G-SHTUKAS

GYUJIN OH

Contents

I will introduce various objects that will play basic roles in the proof of global Langlands by Vincent Lafforgue. For safety, I assume that every G we consider here is split. Global pertinent definitions: $F = \mathbb{F}_q(X)$ is a function field of a smooth projective geometrically irreducible curve X over $\mathbb{F}_q$, and G is a (split) reductive group over F.

1. Bun_G and Automorphic Forms

1.1. **Automorphic forms.** Suppose for now G is a reductive group over $\mathbb{Q}$. Recall that an automorphic form in a classical sense is a function on $G(\mathbb{Q})\backslash G(\mathbb{A})$ such that

- f is smooth (or L^2 mod center, whatever).
- f is right K-invariant for some compact open subgroup $K \subset G(\mathbb{A})$ ("level").
- f has "moderate growth" (the growth rate when f escapes to infinity, can be defined precisely using Siegel sets).
- f is $Z(\mathfrak{g})$-finite (e.g. forming a finite-dimensional representation via Laplacian for $G = \mathrm{SL}_2$).

We say an automorphic form is cuspidal if it "vanishes at cusps", or more robustly

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f(ng)dn = 0$$

for almost every $g \in G(\mathbb{A})$, for any unipotent radical N of a parabolic subgroup of G. We can imagine ourselves applying the same condition in the function field setting, except two possible ambiguities.

- What's the moderate growth condition? (What's "escaping to infinity"?)
- What's $Z(\mathfrak{g})$-finiteness? (What are "differential equations"?)

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Recall that cuspidality means "vanishing at cusps" so one can imagine there is no growth condition needed in function field case. It turns out that it is indeed true because a cusp form is automatically compactly supported mod center [Har].

For $Z(\mathfrak{g})$-finiteness, recall this is just the admissibility condition at infinity (each K-type occurs finitely many times). Thus one can imagine declaring one place to be infinity and use admissibility (for td groups, recall admissible = any vector has open stabilizer + any compact open fixes finite-dimensional subspace) at that place to be the $Z(\mathfrak{g})$-finiteness. It turns out that this is indeed true (i.e. independent of choice of place), and this is further equivalent to $Z(\mathbb{A})$-finiteness [BJ].

So in our case it is reasonable to define

$$C_{\text{cusp}}(G(F)\backslash G(\mathbb{A})/K\Xi, \overline{\mathbb{Q}_\ell}),$$

the set of locally constant functions satisfying cuspidality condition, be the space of cusp forms of level K where K is a compact open subgroup of $G(\mathbb{A})$ and $\Xi \subset Z(F)\backslash Z(\mathbb{A})$ is a lattice of cofinite volume. **Given a level this is known to be finite** [Har].

1.2. **Moduli stack of G-torsors.** On the other hand the domain of automorphic forms is very related to Bun_G, the moduli stack of G-torsors on X. We define Bun_G as $\mathrm{Bun}_G(S) = \{G\text{-torsor on } X \times_{\mathbb{F}_q} S\}$. Recall that a G-torsor is $P \to X$ such that $P \times_X P \cong G \times_X P$ and has section locally (general definition is with respect to fpqc topology, but here étale topology suffices, as a smooth morphism has an étale local section). Recall we have the Weil uniformization

$$\mathrm{Bun}_G(\mathbb{F}_q) = G(F)\backslash G(\mathbb{A})/G(\mathcal{O}),$$

where the bijection realizes the discrepancies between the generic trivialization and the local trivialization at each closed point of X. Similarly with a level structure

$$\mathrm{Bun}_{G,N}(\mathbb{F}_q) = G(F)\backslash G(\mathbb{A})/K_N,$$

where $\mathrm{Bun}_{G,N}$ classifies G-torsors with level structure (i.e. isomorpism $\mathcal{E}|_N \cong G_N$) and K_N is the congruence subgroup assiciated to N (more precisely if $N = \sum a_i x_i$ then $K_N = \prod_{x \neq x_i} G(\mathcal{O}_x) \times \prod_{x = x_i} \ker(G(\mathcal{O}_{x_i}) \to G(\mathcal{O}_{x_i}/\mathfrak{m}_{x_i}^{a_i})))$.

Remark 1.1. It is actually not literally true that $\mathrm{Bun}_G(\mathbb{F}_q) = G(F)\backslash G(\mathbb{A})/G(\mathcal{O})$. Rather it is just an identification of **underlying class of objects** as $\mathrm{Bun}_G(\mathbb{F}_q)$ is not just a set but really a groupoid. A more precise description would rather be

$$\mathrm{Bun}_G(\mathbb{F}_q) = \coprod_{\mathcal{E}\ G\text{-torsor}/\sim} [\mathrm{pt}/\operatorname{Aut}(\mathcal{E})].$$

However we are deliberately ignoring non-reducedness issues (after all we can take reduction to everything we consider here), so it doesn't really matter.

Remark 1.2. $\pi_0(\mathrm{Bun}_G) \cong \pi_1(G)$ $\left(= \frac{\text{coweight lattice}}{\text{coroot lattice}}\right)$.

Remark 1.3. The cuspidality condition can be thought more geometrically as having vanishing Jacquet functor for all Levis M of parabolics $P \subset G$. Indeed the Jacquet functor for M can be geometrically defined as the cohomological correspondence $\mathrm{Sht}_{G,N,I} \leftarrow \mathrm{Sht}_{P,N,I} \to \mathrm{Sht}_{M,N,I}$ coming from $G \hookleftarrow P \twoheadrightarrow M$ (the stack of P-shtukas is also defined in [Var]).

1.3. **Hecke stack and Hecke correspondence.** We now motivate the notion of Hecke stacks.

Think about the classical action of a (spherical) Hecke algebra; it is a convolution. To "unfold" convolution we can think of the following identification

$$G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x) = G(F_x)\backslash(G(F_x)/G(\mathcal{O}_x) \times G(F_x)/G(\mathcal{O}_x)),$$

given by $[g] \mapsto (1, [g])$. What is this? Note that if $G = \mathrm{GL}_n$, $\mathrm{GL}_n(F_x)/\mathrm{GL}_n(\mathcal{O}_x)$ is the set of lattices, which means $\mathcal{O}_x^{\oplus n} \subset F_x^{\oplus n}$. And the left-action of $G(F_x)$ changes two lattices simultaneously, so $G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x)$ is somehow the set of "relative positions" of two lattices. This can be made precise, because by Cartan decomposition, literally $G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x) = X_*(T)/W = X_*(T)^+$, where $X_*(T)$ is the coweight lattice. We can then conversely **define** the relative position of two lattices (or rather, two elements in affine Grassmannian) to be the dominant coweight corresponding via this identification.

The action of $f \in C_c(G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x))$ on an automorphic form φ should be then interpreted as

$$(f * \varphi)(\mathcal{E}) = \sum_{\mathcal{E},\mathcal{E}' \ G\text{-torsors over } X, \text{ identified outside } x} f(\mathcal{E},\mathcal{E}')\varphi(\mathcal{E}'),$$

where $f(\mathcal{E},\mathcal{E}')$ is the function induced from f of spherical Hecke algebra by just noticing relative positions of $\mathcal{E}_x, \mathcal{E}'_x$. Thus, if we define $\mathcal{H}_{x,\lambda} = \{(\mathcal{E} \overset{\phi}{\dashrightarrow} \mathcal{E}')\}$ where ϕ means a modification at x with relative position λ, the Hecke operator can then be understood that the cohomological correspondence

$$\begin{tikzcd} & \mathcal{H}_{x,\lambda} \arrow[dl, "h_{\leftarrow}"'] \arrow[dr, "h_{\rightarrow}"] & \\ \mathrm{Bun}_G(\mathbb{F}_q) & & \mathrm{Bun}_G(\mathbb{F}_q) \end{tikzcd}$$

where $h_{\leftarrow}(\mathcal{E} \overset{\phi}{\dashrightarrow} \mathcal{E}') = \mathcal{E}$ and $h_{\rightarrow}(\mathcal{E} \overset{\phi}{\dashrightarrow} \mathcal{E}') = \mathcal{E}'$; here pushforward $(h_{\rightarrow})_*$ in this "cohomological correspondence" means summing over the fiber. We will call this $T_{\lambda,x}$.

2. G-shtukas

2.1. ***G*-shtukas, Bun_G and Hecke stacks.** Recall that we have defined Drinfeld shtukas a few talks ago as follows.

Definition 2.1 (Drinfeld shtukas). *Let S be any scheme over X. A (left)* Drinfeld shtuka *over S is $((x_i)_{i=1,2}, \mathcal{E}_0 \xrightarrow{\phi_1} \mathcal{E}_1 \xrightarrow{\phi_2} {}^\tau\mathcal{E}_0)$ where $x_1, x_2 \in X(S)$, $\mathcal{E}_0, \mathcal{E}_1$ rank n vector bundles on $X \times_{\mathbb{F}_q} S$, ${}^\tau\mathcal{E}_0 := (\mathrm{id}, \tau)^*\mathcal{E}_0$ with a diagram*

$$\begin{tikzcd} \mathcal{E}_0 \arrow[dr, hook, "\phi_1"] & \\ & \mathcal{E}_1 \\ {}^\tau\mathcal{E}_0 \arrow[ur, hook, "\phi_2^{-1}"'] & \end{tikzcd}$$

such that

- $\operatorname{coker} \phi_1$ *is supported on $\Gamma_{x_1} \subset X \times S$ and it is a line bundle on support,*
- $\operatorname{coker} \phi_2^{-1}$ *is supported on $\Gamma_{x_2} \subset X \times S$ and it is a line bundle on support.*

We can unfold this definition into the collection of:

- $\mathcal{E}_0 \overset{\phi_1}{\dashrightarrow} \mathcal{E}_1 \overset{\phi_2}{\dashrightarrow} \mathcal{E}_2$ such that ϕ_1 is a modification at x_1 of relative position $(1,0)$ ("St") and ϕ_2 is a modification at x_2 of relative position $(0,-1)$ ("St*"),
- ${}^\tau\mathcal{E}_0 \cong \mathcal{E}_2$.

Thus one can imagine a diagram of form

$$\begin{tikzcd}
\mathrm{Sht}^{\{1\}\amalg\{2\}}_{\mathrm{GL}_2,\{1,2\},\mathrm{St}\boxtimes\mathrm{St}^*}(\mathbb{F}_q) \arrow[r, "(\mathcal{E}_0,\mathcal{E}_1,{}^\tau\mathcal{E}_0)"] \arrow[d] & (\mathcal{E}_0 \arrow[r, dashed, "\phi_1"] & \mathcal{E}_1 \arrow[r, dashed, "\phi_2"] \arrow[d, "(\mathcal{E}_0,\mathcal{E}_2)"] & \mathcal{E}_2) \\
\mathrm{Bun}_G(\mathbb{F}_q) \arrow[rr, "(\mathrm{id},\mathrm{Frob})"] & & \mathrm{Bun}_G(\mathbb{F}_q)\times\mathrm{Bun}_G(\mathbb{F}_q) &
\end{tikzcd}$$

The top right corner looks very similar to our interpretation of Hecke correspondence. The philosophy of Beilinson-Drinfeld affine Grassmannian and Beauville-Laszlo theorem says that, one can define a global object by letting the point of modification to vary. Combining this, we can define the notion of iterated Hecke stacks/iterated G-shtukas as follows.

Definition 2.2 (Iterated Hecke stack). *The* iterated Hecke stack $\mathrm{Hk}^\alpha_{G,I,W}$ *is, given I a finite set, W an I-tuple of dominant coweights of G (or equivalently a representation of $(\widehat{G})^I$), and a map $\alpha : I \to \{1,\cdots,n\}$ (one should think of this as a partition), a functor such that, for an $\mathbb{F}_q$-scheme S, $\mathrm{Hk}^\alpha_{G,I,W}(S)$ is the set of*

- *$(\mathcal{E}_0,\cdots,\mathcal{E}_n)$, all elements of $\mathrm{Bun}_G(S)$,*
- *$(x_i)_{i\in I}$, all points of $X(S)$,*
- *$\mathcal{E}_0 \overset{\varphi_1}{\dashrightarrow} \mathcal{E}_1 \cdots \mathcal{E}_{n-1} \overset{\varphi_n}{\dashrightarrow} \mathcal{E}_n$, where φ_r is a modification at $\cup_{i\in\alpha^{-1}(r)}\Gamma_{x_i}$ such that the relative position at x_i is bounded above by the dominant coweight of W_i.*

Note that it has obvious maps $\mathrm{Hk}^\alpha_{G,I,W} \to X^I$ and $h_i : \mathrm{Hk}^\alpha_{G,I,W} \to \mathrm{Bun}_G$.

Definition 2.3 (Iterated shtukas). *The moduli of* iterated shtukas $\mathrm{Sht}^\alpha_{G,I,W}$ *is defined by the cartesian square*

$$\begin{tikzcd}
\mathrm{Sht}^\alpha_{G,I,W} \arrow[r] \arrow[d] & \mathrm{Hk}^\alpha_{G,I,W} \arrow[d, "(h_0,h_n)"] \\
\mathrm{Bun}_G \arrow[r, "(\mathrm{id},\mathrm{Frob})"] & \mathrm{Bun}_G \times_{\mathbb{F}_q} \mathrm{Bun}_G
\end{tikzcd}$$

This also has a morphism of legs $\mathfrak{p}^\alpha_{G,I,W} : \mathrm{Sht}^\alpha_{G,I,W} \to X^I$.

Similarly we can define the **iterated Beilinson-Drinfeld Grassmannian** $\mathrm{Gr}^\alpha_{G,I,W}$ as the functor parametrizing not only the data of $\mathrm{Hk}^\alpha_{G,I,W}$ but also the trivialization of $\mathcal{E}_n$.

Some more super/subscripts to add:

- We can impose the level structure on everything by replacing the curve X with $X\backslash N$. So we now have $\mathrm{Sht}^\alpha_{G,N,I,W}$.
- We can truncate Bun_G and everything above via **Harder-Narasimhan filtration**, $\mathrm{Bun}_G^{\leq\mu}$ for μ a dominant coweight of G^{ad}. I can spell out the general definition if you want but the point is that it is some kind of filtration such that, for any μ, $\mathrm{Bun}_G^{\leq\mu}$ is an open substack of Bun_G, and if $\deg(N)$ is big enough, $\mathrm{Bun}_{G,N}^{\leq\mu}$ is a countable disjoint union of quasi-projective schemes. If you want, let Bun_G^ν be the connected component associated to $\nu \in \pi_1(G)$, then $\mathrm{Bun}_{G,N}^{\leq\mu,\nu} = \mathrm{Bun}_{G,N}^{\leq\mu} \cap \mathrm{Bun}_G^\nu$ is a quasi-projective scheme.
  - Precise definition, at least when the derived group of G is simply connected: $\mathrm{Bun}_{G,N}^{\leq\mu}(S)$ is the collection of points $\mathcal{E} \in \mathrm{Bun}_{G,N}(S)$ such that, for every geometric point s of S, every dominant weight λ of T and for every B-torsor $\mathcal{B}$ on $X \times \{s\}$ such that $(\mathcal{B}\times G)/P \cong \mathcal{E}$ ("B-structure"), $\deg(\mathcal{B}_\lambda) \leq \langle\lambda,\mu\rangle$, where $\mathcal{B}_\lambda$ is the vector bundle associated to $\mathcal{B}$ twisted by the algebraic representation of B with highest weight λ. The general definition can be found in [Beh].

We can then pullback this to everything, Hecke and Sht. So the ultimate decoration would be

$$\mathrm{Sht}^{\alpha,\leq\mu}_{G,N,I,W}.$$

Like $\mathrm{Bun}^{\leq\mu}_{G,N}$, the other two stacks, if truncated and modded out by Σ, become finite type Deligne-Mumford stacks, and even schemes if $\deg(N)$ is large enough (depending on μ).

- On the other hand, we can abstractly think $\mathrm{Sht}^{\alpha}_{G,I}$ as an ind-stack, which is just the data of all $\mathrm{Sht}^{\alpha}_{G,I,W}$'s. This is quite harmless because the interaction of Harder-Narasimhan filtration with other operators turns out to be mild (changes filtration by a bounded amount).

We record a relevant proposition from V. Lafforgue's paper.

Proposition 2.1 ([Laf, Proposition 2.6],[Var, Proposition 2.16])**.** *The following are true.*

- $\mathrm{Sht}^{\alpha}_{G,N,I,W}$ *is a locally finite type Deligne-Mumford stack over* $(X\backslash N)^I$*.*
- *If* N *is nonempty,* $\mathrm{Sht}^{\alpha,\leq\mu}_{G,N,I,W}$ *is the quotient of a quasiprojective* $(X\backslash N)^I$*-scheme by a finite group. If* N *is empty, it is so if restricted to* U^I *for any* U *a proper open subset of* X*.*
- *If* $J\subset I$ *such that* $W_i=0$ *for* $i\in I\backslash J$*,* $\mathrm{Sht}^{\alpha}_{G,N,I,W}\cong (X\backslash N)^{I\backslash J}\times \mathrm{Sht}^{\alpha|_J}_{G,N,J,W|_J}$*.*
- $\mathrm{Sht}^{\alpha}_{G,N,I,W}$ *is nonempty if and only if* $[\sum_{i\in I}W_i]=0$ *as elements in* $\pi_1(G)$ *(We call it* ***admissible****).*

Example 2.1. Consider the moduli of shtukas with no legs. Then the corresponding Hecke stack is just $\mathrm{Bun}_{G,N}$, so $\mathrm{Sht}_{G,N,\emptyset}=(\mathrm{Bun}_{G,N})^{\mathrm{Frob}}$ as stacks. On the other hand, for any locally finite type Deligne-Mumford stack $Y/\mathbb{F}_q$, Y^{Frob} is a discrete constant stack of value $Y(\mathbb{F}_q)$. One can think of this as a generalization of Katz's theorem, which says

$$\{\text{unit root crystals (i.e. } M\cong \mathrm{Frob}^* M)\}=\{\mathbb{F}_q\text{-étale local systems}\};$$

the identification from left to right is done by taking the equalizer of the given isomorphism and the relative Frobenius. Thus, $\mathrm{Sht}_{G,N,\emptyset}$ is the discrete constant stack of values in $\mathrm{Bun}_{G,N}(\mathbb{F}_q)=G(F)\backslash G(\mathbb{A})/K_N$ (after ignoring nilpotence, of course).

2.2. **Sheaves on the moduli of shtukas.** If one believes that the moduli spaces of shtukas are true analogues of Shimura varieties, one should believe that the Langlands correspondence should be realized in the intersection cohomology of those. Thus it is very natural to study

$$\mathcal{H}^{\alpha,\leq\mu}_{G,N,I,W}:=(\mathfrak{p}^{\alpha,\leq\mu}_{G,N,I,W})_!(\mathrm{IC}_{\mathrm{Sht}^{\alpha,\leq\mu}_{G,N,I,W}/\Sigma}),$$

where pushforward is taken in a derived sense. The above example then says that $\mathcal{H}_{G,N,\emptyset}|_{\overline{\mathbb{F}_q}}:=\varinjlim_{\mu}\mathcal{H}^{\leq\mu}_{G,N,\emptyset}|_{\overline{\mathbb{F}_q}}$ is the space of automorphic forms. Once we equip Hecke correspondence on these sheaves, this will truly become the Hecke module of automorphic forms.

2.3. **Local models.** A very useful fact, and the primary reason why we introduced partition of indices, is that $\mathcal{H}^{\alpha,\leq\mu}_{G,N,I,W}$ does not depend on the partition α. This is because the natural maps $\mathrm{Hk}^{\alpha}_{G,N,I,W}\to\mathrm{Hk}^{\mathrm{coll}\circ\alpha}_{G,N,I,W}$ and the corresponding map for shtukas are (stratified) small (recall a proper generically finite map $\pi:X\to Y$ is small if $\mathrm{codim}_Y\{y\in Y\mid \dim f^{-1}(y)\geq d\}\geq 2d$ for all d), where $\mathrm{coll}:\{1,\cdots,n\}\to\{1\}$ amounts to gathering all partitions into one; recall that the IC sheaf proper pushforwards to the IC sheaf through a small map. The smallness comes from the smallness of $\mathrm{Gr}^{\alpha,\leq\mu}_{G,N,I,W}\to\mathrm{Gr}^{\mathrm{coll}\circ\alpha,\leq\mu}_{G,N,I,W}$ and the realization of Sht as a local model

$$\mathrm{Sht}^{\alpha}_{G,I,W}\to\mathrm{Gr}^{\alpha}_{G,I,W}/G_{\Sigma\infty x_i},$$

where $G_{\Sigma\infty x_i}$ is the restriction of G on the formal neighborhood $\sum\infty x_i$ of $\Gamma_x=\cup\Gamma_{x_i}$, viewed as an S-group scheme (i.e. Weil restriction of scalars). What is this map? By Beauville-Laszlo, $\mathrm{Gr}^{\alpha}_{G,I,W}$ can be thought as parametrizing G-torsors on the formal neighborhood $\Sigma\infty x_i$. So in particular any G-torsor is trivializable over this formal neighborhood. Now let $G_{\sum n_i x_i} \to X^I$ be the smooth group scheme such that $G_{\sum n_i x_i}(S) = \{(x_i, g_i) \mid (x_i) \in X^I(S), g_i \in G(\Gamma_{\sum n_i x_i})\}$ where $\Gamma_{\sum n_i x_i}$ is the obvious thickening of Γ_{x_i}'s (or similarly $G_{\sum n_i x_i}$ is the Weil restriction of scalar of $G/\Gamma_{\sum n_i x_i}$ to S). Then the freedom on the choice of trivialization is precisely a $G_{\Sigma \infty x_i}$-torsor, so the map is well-defined. This factors through a finite level, i.e. a $G_{\Sigma n_i x_i}$-action for some finite n_i's, as $(\mathcal{E}, (x_i), \psi) \in \mathrm{Gr}_{I,W}(S)$ if and only if for all dominant weight λ, $\psi(\mathcal{E}_\lambda) \subset G_\lambda(\sum \langle \lambda, W_i \rangle \Gamma_{x_i})$ (cf. [Gai]). It turns out that by a similar reasoning we have an isomorphism (!)

$$\mathrm{Hk}^{\alpha}_{G,I,W} \xrightarrow{\sim} (\mathrm{Gr}^{\alpha}_{G,I,W} \times_{X^I} \mathrm{Bun}_{G,\sum n_i x_i})/G_{\sum n_i x_i}$$

for large enough n_i's. One could then think of the local model of moduli of shtukas coming from really this identification of the Hecke stack.

The smallness of a morphism of changing partitions then comes from that

- the local models morphism is smooth of relative dimension $\dim G_{\sum n_i x_i}$ (which in particular does not depend on the partition),
- the convolution of usual (i.e. local) affine Grassmannian is semi-small (necessary in even defining the convolution product?),
- and there is an **extra dimension of base curve** which makes sense via Beauville-Laszlo.

Example 2.2. Consider the map

$$\mathrm{Hk}^{\{1\} \amalg \{2\}}_{\mathrm{GL}_2, \{1,2\}, \mathrm{St} \boxtimes \mathrm{St}^*} \to \mathrm{Hk}^{\{1,2\}}_{\mathrm{GL}_2, \{1,2\}, \mathrm{St} \boxtimes \mathrm{St}^*},$$

which corresponds to the Drinfeld case. It is a map that sends $(x_1, x_2, \mathcal{E}_0 \hookrightarrow \mathcal{E}_1 \hookleftarrow \mathcal{E}_2)$ to $(x_1, x_2, \mathcal{E}_0 \dashrightarrow \mathcal{E}_2)$ where the modification is at x_1, x_2.

We want to know the fibers of these. Over (x_1, x_2), if $x_1 \neq x_2$, then $\mathcal{E}_1$ is uniquely determined by $\mathcal{E}_0, \mathcal{E}_2$, because we can see this as finding a lattice that contains two lattices inside the vector space at the generic fiber. If $x_1 = x_2$, morally determining a point in the fiber of partitioned shtukas amounts to

(1) choosing $\mathcal{E}_0$,
(2) choosing $\mathcal{E}_0 \subset \mathcal{E}_1$ ($=\mathrm{Gr}(1,2)^\vee = \mathbb{P}^1$),
(3) and choosing $\mathcal{E}_1 \supset \mathcal{E}_2$ ($=\mathrm{Gr}(1,2) = \mathbb{P}^1$).

So the fiber over a point of the diagonal inside X^2 should be $\mathrm{Bun}_{\mathrm{GL}_2}$ times a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ (note changing the whole sequence does not change relative position, so the Bun_G-contribution is split).

On the other hand, in the moduli of unpartitioned shtukas, the choice of $\mathcal{E}_1$ is unique even when $\mathcal{E}_0 \neq \mathcal{E}_2$, by the same reason this time looking at two lattices at $x_1 = x_2$. But if $\mathcal{E}_0 \cong \mathcal{E}_2$, there is a freedom of $\mathbb{P}^1$ for choosing $\mathcal{E}_1$. So the above map between Hecke stacks should be an isomorphism outside $x_1 = x_2$, and Bun_G times a contraction of $\mathbb{P}^1$ in $\mathbb{P}^1$-bundle when over $\mathbb{P}^1$ over the diagonal.

Remark 2.1. Vaguely this is analogous to the notion of local model in the Shimura variety case, which in an ideal situation should be about having a diagram of form

$$\begin{tikzcd} & \widetilde{\mathrm{Sh}} \arrow[dl] \arrow[dr] & \\ \mathrm{Sh} & & M^{\mathrm{loc}} \end{tikzcd}$$

where, for a suitable (in particular flat) integral model of the given Shimura variety Sh of parahoric level at p, there is an $\mathcal{O}_E$-scheme $\widetilde{\mathrm{Sh}}$ which is a torsor over Sh under the (base change of) Bruhat-Tits group scheme $\mathcal{G}$ (some canonical group scheme for the parahoric group which is the level at p for this Shimura variety), and $\widetilde{\mathrm{Sh}} \to M^{\mathrm{loc}}$ is a smooth morphism of relative dimension $\dim G$, where M^{loc} should arise as the Zariski closure of some Schubert cell in the (mixed characteristic) affine Grassmannian (cf. [RPS], [PZ]). In particular, one gets a smooth morphism $\mathrm{Sh} \to [M^{\mathrm{loc}}/\mathcal{G}]$ of relative dimension $\dim G$.

For example, in [dJ], this is realized (all over $\operatorname{Spec}\mathbb{Z}$, though) for $\mathrm{GSp}_{2g}(\mathbb{Q})$ with $\Gamma_0(p)$-level structure, where

- Sh is the moduli space (finite type Deligne-Mumford stack over $\operatorname{Spec}(\mathbb{Z})$) of principally polarized abelian varieties with $\Gamma_0(p)$-level structure (i.e. flag of subgroup schemes $0 \subset H_1 \subset \cdots \subset H_g \subset A[p]$ such that $\#H_i = p^i$ and H_g is Lagrangian with respect to the Weil pairing induced by the polarization),
- $\widetilde{\mathrm{Sh}}$ is the universal abelian variety over Sh,
- and M^{loc} is realized as some Schubert cell inside the product of classical Grassmannians (over $\mathbb{Z}$) such that $\widetilde{\mathrm{Sh}} \to M^{\mathrm{loc}}$ is just

(chain of p-isogenies of abelian varieties) $\mapsto$ (Fil^1 of Dieudonné modules of abelian varieties in the chain).

2.4. **Geometric Satake and more general sheaves on the moduli of shtukas.** We want to naturally extend the notion of $\mathcal{H}^{\alpha}_{G,N,I,W}$ to any $W \in \operatorname{Rep}(\widehat{G}^I)$ not necessarily irreducible. For this purpose it is quite natural to try to use the **geometric Satake**, which is sort of a general construction of a functor that extends $W \mapsto \mathrm{IC}_W$ (i.e. IC-sheaf of the Schubert cell). More precisely, the geometric Satake functor is some nice (e.g. exact, faithful, ...) $\otimes$-functor $\mathrm{Sat}_I : \operatorname{Rep}(\widehat{G}^I) \to \mathrm{Sat}(\mathrm{Gr}_{X^I}) \subset \mathrm{Perv}(\mathrm{Gr}_{X^I})$. Here, the Satake category $\mathrm{Sat}(\mathrm{Gr}_{X^I})$ can be thought as "L^+G-equivariant perverse sheaves", which should be made precise in the later lectures. In particular, for irreducible $W \in \operatorname{Rep}(\widehat{G}^I)$, this should give IC_W as expected.

If we let $\mathrm{Gr}_{G,N,I,W} = \cup_{\underline{W} \subset W \text{ irreducible}} \mathrm{Gr}_{G,N,I,\underline{W}}$ for general $W \in \operatorname{Rep}(\widehat{G}^I)$, then $\mathrm{Sat}_I(W)$ is supported on $\mathrm{Gr}_{G,N,I,W}$. Also letting $\mathrm{Sht}_{G,N,I,W} = \cup_{\underline{W} \subset W \text{ irr.}} \mathrm{Sht}_{G,N,I,\underline{W}}$, we still have a local model diagram $\varepsilon : \mathrm{Sht}_{G,N,I,W} \to \mathrm{Gr}_{G,N,I,W}/G_{\sum n_i x_i}$ for $n_i \gg 0$, which is smooth of relative dimension $\dim_{X^I}(G_{\sum n_i x_i})$. Thus, up to appropriate twist and degree shift, we can obtain a perverse sheaf $\mathcal{F}_{G,N,I,W} = \varepsilon^* \mathrm{Sat}_I(W)[sth](sth)$. Now we can define $\mathcal{H}^{\alpha,\le W}_{G,N,I,W}$ to be the (derived) proper pushforward of this sheaf via leg morphism. This is also independent of the partition α as we have observed before.

As the geometric Satake is functorial, this sheaf also inherits nice functoriality. For example, for any $u : W \to W'$ a morphism of $\widehat{G}^I$-representations, we get the associated morphism $\mathcal{H}(u) : \mathcal{H}_{G,N,I,W} \to \mathcal{H}_{G,N,I,W'}$. Moreover, as the geometric Satake respects fusion, $\mathcal{H}_{G,N,I,W}$ respects it as well. To be more precise, for any $\phi : I \twoheadrightarrow J$ a surjective map between two finite sets, we have a natural embedding $\Delta_\phi : X^J \hookrightarrow X^I$, and the geometric Satake satisfies, for any $W \in \operatorname{Rep}(\widehat{G}^I)$, that $\Delta_\phi^* \mathrm{Sat}_I(W)$ is canonically isomorphic to $\mathrm{Sat}_J(W^\phi)$, where W^ϕ is W seen as $\widehat{G}^J$-representation via ϕ. From this, we get the **coalescence isomorphism** $\chi_\phi : \Delta_\phi^* \mathcal{H}_{G,N,I,W} \xrightarrow{\sim} \mathcal{H}_{G,N,I,W^\phi}$.

2.5. **Hecke correspondence and partial Frobenius.** Now we want to define two extremely important operations on the cohomology of moduli of shtukas. The first is the Hecke correspondence. Recall that the Hecke stack of Bun_G, which is the Hecke stack we defined above, classifies modifications between two G-torsors; thus, if there is a Hecke correspondence of shtukas that is realized as a cohomological correspondence, this should parametrize modifications between two G-shtukas.

Definition 2.4 (Hecke correspondence for $\mathrm{Sht}^{(I_1,\cdots,I_k)}_{G,N,I,W}$)**.** *Let $W \in X_*^+(T)^I$ be a collection of dominant coweights, and let $g \in G(\mathbb{A})$ be unramified at N. Let the collection of bad places be denoted as $\mathcal{S}$. Then we define $\Gamma_N(g)$ (**which should correspond to the Hecke operator** $\mathbb{1}_{K_N g K_N}$) be the stack such that $\Gamma_N(g)(S)$ classifies*

- *legs $(x_i)_{i\in I}$, $x_i : S \to (X \backslash (|N| \cup \mathcal{S}))$,*

- *the diagram*

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r, "\phi_1"] \arrow[d, "\kappa"'] & \mathcal{E}_1 \arrow[r, "\phi_2"] & \cdots \arrow[r, "\phi_{k-1}"] & \mathcal{E}_{k-1} \arrow[r, "\phi_k"] & {}^\tau\mathcal{E}_0 \arrow[d, "{}^\tau\kappa"] \\
\mathcal{E}'_0 \arrow[r, "\phi'_1"'] & \mathcal{E}'_1 \arrow[r, "\phi'_2"'] & \cdots \arrow[r, "\phi'_{k-1}"'] & \mathcal{E}'_{k-1} \arrow[r, "\phi'_k"'] & {}^\tau\mathcal{E}'_0
\end{tikzcd}$$

such that the two rows both belong in $\mathrm{Sht}^{(I_1,\cdots,I_k)}_{G,N,I,W}(S)$*, and* κ *is a modification at good places of relative position defined by* g*.*

This is such a big chunk that, although it is a morally correct definition, it is not obvious that this defines a cohomological correspondence. It does, but it is not clear at all. Hopefully later talks will clarify that at least some Hecke operators (namely those denoted as "$T(h_{V,v})$" in [Laf]) are well-defined.

Remark 2.2. There is another slick way of seeing Hecke correspondence. For simplicity we only consider unpartitioned shtukas, and suppress W and consider everything as ind-stacks. Pick a good place v, then the data **at** v of shtukas with legs **away from** v is just an isomorphism between a formal G-torsor with its Frobenius twist. Thus this data is precisely $[pt/G(\mathcal{O}_v)]$ (again, similar to the discussion regarding shtukas with no legs and local model diagram, extra choice of isomorphism gives this quotient). This evaluation map fits into a cartesian diagram

$$\begin{tikzcd}
\mathrm{Sht}_{G,N\cup\infty v,I}|_{(X-v)^I} \arrow[r] \arrow[d] & pt \arrow[d] \\
\mathrm{Sht}_{G,N,I}|_{(X-v)^I} \arrow[r, "\mathrm{ev}_v"] & {[pt/G(\mathcal{O}_v)]}
\end{tikzcd}$$

where ∞v means either the union of nv for all $n \geq 0$, or level structure over the formal neighborhood of v. Changing level structure at v gives $G(\mathcal{O}_v)$-action on $\mathrm{Sht}_{G,N\cup\infty v,I}|_{(X-v)^I}$, but this extends to $G(F_v)$-action naturally, thanks also to Beauville-Laszlo; changing trivializations over punctured formal disc is the desired $G(F_v)$-action. Then we can define $\mathrm{Sht}_{G,N\cup\infty v,I}|_{(X-v)^I} \times^{G(\mathcal{O}_v)} G(F_v)/G(\mathcal{O}_v)$ and two natural maps to $\mathrm{Sht}_{G,N,I}|_{(X-v)^I}$, where $h_{\leftarrow}$ is the first projection and $h_{\rightarrow}$ is the action of the second factor on the first factor. Both are quite clearly ind-finite-étale, because $G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v)$ is finite. The corresponding cohomological correspondence is indeed the correct Hecke correspondence. This is however defined only over the open curve $(X-v)^I$, and that this extends to the whole curve is seen only via $S = T$ theorem of [Laf].

Another important construction is the partial Frobenius. The definition is simple, as the partial Frobenius morphism exists as morphisms between the moduli spaces of shtukas. Namely, one can think of

$$\mathrm{Frob}^{(I_1,\cdots,I_r)}_{I_1} : \mathrm{Sht}^{(I_1,\cdots,I_r)}_{G,N,I,W} \to \mathrm{Sht}^{(I_2,\cdots,I_r,I_1)}_{G,N,I,W}$$

sending $(\mathcal{E}_0 \to \cdots \to \mathcal{E}_r \cong {}^\tau\mathcal{E}_0)$ to $(\mathcal{E}_1 \to \cdots \to \mathcal{E}_r \to {}^\tau\mathcal{E}_1 \cong {}^\tau\mathcal{E}_1)$. This is a universal homeomorphism as successive composition of partial Frobenii in any cyclic order gives the usual Frobenius which is a universal homeomorphism. Thus by proper base change $(\mathrm{Frob}^{(I_1,\cdots,I_r)}_{I_1})^*\mathcal{F}^{(I_2,\cdots,I_r,I_1)}_{G,N,I,W} \cong \mathcal{F}^{(I_1,\cdots,I_r)}_{G,N,I,W}$ and the same holds for $\mathcal{H}^{\alpha}_{G,N,I,W}$.

Remark 2.3. Both Hecke correspondence and partial Frobenius interact nontrivially with Harder-Narasimhan filtration and in particular changes it. However it changes by a bounded amount, so still deliberately ignoring finite-typeness issue can be justified (cf. [Laf, 0.6, 0.7]).

2.6. **Cusp forms = Hecke-finite automorphic forms.** We now end with how to realize cusp forms from Hecke action. Namely, it is generally expected that the cuspidal cohomology (i.e., the common kernel of all Jacquet functors) is the same as the "Hecke-finite part" of the cohomology, the part where the Hecke algebra generates finite type module over the ring of integers. One direction, showing that cusp forms are Hecke-finite, is easy, as we know the space of cusp forms, or generally $\mathcal{H}_{G,N,I,W}$, when truncated properly, is finite type. The other direction, conjectured in [Laf], is proved in [Xue] for "rational Hecke-finite part", i.e. those which are Hecke-finite modulo torsion. However in the setting of actual automorphic forms, namely for the 0-th cohomology sheaf of moduli of shtukas with no legs, it is proven in [Laf] that the two are the same, and from there *loc. cit.* works solely with Hecke-finite cohomology, which is easier to deal with.

Proof that Hecke-finite automorphic forms are cusp forms ([Laf]*,* [Xue]*).* Suppose that f is Hecke-finite but not cuspidal. Then there is some proper Levi $M \subset P$ such that the map $f_P(g) := \int_{U(F)\backslash U(\mathbb{A})} f(ug)$ is nonzero. The primary reason that Bun_G (and thus Sht_G) fails to be finite-type is because there may be infinitely many connected components, and the existence of nontrivial center is the major reason behind this. Namely, one can define the "degree morphism" (coined as so because this is literally the degree of line bundles when $G = \mathbb{G}_m$)

$$\mathrm{Bun}_G \to \mathrm{Bun}_{G^{\mathrm{ab}}} \to \pi_0(\mathrm{Bun}_{G^{\mathrm{ab}}}) \xrightarrow{\sim} X_*(G^{\mathrm{ab}}) \to X_*(G^{\mathrm{ab}})_{\mathbb{Q}} \cong X_*(Z(G))_{\mathbb{Q}},$$

where $\pi_0(\mathrm{Bun}_{G^{\mathrm{ab}}}) \xrightarrow{\sim} X_*(G^{\mathrm{ab}})$ is the canonical isomorphism induced by $\pi_0(\mathrm{Bun}_{G^{\mathrm{ab}}}) \xrightarrow{\chi} \pi_0(\mathrm{Bun}_{\mathbb{G}_m}) \xrightarrow{\deg} \mathbb{Z}$ for all $\chi \in X^*(G^{\mathrm{ab}})$. We can consider the same thing for M. In particular, Bun_M inherits the action of Ξ and the union of components in a fiber of $\mathrm{Bun}_M \xrightarrow{\deg_M} X_*(Z(M))_{\mathbb{Q}} \to X_*(Z(M)/Z(G))_{\mathbb{Q}}$ is Ξ-stable. Let Bun_M^{ν} be the fiber of $\nu \in X_*(Z(M)/Z(G))_{\mathbb{Q}}$ via this composition. Then it turns out that, given dominant coweight μ of G^{ad}, the locus of $\nu \in X^*(Z(M)/Z(G))_{\mathbb{Q}}$ where $\mathrm{Bun}_M^{\leq\mu,\nu}$ is nonempty is supported in some cone. By choosing any good place v and $g \in Z(M)(F_v)$ such that $g \notin Z(M)(\mathcal{O}_v)Z(G)(F_v)$, as it lands in a nonzero element in $X_*(Z(M)/Z(G))_{\mathbb{Q}}$, the Hecke action $T(g)$ should inevitably "shift" the cone into certain direction. In particular either g or g^{-1} shifts the cone so that the new cone has some area that does not belong to the original cone. So, for the given f, as it has some connected component of Bun_M/Ξ as a support (nonvanishing Jacquet functor!), the Hecke action of a sufficiently high power of g or g^{-1} should give a new connected component as a support, and this forever continues, so $\langle T(g^{\pm nN})f\rangle_{n\geq 1}$, for some large enough N, generates infinite type module. □

References

[Beh] K. Behrend, *Semi-stability of reductive group schemes over curves*, Math. Ann. **301** (1995).
[BJ] A. Borel, H. Jacquet, *Automorphic forms and automorphic representations*, in Corvallis Proceedings Part I. 1979.
[dJ] A. J. de Jong, *The moduli spaces of principally polarized abelian varieties with* $\Gamma_0(p)$*-level structure*, J. Alg. Geom. **2** (1993).
[Gai] D. Gaitsgory, *Construction of central elements in the affine Hecke algebra via nearby cycles*, Invent. Math. **144** (2001).
[Har] G. Harder, *Chevalley Groups Over Function Fields and Automorphic Forms*, Ann. of Math. **100** (1974).
[Laf] V. Lafforgue, *Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale*, Preprint. 2012.
[PZ] G. Pappas, X. Zhu, *Local models of Shimura varieties and a conjecture of Kottwitz*, Invent. Math. **194** (2013).
[RPS] M. Rapoport, G. Pappas, B. Smithling, *Local models of Shimura varieties, I. Geometry and combinatorics,* in I. Morrison, G. Farkas eds., Handbook of Moduli, vol. III, International Press, 2013.
[Var] Y. Varshavsky, *Moduli spaces of principal F-bundles*, Selecta Math. **10** (2004).
[Xue] C. Xue, *Cuspidal cohomology of stacks of shtukas*, Preprint. 2018.