C^*(X) \to 0,$$

one has $\mathrm{TC}_2^*(\mathfrak{X}) \cong C^{n+1-*}(X)^\vee$.

What can we say from Deligne's Weil I, II about Lefschetz conjectures in this setting? We have the Hard Lefschetz for smooth proper varieties. To be more precise, we know the following.

- The Hard Lefschetz holds for $R^{2i}\pi_{U,*}\mathbb{Q}_\ell(i)$ with center of symmetry $\frac{n}{2}$, for any U over which π is smooth.
- This implies that $C^i(X)$ satisfies the Hard Lefschetz with center of symmetry $\frac{n}{2}$.
- Note that $\mathrm{TC}^i(X)^0 = H^1(B_{\overline{k}}, j_{U,*}j_U^* R^{2i-1}\pi_*\mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}/k)}$, for again $U \subset B$ any open subset over which π is smooth. From this we can see that the Hard Lefschetz holds for $\mathrm{TC}^i(X)^0$ with center of symmetry $\frac{n+1}{2}$.
- As $\mathrm{TC}_2^*(\mathfrak{X}) \cong C^{n+1-*}(X)^\vee$, $\mathrm{TC}_2^*(\mathfrak{X})$ satisfies the Hard Lefschetz with center of symmetry $\frac{n}{2}+1$.

Thus, the top exact row and the rightmost vertical column of the usual commutative diagram uniquely split as $\mathbb{Q}_\ell[\mathbb{L}]$-modules. This shows that the intersection pairing on $\mathrm{TC}^*(\mathfrak{X})^0$ gives a perfect pairing on $\mathrm{TC}^*(X)$. Let $\mathrm{TC}_{\mathbb{L}}^*(\mathfrak{X})^0$ be the canonical lifting of $\mathrm{TC}^*(X)^0$ inside $\mathrm{TC}^*(\mathfrak{X})^0$.

11. Hard Lefschetz Conjectures and Harmonic Cycles

To produce a fully analogous situation, we need a canonical splitting of the leftmost column and/or the bottom row of the usual commutative diagram. We have the Hard Lefschetz for three of the four corners of the diagram, so what we need is the Hard Lefschetz for $B^*(\mathfrak{X}) = C_1^*(\mathfrak{X})$.

Conjecture 11.1 (Hard Lefschetz for $B^*(\mathfrak{X})$). *For $i \leq (n+1)/2$, $\mathbb{L}^{n+1-2i} : B^i(\mathfrak{X}) \to B^{n+1-i}(\mathfrak{X})$ is an isomorphism.*

Remark 11.1. Because of our choice of coefficient, $\mathbb{Q}_\ell$, it is not clear what would "positivity" mean in this setting. Therefore there is no analogue for the Hodge Standard Conjectures.

As usual, we get a full decomposition when we assume the above conjecture.

Theorem 11.1. *Assume the Hard Lefschetz for $B^*(\mathfrak{X})$, Conjecture 11.1. Then, the exact sequences*

$$0 \to \mathrm{TC}_2^*(\mathfrak{X}) \to \mathrm{TC}_1^*(\mathfrak{X}) \to B^*(\mathfrak{X}) \to 0,$$
$$0 \to B^*(\mathfrak{X}) \to C^*(\mathfrak{X}) \to C^*(X) \to 0,$$

of $\mathbb{Q}_\ell[\mathbb{L}]$-modules are uniquely split.

Denote the canonical liftings of $\mathrm{TC}_1^*(\mathfrak{X}) \to B^*(\mathfrak{X})$ and $C^*(\mathfrak{X}) \to C^*(X)$ as $B_\mathbb{L}^*(\mathfrak{X})$ and $C_\mathbb{L}^*(\mathfrak{X})$, respectively. By analogy we can call $C_\mathbb{L}^*(\mathfrak{X})$ as the space of **harmonic (curvature) forms**.

Similarly as before, note that the Hard Lefschetz for $B^*(\mathfrak{X})$ implies that $\mathrm{TC}^*(\mathfrak{X}) = B_\mathbb{L}^*(\mathfrak{X}) \oplus \mathrm{TC}_\mathbb{L}^*(\mathfrak{X})^0 \oplus \mathrm{TC}_2^*(\mathfrak{X}) \oplus C_\mathbb{L}^*(\mathfrak{X})$ satisfies the Hard Lefschetz with center of symmetry $\frac{n+1}{2}$. Indeed, both $B_\mathbb{L}^*(\mathfrak{X}) \cong B^*(\mathfrak{X})$ and $\mathrm{TC}_\mathbb{L}^*(\mathfrak{X})^0 \cong \mathrm{TC}^*(X)^0$ already enjoy the Hard Lefschetz with center of symmetry $\frac{n+1}{2}$, and the remaining piece is $\mathrm{TC}_2^*(\mathfrak{X}) \oplus C_\mathbb{L}^*(\mathfrak{X}) \cong C^{*-1}(X) \oplus C^*(X)$, so that

$$\mathbb{L}^{n+1-2i} : C^{i-1}(X) \oplus C^i(X) \to C^{n-i}(X) \oplus C^{n+1-i}(X) \cong C^i(X) \oplus C^{i-1}(X),$$

is an isomorphism by the Hard Lefschetz for $C^*(X)$. Thus, one can construct Λ as an appropriate adjoint of $\mathbb{L}$ on $\mathrm{TC}^*(\mathfrak{X})$.

Then, define $E_\mathbb{L}^*(\mathfrak{X}) = \mathfrak{sl}_2 \cdot \mathrm{TC}_2^*(\mathfrak{X})$. Due to the exactly same reasoning as in the case of global pairing, $E_\mathbb{L}^*(\mathfrak{X})$ is the $\mathbb{Q}_\ell[\mathbb{L}]$-orthogonal complement of $B_\mathbb{L}^*(\mathfrak{X}) \oplus \mathrm{TC}_\mathbb{L}^*(\mathfrak{X})^0$. Equivalently, it is the minimal $\mathbb{Q}_\ell[\mathbb{L}, \Lambda]$-submodule of $\mathrm{TC}^*(\mathfrak{X})$ that fits into an exact sequence of $\mathbb{Q}_\ell[\mathbb{L}]$-modules

$$0 \to \mathrm{TC}_2^*(\mathfrak{X}) \to E_\mathbb{L}^*(\mathfrak{X}) \to C_\mathbb{L}^*(\mathfrak{X}) \to 0.$$

Thus, one has a canonical decomposition of $\mathrm{TC}^*(\mathfrak{X})$ into $\mathbb{Q}_\ell[\mathbb{L}, \Lambda]$-modules as

$$\mathrm{TC}^*(\mathfrak{X}) = B_\mathbb{L}^*(\mathfrak{X}) \oplus \mathrm{TC}_\mathbb{L}^*(\mathfrak{X})^0 \oplus E_\mathbb{L}^*(\mathfrak{X}).$$

As before, we can then define the canonical lifting of $E_\mathbb{L}^*(\mathfrak{X}) \to C_\mathbb{L}^*(\mathfrak{X})$, the space of **harmonic cycles**, as

$$F_\mathbb{L}^*(\mathfrak{X}) := \mathbb{Q}_\ell[\mathbb{L}](\Lambda(\mathrm{TC}_2^*(\mathfrak{X})_0)),$$

where Λ in the above expression is the Λ of $\mathrm{TC}^*(\mathfrak{X})$, and $\mathrm{TC}_2^*(\mathfrak{X})_0$ is the space of primitive classes in $\mathrm{TC}_2^*(\mathfrak{X})$, i.e. $\ker \Lambda$ for the Λ of $C^{n+1-*}(X)^\vee$ (which is again incompatible with the Λ of $\mathrm{TC}^*(\mathfrak{X})$!).

12. Interlude: Perverse Sheaves and Pairing (Lecture by Weizhe Zheng)

We hope to see that seeing our cohomology groups as (perverse) sheaves on the base curve helps us understand what is going on. We keep the notations as in the previous section.

Many things simplify as we are working over a curve. For example, a constructible sheaf $F \in Shv_{cons}(B, \mathbb{Q}_\ell)$ is a collection of the following data.

- $U \subset B$ a dense open subset,
- $F_{\overline{\eta}}$, a $\mathbb{Q}_\ell$-vector space, equipped with an action by $\pi_{1,\text{ét}}(U)$ (a quotient of $\mathrm{Gal}(\overline{\eta}/\eta)$), which corresponds to the stalk at the geometric generic point $\overline{\eta}$,
- $F_{\overline{s}}$ for each $s \in B - U$, equipped with an action by $\mathrm{Gal}(\overline{s}/s)$, which corresponds to the stalk at $\overline{j}$,
- and specialization maps $\mathrm{sp}_s : F_{\overline{s}} \to F_{\overline{\eta}}^{I(\overline{s})}$, equivariant with respect to group actions on both sides (here $I(\overline{s})$ is the inertia group).

We want to consider the derived category of constructible sheaves, and specifically consider $R\pi_*\mathbb{Q}_\ell$. As we started with a regular proper model, the Verdier duality shows that $R\pi_*\mathbb{Q}_\ell$ is Verdier-dual to itself, up to twist and shift.

On the other hand, constructible sheaves, namely complexes of constructible sheaves concentrated in degree 0, are not preserved by the Verdier duality, even after shifts. To deal with this problem one considers the category of **perverse sheaves** $Perv(B, \mathbb{Q}_\ell)$ (we only consider middle perversity throughout the lecture). A general definition of perverse sheaves involves conditions on the dimension of supports, but this greatly simplifies on a curve and we have a simpler description in the flavor of the above description of constructible sheaves; namely, a complex of constructible sheaves $L \in D^b_{cons}(B, \mathbb{Q}_\ell)$ is a perverse sheaf if the following holds.

- L is supported in degrees $[-1, 0]$.
- $\dim \operatorname{Supp} \mathcal{H}^0(L) \leq 0$.
- For $F = \mathcal{H}^{-1}(L)$, the specialization maps $\mathrm{sp}_s : F_{\overline{s}} \to F_{\overline{\eta}}^{I(\overline{s})}$ are injective.

Example 12.1. (1) A constructible ℓ-adic sheaf F on B with $\dim \operatorname{Supp}(F) = 0$ is perverse.
(2) For a lisse ℓ-adic sheaf $\mathscr{L}$ on $j : U \hookrightarrow B$, $j_!\mathscr{L}[1], j_*\mathscr{L}[1], Rj_*\mathscr{L}[1]$ are all perverse.

The category of perverse sheaves is **abelian**, being the heart of a t-structure.

Remark 12.1. You have to have different intuition for the category of perverse sheaves: let $j : U \hookrightarrow B$ and $i : B \backslash U \to B$. Then in the usual category of sheaves

$$0 \to j_!\mathscr{L} \to j_*\mathscr{L} \to i_*i^*j_*\mathscr{L} \to 0,$$

is exact. On the other hand, $i_*i^*j_*\mathscr{L}$, being a skyscraper sheaf, is perverse, and so are $j_!\mathscr{L}[1]$ and $j_*\mathscr{L}[1]$. Thus

$$0 \to i_*i^*j_*\mathscr{L}[1] \to j_!\mathscr{L}[1] \to j_*\mathscr{L}[1] \to 0,$$

is exact in the category of perverse sheaves (the cokernel becomes the kernel!).

Perverse sheaves are simpler than constructible sheaves in the following sense.

Theorem 12.1 ([BBD]). *Every perverse sheaf is of finite length. Simple objects in the perverse category are just pushforwards of lisse sheaves at a point or lisse sheaves, shifted by 1, over an open subset.*

Our case is even better, because we can use Deligne's theory of weights.

Theorem 12.2 (Weil II). *Let X be a finite type separated $k = \mathbb{F}_q$-scheme. The action of Frobenius on $H^i_c(X_{\overline{k}}, \mathbb{Q}_\ell)$ are Weil numbers of weight $w \in \mathbb{Z}_{\leq i}$.*

We can extend the notion of weight to a bounded complex of constructible sheaves.

Definition 12.1. *A bounded complex of constructible sheaves L is of weight $\leq w$ if, for all $s \in |C|$, the Frobenius eigenvalues on H^iL_s are Weil numbers of weight $\leq w + i$. Similarly, L is of weight $\geq w$ if its Verdier dual $D_c(L)$ is of weight $\leq -w$ (not a similar statement; this is about shriek-restrictions).*

For example, by Weil II, our $R\pi_!\mathbb{Q}_\ell = R\pi_*\mathbb{Q}_\ell$ is pure of weight 0.

In general, one can express the weight conditions using **perverse cohomology sheaves**. Namely, L is of weight $\leq w$ ($\geq w$, $= w$, resp.) if and only if $\mathcal{H}^i_{perv}L$ is of weight $\leq w$ ($\geq w$, $= w$, resp.).

Taking the sheaf cohomology on the distinguished triangle $\tau^{\leq i}_{perv}L \to L \to \tau^{\geq i+1}_{perv}L \xrightarrow{+1}$, one gets $0 \to \mathcal{H}^0\mathcal{H}^i_{perv}L \to \mathcal{H}^iL \to \mathcal{H}^{-1}\mathcal{H}^{i+1}_{perv}L \to 0$, exact in $Shv_{cons}(B, \mathbb{Q}_\ell)$. As we are over a curve, these determine $\mathcal{H}^*_{perv}L$.

Theorem 12.3 (Decomposition Theorem, [BBD])**.** *For L a complex of constructible sheaves pure of weight w, $L_{\overline{k}} \cong \oplus_i \mathcal{H}^i_{perv} L_{\overline{k}}[-i]$. In particular, if L is furthermore perverse, by Galois descent $L = j_*\mathscr{L}[1] \oplus \oplus_{s \in C-U} i_{s*} V_s$ for some lisse $\mathbb{Q}_\ell$-sheaf $\mathscr{L}$ over $U \subset C$ and finite dimensional $\mathbb{Q}_\ell$-vector spaces V_s for each $s \in C - U$.*

Theorem 12.4 (Relative Hard Lefschetz)**.** *For $\pi : X \to C$ a projective morphism and $\mathscr{L}$ a relatively ample line bundle, cupping with $c_1(\mathscr{L})^i$ gives an isomorphism*

$$\cup c_1(\mathscr{L})^i : R^{n+1-i}_{perv}\pi_*\mathbb{Q}_\ell \xrightarrow{\sim} R^{n+1+i}_{perv}\pi_*\mathbb{Q}_\ell,$$

where $R^i_{perv}\pi_* = H^i_{perv} R\pi_*$.

Remark 12.2. Using Chow's lemma, we have a noncanonical decomposition $R\pi_*\mathbb{Q}_\ell \cong \oplus_i R^i_{perv}\pi_*\mathbb{Q}_\ell[-i]$ for π proper.

12.1. **Global pairing.** We have a perverse Leray spectral sequence:

$$E_2^{p,q} = H^p(B_{\overline{k}}, R^q_{perv}\pi_*\mathbb{Q}_\ell) \Rightarrow H^{p+q}(\mathfrak{X}_{\overline{k}}, \mathbb{Q}_\ell).$$

By weight reasons, this must be degenerate at E_2. Thus we get a filtration

$$H^i = F^{-1}H^i \supset F^0 H^i \supset F^1 H^i \supset F^2 H^i = 0.$$

Let $F^j H^i / F^{j+1} H^i =: G^j H^i$. Note

$$R^q_{perv}\pi_*\mathbb{Q}_\ell = j_{\eta *} R^{q-1}\pi_{\eta *}\mathbb{Q}_\ell[1] \oplus M^q,$$

where M^q is the skyscraper part. Thus,

$$G^{-1}H^i = H^0(B_{\overline{k}}, j_{\eta *}R^i\pi_{\eta *}\mathbb{Q}_\ell) = H^i(\mathfrak{X}_{\overline{\eta}}, \mathbb{Q}_\ell)^{\mathrm{Gal}(\overline{\eta}/\eta)},$$
$$G^1 H^i = H^2(B_{\overline{k}}, j_{\eta *}R^{i-2}\pi_{\eta *}\mathbb{Q}_\ell),$$
$$G^0 H^i = G^0_\eta \oplus G^0_{\mathrm{sing}},$$

where $G^0_{\mathrm{sing}} = \oplus_s M^i_s$, summed over singular points, and $G^0_\eta = H^1(B_{\overline{k}}, j_{\eta *}R^{i-1}\pi_{\eta *}\mathbb{Q}_\ell)$. The Verdier duality on this swaps $G^{-1}H^i$ and $G^1 H^i$, and acts on both $G^0_\eta, G^0_{\mathrm{sing}}$, giving perfect pairings on them. The perfect pairing $G^0_\eta H^{n+1-i} \otimes G^0_\eta H^{n+1+i} \to \mathbb{Q}_\ell(-n-1)$ is Beilinson's height pairing, whereas $G^0_{\mathrm{sing}} H^{n+1-i} \otimes G^0_{\mathrm{sing}} H^{n+1+i} \to \mathbb{Q}_\ell(-n-1)$ is what's done in our class ("B^*"-part), the only part depending on the model $\mathfrak{X}$.

Remark 12.3. Taking the cohomology at s to the exact sequence

$$0 \to M^q \to R^q\pi_*\mathbb{Q}_\ell \to j_{\eta *}R^q\pi_{\eta *}\mathbb{Q}_\ell \to 0,$$

we get

$$0 \to M^q_s \to H^q(\mathfrak{X}_{\overline{s}}, \mathbb{Q}_\ell) \xrightarrow{\mathrm{sp}} H^q(\mathfrak{X}_{\overline{\eta}}, \mathbb{Q}_\ell)^{I(s)} \to 0.$$

12.2. **Local pairing.** We will just briefly illustrate how the weight-monodromy conjecture will define a local pairing Let R be a strictly henselian dvr, $S = \operatorname{Spec} R$, and $\mathfrak{X}$ a regular proper S-scheme. Then, the weight filtration gives

$$0 \to W^{\geq i} H^{i-1}(\mathfrak{X}_\eta) \to H^i_{\mathfrak{X}_s}(\mathfrak{X}) \xrightarrow{\phi^i} H^i(\mathfrak{X}) = H^i(\mathfrak{X}_s) \to W_{\leq i} H^i(\mathfrak{X}_\eta) \to 0,$$

giving a pairing on $\mathrm{im}(\phi^i)$.

Conjecture 12.1 (Weight-monodromy conjecture)**.** *If $H^i(\mathfrak{X}_{\overline{\eta}})^I$ is of weight $\leq i$,*

$$0 \to H^{i-2}(\mathfrak{X}_{\overline{\eta}})_I(-1) \to H^i_{\mathfrak{X}_s}(\mathfrak{X}) \xrightarrow{\phi^i} H^i(\mathfrak{X}_s) \xrightarrow{sp} H^i(\mathfrak{X}_{\overline{\eta}})^I \to 0.$$

In particular, the pairing exists on the kernel of the specialization map.

This holds for R equicharacteristic, or R mixed characteristic with $\mathfrak{X}$ complete intersection in a smooth toric variety.

Remark 12.4. The Hard Lefschetz in this setting would be the existence of canonical splitting $H^i(\mathfrak{X}_{\overline{\eta}})^I \to H^i(\mathfrak{X}_s)$ of the specialization map.

Part 5. Universal Non-archimedean Local Pairing

13. Universal Intersection Theory

We conjecturally defined local heights using a specific regular flat proper model. We would like to illustrate that this can be possibly done by "considering all models at once" and using p-adic geometry.

Let R be a complete discretely valued ring, $K = \operatorname{Frac} R$, $k = R/\mathfrak{m}_R$ and X be a smooth projective K-scheme. Assume for simplicity the resolution of singularities (although alterations are sufficient).

Consider $\varinjlim_{\mathfrak{X}} \widehat{Z}^*(\mathfrak{X})$, running over all regular flat proper models of $\mathfrak{X}$ over R, where we use rational equivalence classes as opposed to numerical equivalence classes we've used before, and the transition maps are shriek pullback maps. Namely,

$$\widehat{Z}^*(\mathfrak{X}) = \{(Z, \alpha) \mid Z \in Z^*(X), \alpha \in \mathrm{CH}_{n+1-*}(\mathfrak{X}_k)\},$$

or "$Z^*(\mathfrak{X})$/rational equivalence in $\mathfrak{X}_k$". Indeed, pullback maps are well-defined because models we consider are regular, so a regular map between them is locally complete intersection, which factors as a smooth map followed by a regular embedding. Smooth shriek is OK; regular embedding means that normal sheaf is a vector bundle, so we can deform to the case of normal bundle. For more details, see [Fu98, §17].

We denote $\varinjlim_{\mathfrak{X}} \widehat{Z}^*(\mathfrak{X})$ as $\widehat{Z}^*(X)$ (**cohomological cycles**). Then, we have an exact sequence

$$0 \to \widehat{Z}_1^*(X) \to \widehat{Z}^*(X) \to Z^*(X) \to 0,$$

where $\widehat{Z}_1^*(X) = \varinjlim \mathrm{CH}_{n+1-*}(\mathfrak{X}_k)$. To be consistent with the analogy we denote $\mathrm{CH}_{n+1-*}(\mathfrak{X}_k)$ also as $\widehat{Z}_1^*(\mathfrak{X})$.

Recall that for a given model, we had a section to $\widehat{Z}^*(\mathfrak{X}) \to Z^*(X)$, namely taking the Zariski closure. Is there an analogue for the "all models" definition? Not really in the current setting, as taking Zariski closure is compatible with respect to pushforwards but not with pullbacks. Thus, we are led to consider $\varprojlim \widehat{Z}^*(\mathfrak{X})$, with transition maps being **pushforwards**, and this contains $\varinjlim \widehat{Z}^*(\mathfrak{X})$ because if you pullback and pushforward you get the thing you started with. On this level you have a splitting, $\varprojlim \widehat{Z}^*(\mathfrak{X}) = Z^*(X) \oplus \varprojlim \widehat{Z}_1^*(\mathfrak{X})$. The component in $\varprojlim \widehat{Z}_1^*(\mathfrak{X})$ is called the **Green current** in this setting. We denote $\varprojlim \widehat{Z}^*(\mathfrak{X})$ as $\widehat{Z}_{n+1-*}(X)$ (**homological cycles**). Sometimes we denote $\varprojlim \widehat{Z}_1^*(\mathfrak{X})$ as $\widehat{Z}_{1,n+1-*}(X)$.

Example 13.1. Let $X = \mathbb{P}^1$. Every proper regular model of $\mathbb{P}^1$ is a blowup of $\mathbb{P}^1_{\mathcal{O}_K}$. Then for any blow-up $\pi : \mathfrak{X} \to \mathbb{P}^1_{\mathcal{O}_K}$, $\pi^!\infty = \infty + A_{\mathfrak{X}}$, which comes from exceptional divisor. These $\{A_{\mathfrak{X}}\}$ form a family compatible with pushforwards and this is a Green current.

A hint towards a definition of curvature map can be seen as follows. Given $\alpha \in \widehat{Z}_1^*(X)$ and $P \in \mathbb{P}^1(\overline{K})$, we define $f_\alpha(P) := \frac{1}{[\overline{P}:\operatorname{Spec}\mathcal{O}_K]}\alpha \cdot \overline{P} \in \mathbb{Q}$, where $\overline{P}$ is the Zariski closure of P in $\mathfrak{X}$. This does not depend on the model, so this gives a map $\widehat{Z}_1^*(X) \to \operatorname{Map}(\mathbb{P}^1(\overline{K}), \mathbb{Q})$.

How should we define an intersection pairing $\widehat{Z}^p(X) \times \widehat{Z}^q(X) \to \widehat{Z}^{p+q}(X)$ in general? We can try to embed our situation into $\widehat{Z}_{n+1-*}(X)$ and contemplate on how to intersect Green currents. Many constructions of the Kähler setting actually have anaolgues in this framework:

Definition 13.1 (Dirac delta)**.** *Given $Y \in Z^*(X)$, we define $\delta_Y \in \varprojlim_{\mathfrak{X}} \mathrm{CH}^*(\mathfrak{X}_k)$ by the sequence $(\overline{Y} \cap [\mathfrak{X}_k])$ running over all regular proper flat models $\mathfrak{X}$.*

Definition 13.2 ($\partial\overline{\partial}$-operator)**.** *The* ***$\partial\overline{\partial}$-operator*** *$\partial\overline{\partial} : \widehat{Z}_{1,n+1-*}(X) = \varprojlim_{\mathfrak{X}} \mathrm{CH}_{n+1-*}(\mathfrak{X}_k) \to \varprojlim_{\mathfrak{X}} \mathrm{CH}^*(\mathfrak{X}_k)$ is defined as the inverse limit of* $\mathrm{CH}_{n+1-*}(\mathfrak{X}_k) \to \mathrm{CH}_{n+1-*}(\mathfrak{X}) \cong \mathrm{CH}^*(\mathfrak{X}) \to \mathrm{CH}^*(\mathfrak{X}_k)$.

Definition 13.3 (Curvature map)**.** *Given* $\widehat{Z} = (Z, g) \in \widehat{Z}_{n+1-*}(X) = Z^*(X) \oplus \widehat{Z}_{1,n+1-*}(X)$*, we define* $\omega(\widehat{Z}) := \delta_Z + \partial\overline{\partial} g \in \varprojlim_{\mathfrak{X}} \mathrm{CH}^*(\mathfrak{X}_k)$.

Lemma 13.1. *If* $\widehat{Z} \in \widehat{Z}_{n+1-*}(X)$ *is an element of* $\widehat{Z}^*(X) \subset \widehat{Z}_{n+1-*}(X)$*, then* $\omega(\widehat{Z})$ *is* ***smooth****, namely* $\omega(\widehat{Z}) \in \varinjlim_{\mathfrak{X}} \mathrm{CH}^*(\mathfrak{X}_k) \subset \varprojlim_{\mathfrak{X}} \mathrm{CH}^*(\mathfrak{X}_k)$.

Proof. If $\widehat{Z}$ is an element of $\widehat{Z}^*(X)$, you can pullback to $\mathfrak{X}_k$ to start with. ☐

Given the lemma, we can (surprisingly, without assuming anything!) define the intersection pairing as follows.

Definition 13.4 (Intersection pairing on $\widehat{Z}^*(X)$)**.** *For* $p + q = n + 1$ *and* $(Z_1, g_1) \in \widehat{Z}^p(X) \subset \widehat{Z}_q(X), (Z_2, g_2) \in \widehat{Z}^q(X) \subset \widehat{Z}_p(X)$*, we define* $(Z_1, g_1) \cdot (Z_2, g_2) := g_1\omega(Z_2, g_2) + g_2\delta_{Z_1}$*. To be more precise, it is the sum of two intersection numbers, paired on* $\mathrm{CH}^*(\mathfrak{X})$*, namely*

$$g_1 \in \varprojlim_{\mathfrak{X},\text{push}} \mathrm{CH}_q(\mathfrak{X}_k) \xrightarrow{\text{push}} \varprojlim_{\mathfrak{X},\text{push}} \mathrm{CH}_q(\mathfrak{X}) \cong \varprojlim_{\mathfrak{X},\text{push}} \mathrm{CH}^p(\mathfrak{X}),$$

paired with

$$\omega(Z_2, g_2) \in \varinjlim_{\mathfrak{X},\text{pull}} \mathrm{CH}^q(\mathfrak{X}_k) \xrightarrow{\text{pull}} \varinjlim_{\mathfrak{X},\text{pull}} \mathrm{CH}^q(\mathfrak{X}) \subset \varprojlim_{\mathfrak{X},\text{push}} \mathrm{CH}^q(\mathfrak{X}),$$

plus

$$g_2 \in \varprojlim_{\mathfrak{X},\text{push}} \mathrm{CH}_p(\mathfrak{X}_k) \xrightarrow{\text{push}} \varprojlim_{\mathfrak{X},\text{push}} \mathrm{CH}_p(\mathfrak{X}) \cong \varprojlim_{\mathfrak{X},\text{push}} \mathrm{CH}^q(\mathfrak{X}),$$

paired with

$$\delta_{Z_1} = (\overline{Z_1} \cap [\mathfrak{X}_k]) \in \varprojlim_{\mathfrak{X},\text{push}} \mathrm{CH}^p(\mathfrak{X}).$$

This is very general but at the same time very abstract and vague. What we have only used are some formal properties. We can thus define a **universal intersection theory** given similar spaces with desired formal properties.

Definition 13.5 (Universal intersection theory)**.** *A general setting for the* ***universal intersection theory*** *is as follows. For every proper regular model* $\mathfrak{X}$*, we are given with*

- *a graded ring* $A^*(\mathfrak{X}_k)$*,*
- *an* $A^*(\mathfrak{X}_k)$*-module* $A_*(\mathfrak{X}_k)$*,*
- *the* ***$\partial\overline{\partial}$-operator*** $\partial\overline{\partial} : A_*(\mathfrak{X}_k) \to A^{n+1-*}(\mathfrak{X}_k)$*,*
- *the* ***Dirac delta operator*** $\delta : \mathrm{CH}^*(X) \to A^*(\mathfrak{X}_k)$*,*

equipped with pushforward/pullback maps, such that pushforward $\circ$ *pullback through the same map is the identity in any cases. Given the above, we define*

- *the* ***space of homological cycles*** $\widehat{Z}_{n+1-*}(X) := Z^*(X) \times \varprojlim_{\mathfrak{X},\text{push}} A_{n+1-*}(\mathfrak{X}_k)$*,*
- *the* ***curvature map*** $\omega : \widehat{Z}_{n+1-*}(X) \to \varprojlim_{\mathfrak{X},\text{push}} A^*(\mathfrak{X}_k)$ *as* $\omega(Z, g) := \partial\overline{\partial} g + \delta_Z$*,*
- *and the* ***space of cohomological cycles*** $\widehat{Z}^*(X) \subset \widehat{Z}_{n+1-*}(X)$ *as the set of* (Z, g) *such that* $\omega(Z, g)$ *lies in* $\varinjlim_{\mathfrak{X},\text{pull}} A^*(\mathfrak{X}_k) \subset \varprojlim_{\mathfrak{X},\text{push}} A^*(\mathfrak{X}_k)$.

Ways of defining intersection pairing on $\widehat{Z}^*(X)$ might slightly differ from setting to setting (depending on where the pairings are already defined), but in any cases the general formula of "$(g_1, Z_1) \cdot (g_2, Z_2) := g_1\omega(Z_2, g_2) + g_2\delta_{Z_1}$" should be easy to be applied.

Example 13.2. The example we did in our section is $A^*(\mathfrak{X}_k) = \mathrm{CH}^*(\mathfrak{X}_k)$ and $A_*(\mathfrak{X}_k) = \mathrm{CH}_*(\mathfrak{X}_k)$. Another example that would work for equicharacteristic dvrs is $A^*(\mathfrak{X}_k) = H^{2*}(\mathfrak{X}_0, \mathbb{Q}_\ell(*))$ and $A_*(\mathfrak{X}_k) = H^{2(n+1-*)}_{\mathfrak{X}_k}(\mathfrak{X}, \mathbb{Q}_\ell(n + 1 - *))$.

We suspect that, by climbing up the tower of models, we are always guaranteed to find a "good" representative to be intersected, in the following sense.

Conjecture 13.1. *Assume the appropriate resolution of singularities. Then,* $\widehat{Z}^*_1(X)$ *is generated by vertical divisors, and* $\widehat{Z}_{1,n+1-*}(X)$ *is generated by intersection of divisors. Or, "every cycle in the projective limit is complete intersection."*

This is motivated by its analogue in the Archimedean case.

Theorem 13.1. *Let* $X/\mathbb{C}$ *be a projective variety, consider the collection of birational maps* $\{\mathfrak{X} \to X\}$*, and let* $\widetilde{\mathrm{CH}}^*(X) = \varinjlim_{\mathfrak{X}} \mathrm{CH}^*(\mathfrak{X})$*. Then,* $\widetilde{\mathrm{CH}}^*(X)$ *is generated by* $\widetilde{\mathrm{CH}}^1(X)$*.*

Proof. We know that $\mathrm{CH}^*(X) = K^*(X)$ and that it is generated by Chern class. Thus, it suffices to show that, for any vector bundle $\mathcal{E}$ on X, there is a birational map $\pi : \mathfrak{X} \to X$ such that $\pi^*\mathcal{E}$ has a complete flag (which will imply that $c(\mathcal{E}) = \sum c(\mathscr{L}_i)$). Let $F \to X$ be the moduli of complete flags of $\mathcal{E}$ (over a large variety it has a complete flag; consider succesive $\mathbb{P}^n$-bundles). Take any flag of $\mathcal{E}_\eta$, where η is the generic point of X, which gives $\xi \in F(\eta)$. Take the Zariski closure $\overline{\xi} \to X$. Take the resolution of singularities $\widetilde{X} \to \overline{\xi} \to X$. This is birational, and over that we indeed have a complete flag. ☐

Questions.

(1) Can you prove this for general cohomological cycles on the topological level?
(2) For a compact manifold X, is $H^*(X)$ generated by characteristic classes of vector bundles over X?

14. p-adic Arakelov Theory for Curves

Recall that what was truly used in the Archimedean case for $A^*(\mathfrak{X}_k)$ and $A_*(\mathfrak{X}_k)$ are differential forms, distributions, currents, $\cdots$. Thus, we might want to follow this strategy, hoping that we have a decent enough theory of calculus over nonarchimedean dvrs/fields. At least for a curve this is very true; we will illustrate how such theory can give a quite explicit setup.

Let X be a projective smooth curve over $\mathbb{C}_p$, defined over a discretely valued field K with $\mathcal{O}_K$ a strictly henselian ring. Hopefully we would like to develop a notion like "smooth differential forms/distributions with at worst log singularities over $X(\mathbb{C}_p)$.

Let X^{an} be the rigid analytic space associated to X. We have a natural notion of differential forms $\Omega^*_{X^{\mathrm{an}}}$ which is just the analytification of sheaves of algebraic Kahler differentials over X. We also can think of the sheaf of "functions with log singularities" as

$$\mathcal{O}_{\log, X^{\mathrm{an}}} := \mathcal{O}_{X^{\mathrm{an}}}[\log(\mathcal{O}(X^{\mathrm{an}})^\times)].$$

To develop a calculable notion of smoothness of forms/functions, it is desirable to have a "good cover" of X^{an}. An ideal kind of cover would be $\{U_i\}$ where each U_i is an affinoid with good reduction, each nonempty $U_i \cap U_j$ is an annuli, and there is no triple overlap. One way to construct such cover is to use a semistable regular model of X, which is always guaranteed to exist after a base change by the semistable reduction theorem. Let $\mathfrak{X}$ be a semistable regular model over $\mathcal{O}_K$. A desired cover can be obtained by taking the preimages of the irreducible components of $\mathfrak{X}_k$ via the specialization map $X^{\mathrm{an}} \to \mathfrak{X}_k$. The combinatorial data of irreducible components of $\mathfrak{X}_k$, which is encoded in the **dual graph** $\Gamma(\mathfrak{X})$, can work as a "skeleton" of $X(\mathbb{C}_p)$.

Recall that the dual graph $\Gamma(\mathfrak{X})$ is defined as a graph with irreducible components of $\mathfrak{X}_k$ as vertices and points in the intersection of two components as edges. The dual graph can be given with a metric such that it is independent of the base field. Namely, for any finite base change L/K, as it is totally ramified ($\mathcal{O}_K$ is strictly henselian!), any intersection of two components, which is of form $\{xy = \pi\}$ in (étale) local coordinates, becomes $\{xy = \pi^{[L:K]}\}$ after the base change. After blowing up each point ($[L:K]-1$) times, the singularity will just add ($[L:K]-1$) points in each edge (i.e. dividing each edge into $[L:K]$ smaller edges). Thus giving a normalized length on $\Gamma(\mathfrak{X})$ enables us to gather all specialization maps $X(L) \to \Gamma(\mathfrak{X})$ into a single map $\mathrm{sp} : X(\mathbb{C}_p) \to \Gamma(\mathfrak{X})$, also called the specialization map. In particular, we can talk about $\mathbb{Q}$-points of $\Gamma(\mathfrak{X})$; we denote it as $\Gamma(\mathfrak{X})(\mathbb{Q})$.

Example 14.1. Let's say $X = \mathbb{P}^1$, and let $\mathfrak{X}$ be the blow-up of $\mathbb{P}^1_{\mathcal{O}_K}$ at $0 \in \mathbb{P}^1_k$ (local equation "$xy = p$"). Then the dual graph is just an edge with two endpoints, $0, 1$. Then $\mathrm{sp} : \mathbb{P}^1(\mathbb{C}_p) \to \Gamma(\mathfrak{X})$ will be the map

$$x \mapsto \begin{cases} 0 & \text{if } \mathrm{val}(x) \leq 0 \\ \mathrm{val}(x) & \text{if } 0 < \mathrm{val}(x) < 1 \\ 1 & \text{if } \mathrm{val}(x) \geq 1 \end{cases}.$$

Thus the preimage of γ of the specialization map is a disc if $\gamma = 0, 1$, and is a torus (annulus if you prefer) if $0 < \gamma < 1$.

Using this, we can now define the notion of **smooth functions** on $\mathbb{P}^1(\mathbb{C}_p)$. Let's cover $\Gamma(\mathfrak{X})$ with three open subsets, $U = [0, 2\varepsilon)$, $W = (\varepsilon, 1-\varepsilon)$, $V = (1-2\varepsilon, 1]$. Then, we would like to see $C^\infty(\mathbb{P}^1(\mathbb{C}_p))$ as some subset of $C^\infty(\mathrm{sp}^{-1}(U)) \oplus C^\infty(\mathrm{sp}^{-1}(V)) \oplus C^\infty(\mathrm{sp}^{-1}(W))$. Over $\mathrm{sp}^{-1}(U)$ and $\mathrm{sp}^{-1}(V)$, we can just declare C^∞ to be convergent power series. The open sets $\mathrm{sp}^{-1}(U), \mathrm{sp}^{-1}(V)$ are **wide open subsets** in Coleman's terms, i.e. the complement of a finite union of disjoint closed discs of radius < 1, with no two discs in the same residue disc.

For $C^\infty(\mathrm{sp}^{-1}(W))$, one needs to be careful. If you think about an analogue over $\mathbb{C}$ instead of $\mathbb{C}_p$, this domain is something like a cylinder, so we have a Fourier expansion along the coordinate of S^1-direction. Thus, we can use this analogy here to define the notion of smoothness using some kind of Fourier expansion. Namely, we can define $C^\infty(\mathrm{sp}^{-1}(W))$ to be consisted of functions of the form

$$\sum_{n=-\infty}^{\infty} a_n(\mathrm{val}(z)) \frac{z^n}{|z|^n},$$

where $a_n : (\varepsilon, 1-\varepsilon) \to \mathbb{C}_p$ is a piecewise polynomial functions where the breakpoints are rational numbers and two polynomials have the same value on the breakpoints. These functions will be referred later as **continuous piecewise smooth functions**, although this is not true in a literal sense as nonarchimedean fields have totally disconnected topology (only for the analogy with real analytic case).

Using this, we can define smooth differential forms as well. Extending the analogy with $\mathbb{C}$ ("annulus is a cylinder"), we need two coordinates. In addition to z, we can use for example $t = \log |z|^2$, which can be regarded as a variable on the dual graph $\Gamma(\mathfrak{X})$, or $\overline{z} = |z|^2/z = t$, which satisfies $t = \log z + \log \overline{z}$. If we use t, then 1-forms are of form $f dz + g dt$, and 2-forms are of form $f dz dt$. For example we can integrate a 2-form in the following way,

$$\int f dz dt = \int_{\Gamma(\mathfrak{X})} dt \int_{X/\Gamma(\mathfrak{X})} f dz = \int_{\Gamma(\mathfrak{X})} dt a_0(\mathrm{val}(z)).$$

The above example illustrates that $X(\mathbb{C}_p) \to \Gamma(\mathfrak{X})$ is like "collapsing S^1-direction" (prototypical example being $S^1 \times [0,1] \to [0,1]$), strengthening the metaphor that "$\Gamma(\mathfrak{X})$ is a skeleton of $X(\mathbb{C}_p)$."

As $\Gamma(\mathfrak{X})$ offers a systematic way of covering $X(\mathbb{C}_p)$, it is reasonable to expect that we can recover topological information about $X(\mathbb{C}_p)$ from $\Gamma(\mathfrak{X})$.

Definition 14.1 (Smooth forms on the dual graph). *Let $A^0(\Gamma(\mathfrak{X}))$ be the space of continuous piecewise smooth function on $\Gamma(\mathfrak{X})$ with breaking points in $\Gamma(\mathfrak{X})(\mathbb{Q})$. Let $A^1(\Gamma(\mathfrak{X}))$ be the space of 1-forms with coefficients in $A^0(\Gamma(\mathfrak{X}))$, namely locally of form $f(t)dt$ for $f \in A^0(\Gamma(\mathfrak{X}))$.*

Definition 14.2 (Smooth form on $X(\mathbb{C}_p)$)**.** *Define*

$$A^0_{(\log,)\Gamma(\mathfrak{X})} = \mathrm{sp}_* \mathcal{O}_{(\log,)X^{\mathrm{an}}} \otimes_{\mathbb{C}_p} A^0(\Gamma(\mathfrak{X})),$$

$$A^1_{(\log,)\Gamma(\mathfrak{X})} = \mathrm{sp}_* \Omega^1_{(\log,)X^{\mathrm{an}}} \otimes_{\mathbb{C}_p} A^0(\Gamma(\mathfrak{X})) \oplus \mathrm{sp}_* \mathcal{O}_{(\log,)X^{\mathrm{an}}} \otimes_{\mathbb{C}_p} A^1(\Gamma(\mathfrak{X})),$$

$$A^2_{(\log,)\Gamma(\mathfrak{X})} = \mathrm{sp}_* \Omega^1_{(\log,)X^{\mathrm{an}}} \otimes_{\mathbb{C}_p} A^1(\Gamma(\mathfrak{X})).$$

That we can find a good cover of $X(\mathbb{C}_p)$ from the specialization map easily implies the following

Proposition 14.1. *The complex of sheaves $A^0_{\Gamma(\mathfrak{X})} \to A^1_{\Gamma(\mathfrak{X})} \to A^2_{\Gamma(\mathfrak{X})}$ computes the algebraic de Rham cohomology of X.*

Proof. Take a good cover $\{U_i\}$ of $X(\mathbb{C}_p)$ by the preimage of a "good cover of $\Gamma(\mathfrak{X})$", namely a cover of $\Gamma(\mathfrak{X})$ consisting of

- the 2ε-neighborhoods of all vertices (which are wide open),
- and the middle open intervals of length $1-\varepsilon$ (i.e. $(\varepsilon, 1-2\varepsilon)$'s in Example 14.1) of all edges.

Then, $\mathcal{O}_X(X^0) \to \mathcal{O}_X(X^1) \oplus \Omega^1_X(X^0) \to \Omega^1_X(X^1)$ computes the de Rham cohomology, where $X^0 = \coprod U_i$, $X^1 = \coprod U_i \cap U_j$. It is obvious that the hypercohomology of $A^0_{\Gamma(\mathfrak{X})} \to A^1_{\Gamma(\mathfrak{X})} \to A^2_{\Gamma(\mathfrak{X})}$ computes exactly the same thing. $\square$

Remark 14.1. The hypercohomology of log-complex $A^0_{\log,\Gamma(\mathfrak{X})} \to A^1_{\log,\Gamma(\mathfrak{X})} \to A^2_{\log,\Gamma(\mathfrak{X})}$ has $H^1 = 0$. The difference of this from the complex $A^0_{\Gamma(\mathfrak{X})} \to A^1_{\Gamma(\mathfrak{X})} \to A^2_{\Gamma(\mathfrak{X})}$ is that we basically use $\log z$ instead of z, which has an effect of "passing to the universal cover," analogous to $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times$.

We can develop a theory of p-adic integration from $A^i_{\Gamma(\mathfrak{X})}$'s.

Definition 14.3 ($\partial\overline{\partial}$-operator)**.** *Define $\partial\overline{\partial} : A^0_{\Gamma(\mathfrak{X})} \to A^2_{\Gamma(\mathfrak{X})}$ by*

$$\partial\overline{\partial}(\sum a_n(t) z^n) = \partial \sum a_n'(t) z^n d\log\overline{z} = \left(\sum a_n''(t) z^n + \sum a_n'(t) n z^n\right) d\log z \wedge d\log\overline{z},$$

where the differentiation of a_n's is just the differentiation of polynomials.

Remark 14.2. One can use $\frac{dz\wedge dt}{z} = d\log z \wedge d\log\overline{z}$ in the above equation.

We have already seen how to integrate a 2-form, namely $\int_{X(\mathbb{C}_p)} : A^2_{\Gamma(\mathfrak{X})} \to \mathbb{C}_p$ is just $\int_{\Gamma(\mathfrak{X})}\int_{X(\mathbb{C}_p)/\Gamma(\mathfrak{X})}$, which locally is $\int \sum_{n=-\infty}^{\infty} a_n(\mathrm{val}(z)) \frac{z^n}{|z|^n} dz dt = \int_{\Gamma(\mathfrak{X})} dt \int_{X/\Gamma(\mathfrak{X})} f dz = \int_{\Gamma(\mathfrak{X})} dt a_0(\mathrm{val}(z))$. On the other hand, integrating 1-forms is a little more mysterious. There are two main theories of p-adic integration (of 1-forms) on p-adic curves.

1. **Coleman integration.** This works for affinoids or wide open subsets with good reduction. For such a set U, Coleman integration defines a map

$$\Gamma(U, \Omega^1_U) \times U \times U \to \mathbb{C}_p, (\omega, p, q) \mapsto \int_p^q \omega,$$

   which is, as a function of p, q, analytic in residue discs. It has many nice properties, for example it is additive and has change of variable.
2. **Colmez integration.** It works for meromorphic forms without reduction hypothesis but only for (analytification of) algebraic varieties. It has similar properties as above.

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