HW #10

ALGEBRAIC NUMBER THEORY, GU4043; INSTRUCTOR: GYUJIN OH

Due Tuesday, April 2 by 11:59pm on Gradescope.

Question 1. Let L be a field, and let K_1, K_2 be two commutative L-algebras. We aim to provide several ways to think about the tensor product $K_1 \otimes_L K_2$.

(1) Note that $K_1 \otimes_L K_2$ has natural L-algebra homomorphisms

$$\iota_1 : K_1 \xrightarrow{x \mapsto x \otimes 1} K_1 \otimes_L K_2, \quad \iota_2 : K_2 \xrightarrow{x \mapsto 1 \otimes x} K_1 \otimes_L K_2.$$

Show that $K_1 \otimes_L K_2$ satisfies the **universal property of tensor products of commutative algebras**, as follows. If R is a commutative L-algebra, and if $f_1 : K_1 \to R$, $f_2 : K_2 \to R$ are L-algebra homomorphisms, then there exists a unique L-algebra homomorphism $f : K_1 \otimes_L K_2 \to R$ such that

$$f_1 = f \circ \iota_1, \quad f_2 = f \circ \iota_2.$$

(2) Show that the above universal property uniquely characterizes $K_1 \otimes_L K_2$ as an L-algebra. Namely, show that if a commutative L-algebra S with L-algebra homomorphisms $j_1 : K_1 \to S$ and $j_2 : K_2 \to S$ satisfies the above universal property (i.e. given any two maps $f_1 : K_1 \to R$, $f_2 : K_2 \to R$, there is a unique map $f : S \to R$ such that $f_1 = f \circ j_1$, $f_2 = f \circ j_2$), then $S \cong K_1 \otimes_L K_2$.

(3) Let X be the L-vector space spanned by the basis vectors $v \otimes w$ for any pair of $v \in K_1, w \in K_2$, and endow the L-algebra structure by defining the multiplication to be $(v_1 \otimes w_1)(v_2 \otimes w_2) = (v_1 v_2) \otimes (w_1 w_2)$. Let $I \subset X$ be the L-vector subspace spanned by the following elements:

$$\begin{aligned} I = \langle \{ &(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w \ : \ v_1, v_2 \in K_1, w \in K_2 \}, \\ &\{ v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2 \ : \ v \in K_1, w_1, w_2 \in K_2 \}, \\ &\{ t(v \otimes w) - (tv) \otimes w \ : \ t \in L, v \in K_1, w \in K_2 \}, \\ &\{ t(v \otimes w) - v \otimes (tw) \ : \ t \in L, v \in K_1, w \in K_2 \} \rangle. \end{aligned}$$

Show that $I \subset X$ is an ideal.

(4) Show that the L-algebra X/I, together with the natural maps

$$j_1 : K_1 \xrightarrow{x \mapsto x \otimes 1} X/I, \quad j_2 : K_2 \xrightarrow{x \mapsto 1 \otimes x} X/I,$$

satisfies the universal property of (1). This gives another construction of $K_1 \otimes_L K_2$.

Question 2. Let L be a p-adic local field of characteristic 0.

(1) Using Hensel's lemma, show that a finite field extension K/L is unramified if and only if $K = L(\zeta_n)$ for some $(n, p) = 1$.

(2) If K/L is an unramified extension, and if M is a p-adic local field of characteristic 0, show that KM/LM is unramified.

Question 3. Let $K = \mathbb{Q}(\alpha, i)$, where $\alpha^4 = 2$ and $i^2 = -1$. Note that $K/\mathbb{Q}$ is Galois with $G := \mathrm{Gal}(K/\mathbb{Q}) \cong D_4$, a dihedral group, generated by $s, t \in G$ where

$$s(\alpha) = i\alpha, \quad s(i) = i, \quad t(\alpha) = \alpha, \quad t(i) = -i,$$

so that $s^4 = t^2 = 1$ and $tst^{-1} = s^{-1}$. Note that K contains two particular subfields, $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(i)$.

(1) Show that 2 is totally ramified in both $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(i)$.

(2) Show that $\mathbb{Q}_2(\alpha) \cap \mathbb{Q}_2(i) = \mathbb{Q}_2$.

Hint. Otherwise, $\mathbb{Q}_2(\alpha) \supset \mathbb{Q}_2(i)$, and therefore $\mathbb{Q}_2(\alpha)/\mathbb{Q}_2(i)$, which is a quadratic extension, is automatically Galois. Show that the nontrivial element $\sigma \in \mathrm{Gal}(\mathbb{Q}_2(\alpha)/\mathbb{Q}_2(i))$ must send $\sigma(\alpha) = -\alpha$. This implies that $\mathbb{Q}_2(\sqrt{2}) = \mathbb{Q}_2(i)$. Deduce a contradiction using HW9.

(3) Show that $K_2 := K \otimes_{\mathbb{Q}} \mathbb{Q}_2$ is a field.

(4) Show that $K_2/\mathbb{Q}_2$ is totally ramified. Deduce that 2 is totally ramified in K.

Hint. Suppose not. As $\mathbb{Q}_2(\alpha)/\mathbb{Q}_2$ is totally ramified, it should be the case that $e_{K_2/\mathbb{Q}_2} = 4$ and $f_{K_2/\mathbb{Q}_2} = 2$. Therefore, the maximal unramified extension of $\mathbb{Q}_2$ in K_2 is a quadratic extension of $\mathbb{Q}_2$. Using that $\mathrm{Gal}(K_2/\mathbb{Q}_2) \cong D_4$, enumerate all quadratic subfields of K_2, and show that they are all ramified over $\mathbb{Q}_2$ (use HW9), yielding a contradiction.

(5) Show that any rational prime $p \neq 2$ is unramified in K.

Question 4. What we have learned so far suggests that **absolute values correspond to primes** – from this perspective, the archimedean absolute values should be primes! In this analogy, we regard $\mathbb{R}$ and $\mathbb{C}$ as local fields as well, and they are called either **∞-adic local fields** or **archimedean local fields**. Given a number field K, an embedding $i : K \hookrightarrow \mathbb{C}$ defines an **archimedean prime** of K, where a real embedding defines a **real prime**, and a pair of complex embeddings defines a **complex prime**. The extension $\mathbb{C}/\mathbb{R}$ is considered **ramified**

We will see that many aspects of theory of primes and the local class field theory translate well into the case of archimedean primes and local fields.

(1) Let K/L be an extension of number fields. Using the above perspective, define what it means for an archimedean prime of K to lie over an archimedean prime of L.

(2) Retaining the setup of (1), define what it means for an archimedean prime of L to be unramified in K.

(3) The local Artin map for $\mathbb{R}$ can be defined as

$$\mathrm{Art}_{\mathbb{R}} : \mathbb{R}^\times \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \{\pm 1\}, \quad x \mapsto \frac{x}{|x|}.$$

Show that Part (1) of local Artin reciprocity (Theorem 15.10(1) of the notes) holds.

(4) State and prove the local existence theorem for $L = \mathbb{R}$.