HW #6

ALGEBRAIC NUMBER THEORY, GU4043; INSTRUCTOR: GYUJIN OH

Due Tuesday, February 27 by 11:59pm on Gradescope.

Question 1. Let $K = \mathbb{Q}(\zeta_n)$ with $n > 2$.

(1) Show that there is no real embedding of K.
(2) Show that $K^+ = \mathbb{Q}(\zeta_n + \zeta_n^{-1}) = \mathbb{Q}(\cos(\frac{2\pi}{n}))$ is a subfield of K with $[K : K^+] = 2$.
(3) Show that every embedding $\iota : K^+ \hookrightarrow \mathbb{C}$ is a real embedding.
(4) Show that $\mathcal{O}_{K^+} = \mathbb{Z}[\zeta_n + \zeta_n^{-1}]$.

Question 2. Let $K = \mathbb{Q}(\sqrt{6})$, so that $\mathcal{O}_K = \mathbb{Z}[\sqrt{6}]$. We would like to show that $h_K = 1$, i.e. $\mathbb{Z}[\sqrt{6}]$ is a principal ideal domain.

(1) Use the Minkowski's bound to show that any ideal class has an integral ideal representative $\mathfrak{a}$ with $N(\mathfrak{a}) \leq 2$.
(2) Show that there is a unique prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ ideal lying over (2).
(3) Show that $\mathfrak{p}$ is principal by showing that $\mathfrak{p} = (2 + \sqrt{6})$. Conclude that $h_K = 1$.

Question 3. Let $K = \mathbb{Q}(\sqrt{10})$, so that $\mathcal{O}_K = \mathbb{Z}[\sqrt{10}]$. We would like to show that $h_K = 2$.

(1) Use the Minkowski's bound to show that any ideal class has an integral ideal representative $\mathfrak{a}$ with $N(\mathfrak{a}) \leq 3$.
(2) Show that there is a unique prime ideal $\mathfrak{p}_2 \subset \mathcal{O}_K$ lying over (2), with $N(\mathfrak{p}_2) = 2$.
(3) Show that there is no element $\alpha \in \mathcal{O}_K$ with norm ± 2. Conclude that $\mathfrak{p}_2$ is not a principal ideal, and its ideal class in $\mathrm{Cl}(K)$ is a nontrivial order 2 element.

   Hint. Use that $\left(\frac{\pm 2}{5}\right) = -1$.
(4) Show that (3) splits completely in K, with $(3) = \mathfrak{p}_3\mathfrak{p}_3'$, so that $N(\mathfrak{p}_3) = N(\mathfrak{p}_3') = 3$.
(5) Using that $N_{K/\mathbb{Q}}(4 + \sqrt{10}) = 6$, deduce that the ideal classes of $\mathfrak{p}_3$ and $\mathfrak{p}_3'$ are both the same as the ideal class of $\mathfrak{p}_2$. Conclude that $h_K = 2$.

   Hint. After possibly switching $\mathfrak{p}_3$ and $\mathfrak{p}_3'$, $\mathfrak{p}_2\mathfrak{p}_3 = (4 + \sqrt{10})$, so $[\mathfrak{p}_2]^{-1} = [\mathfrak{p}_3]$ in $\mathrm{Cl}(K)$.

Question 4. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{5})$. We would like to show that $h_K = 1$.

(1) Using the Minkowski's bound to show that any ideal class has an integral ideal representative $\mathfrak{a}$ with $N(\mathfrak{a}) \leq 3$.

   Hint. Use $K = \mathbb{Q}(\sqrt{2})\mathbb{Q}(\sqrt{5})$ to compute $\mathrm{disc}(K)$.
(2) Using how (2) factorizes in $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$, show that the prime ideal factorization of (2) in K is

$$(2) = \mathfrak{p}_2^2,$$

for a unique prime ideal $\mathfrak{p}_2 \subset \mathcal{O}_K$ lying over 2. Deduce that $N(\mathfrak{p}_2) = 4$.

(3) Using how (3) factorizes in $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$, show that 3 does not split completely in K. Deduce that there is no integral prime ideal of $\mathcal{O}_K$ with norm exactly equal to 3. Deduce that $h_K = 1$.

(4) We saw in Question 3 that $h_{\mathbb{Q}(\sqrt{10})} = 2$. In fact, $\mathbb{Q}(\sqrt{2}, \sqrt{5})$ is the **Hilbert class field** of $\mathbb{Q}(\sqrt{10})$; the Hilbert class field of a number field F is the extension H_F/F of degree h_F satisfying very nice properties, which we will learn in the section on Artin reciprocity law. One particularly pleasing property is that every non-principal ideal $\mathfrak{a} \subset \mathcal{O}_F$ becomes a principal ideal in the Hilbert class field H_F of F (i.e., $\mathfrak{a}\mathcal{O}_{H_F}$ is a principal ideal).

We saw in Question 3(3) that the unique prime ideal of $\mathbb{Q}(\sqrt{10})$ lying over 2 is not a principal ideal. Show that this ideal generates a **principal ideal** over $\mathbb{Q}(\sqrt{2}, \sqrt{5})$.