# HW #9

ALGEBRAIC NUMBER THEORY, GU4043; INSTRUCTOR: GYUJIN OH

Due Tuesday, March 26 by 11:59pm on Gradescope.

**Question 1.** Let $K$ be a valued field with a non-archimedean absolute value $|\cdot|$.

(1) Let $D(a,r)$ be the open disk of radius $r>0$ centered at $a\in K$. For any $b\in D(a,r)$, show that $D(a,r)=D(b,r)$ (i.e. any point in an open disk is its center).
(2) Show that $|\cdot|:K\to\mathbb{R}_{\geq 0}$ is continuous.
(3) If $|\cdot|$ is discrete, show that any open disk is closed.
(4) If $K$ is furthermore complete, show that the infinite sum $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} a_n=0$.

**Question 2.**

(1) Find all integers $a\in\mathbb{Z}$ such that $5X^2=a$ has a solution in $\mathbb{Z}_5=\mathcal{O}_{\mathbb{Q}_5}$.

**Hint.** Use Hensel's lemma.

(2) Show that $f(X)=X^4-2X^3-X^2+6$ is irreducible in $\mathbb{Q}[X]$.

**Hint.** Use the 2-adic Newton polygon and Gauss's lemma to deduce that, if $f(X)$ is not irreducible, $f(X)=g(X)h(X)$ for $g(X),h(X)\in\mathbb{Z}[X]$ of degree 2. You may want to use the information obtained from Hensel's lemma mod 2 and mod 3.

**Question 3.** Let $p>2$ be a rational prime, and let $v_p$ be the normalized discrete valuation on $\mathbb{Q}_p$ (i.e. $v_p(p)=1$; cf. HW8).

(1) Show that, for $n\geq 1$,
$$v_p(n!)=\sum_{k=1}^{\infty}\left\lfloor \frac{n}{p^k}\right\rfloor<\frac{n}{p-1}.$$
(2) Let $K$ be a finite extension of $\mathbb{Q}_p$, equipped with the extension of $v_p$. Let $\pi$ be a uniformizer, and let $e=e_{K/\mathbb{Q}_p}$, so that $v_p(\pi)=\frac{1}{e}$. Show that, for $x\in\pi^r\mathcal{O}_K$ with $r>\frac{e}{p-1}$, the infinite sum
$$e^x:=\sum_{n=0}^{\infty}\frac{x^n}{n!},$$
converges to an element in $1+\pi^r\mathcal{O}_K$.
(3) Under the same setup as (2), show that the infinite sum
$$\log(1+x):=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n},$$
converges to an element in $\pi^r\mathcal{O}_K$.

(4) Using (2) and (3), show that the multiplicative group $(1+\pi^r\mathcal{O}_K, \times)$ and the additive group $(\pi^r\mathcal{O}_K, +)$ are isomorphic to each other.

**Question 4.** Let $K/L$ be an extension of $p$-adic local fields of degree $n$. We say that $K/L$ is **tamely ramified** if $(p, e_{K/L}) = 1$. Otherwise, i.e. if $p|e_{K/L}$, we say that $K/L$ is **wildly ramified**. In this question, we want to show that totally ramified extensions that are tamely ramified (**totally tamely ramified** in short) has a simpler description.[1]

(1) Suppose $(n, p) = 1$, and let $\pi_L$ be a uniformizer of $L$. Show that $K := L(\pi_L^{1/n})$ is a totally tamely ramified extension of $L$.

(2) Suppose that $K/L$ is totally tamely ramified (so that $(n, p) = 1$). Let $\pi_K$ be a uniformizer of $K$. Show that any element $x \in 1 + \pi_K\mathcal{O}_K$ has an $n$-th root in $K$.

**Hint.** Use Hensel's lemma; $x \pmod{\pi_K} = 1$ has an obvious $n$-th root.

(3) In the setup of (2), show that there exists a unit $u \in \mathcal{O}_K^\times$ such that $\left(\frac{\pi_K}{u}\right)^n \in L$. Deduce that $K = L(\pi_L^{1/n})$ for some uniformizer $\pi_L$ of $L$.

**Hint.** A priori, $\pi_K^n = u'\pi_L$ for a uniformizer $\pi_L$ of $L$ and a unit $u' \in \mathcal{O}_K^\times$. Show that one can choose a different uniformizer of $L$ so that $u' \equiv 1 \pmod{\pi_K}$. Then, use (2).

**Question 5.**

(1) Let $p$ be an odd rational prime. Show that an element $x = p^n u \in \mathbb{Q}_p$, $n \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^\times$, is a square in $\mathbb{Q}_p$, if and only if $n$ is even and $u$ is a square mod $p$.

(2) Show that an element $x = 2^n u \in \mathbb{Q}_2$, $n \in \mathbb{Z}$ and $u \in \mathbb{Z}_2^\times$, is a square in $\mathbb{Q}_2$, if and only if $n$ is even and $u \equiv 1 \pmod 8$.

(3) Show that there are in total $7$ isomorphism classes of quadratic extensions of $\mathbb{Q}_2$ and $3$ isomorphism classes of quadratic extensions of $\mathbb{Q}_p$ for $p$ odd. How many are ramified?

**Hint.** For any field $K$ of characteristic $\neq 2$, isomorphism classes of quadratic extensions of $K$ are in bijection with non-trivial elements of $K^\times/(K^\times)^2$.

[1] This question tells us that the Eisenstein polynomial can be taken to be $X^n - \pi_L$ for a uniformizer $\pi_L$. A field extension obtained by adjoining an $n$-th root of an element downstairs is called a **Kummer extension**.