HW #12

ALGEBRAIC NUMBER THEORY, GU4043; INSTRUCTOR: GYUJIN OH

Due Tuesday, April 16 by 11:59pm on Gradescope.

Question 1. Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field with $d > 1$ is a square-free integer with $d \equiv 1 \pmod 4$, i.e. $\mathrm{disc}(K) = d$, and $\mathcal{O}_K = \mathbb{Z}[\frac{\sqrt{d}+1}{2}]$.

(1) Show that the continued fraction of

$$\left\lfloor \frac{\sqrt{d}+1}{2} \right\rfloor + \frac{\sqrt{d}-1}{2},$$

is purely periodic.

(2) Let $(a_0; a_1, \cdots)$ be the continued fraction of $\frac{\sqrt{d}-1}{2}$, whose period is ℓ. Show that the fundamental unit of K is $\epsilon = P_{\ell-1} + Q_{\ell-1}\frac{\sqrt{d}+1}{2}$.

Question 2. Let $K = \mathbb{Q}(\sqrt{d}) \subset \mathbb{R}$ be a real quadratic field, with $d > 1$ a square-free integer. The **sign** of K is $N(K) := N_{K/\mathbb{Q}}(\epsilon)$, where ϵ is the fundamental unit of K (i.e. the smallest unit > 1).

(1) Show that $N(K) = -1$ if and only there is a unit whose norm is -1. Deduce that $N(K) = -1$ if and only if the equation $x^2 - \mathrm{disc}(K)y^2 = -4$ has integer solutions $x, y \in \mathbb{Z}$.

(2) If d has a prime factor that is $\equiv 3 \pmod 4$, show that $N(K) = 1$.

(3) Let $\mathfrak{m}$ be the modulus of K such that $\mathfrak{m}_f = 1$ and $\mathfrak{m}_\infty$ is the product of the two real embeddings of K. Show that the natural surjective map $\mathrm{Cl}_K^{\mathfrak{m}} \to \mathrm{Cl}(K)$ is an isomorphism if and only if there exists a unit of norm -1.

(4) Using (2) and (3), deduce that if d has a prime factor that is $\equiv 3 \pmod 4$, then there is an abelian extension L/K that is strictly bigger than H_K and is unramified at every prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ (cf. Question 1, HW11).

Question 3. Let χ be a primitive odd Dirichlet character of modulus m, and let

$$\widetilde{\theta}_\chi(iy) = \sum_{n\in\mathbb{Z}} \chi(n) n \sqrt{y} e^{-\pi n^2 y}, \quad y > 0.$$

(1) Show that, for $\mathrm{Re}(s) > 1$,

$$\pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi) = \int_0^\infty y^{\frac{s}{2}} \frac{\widetilde{\theta}_\chi(iy)}{2} \frac{dy}{y}.$$

(2) Using the Poisson summation formula, show that

$$\widetilde{\theta}_\chi(iy) = -\frac{iG(\chi)}{m\sqrt{y}} \widetilde{\theta}_{\overline{\chi}}\left(\frac{i}{m^2 y}\right).$$

Hint. The Fourier transform of $f(x) = xe^{-\pi x^2}$ is $\widehat{f}(x) = -ixe^{-\pi x^2}$.

Question 4. Let χ be a Dirichlet character. Note that the Euler product expansion of $L(s,\chi)$ implies that, for $\mathrm{Re}(s) > 1$,

$$\log L(s,\chi) = -\sum_{p \text{ prime}} \log(1 - \chi(p)p^{-s}).$$

(1) Show that, for $\mathrm{Re}(s) > 1$,

$$\log L(s,\chi) = \sum_{p \text{ prime}} \sum_{n=1}^{\infty} \frac{\chi(p)^n p^{-ns}}{n},$$

and the double infinite sum on the right hand side is absolutely convergent.

(2) Show that there exists a constant $C > 0$ such that for any χ and $\mathrm{Re}(s) > 1$,

$$\left| \log L(s,\chi) - \sum_{p \text{ prime}} \chi(p)p^{-s} \right| < C.$$

(3) Let $n > 1$, and let $a \in (\mathbb{Z}/n\mathbb{Z})^\times$. Show that

$$\left| \frac{1}{\varphi(n)} \left(\sum_{\chi \in \widehat{(\mathbb{Z}/n\mathbb{Z})^\times}} \overline{\chi(a)} \log L(s,\chi) \right) - \left(\sum_{p \text{ prime},\, p \equiv a \pmod{n}} p^{-s} \right) \right| < C.$$

Deduce that there are infinitely many primes that are $\equiv a \pmod{n}$.

Hint. Show that $\lim_{s \to 1^+} \sum_{p \text{ prime},\, p \equiv a \pmod{n}} p^{-s}$ diverges.