KOSZUL DUALITY AND CATEGORICAL ARCHIMEDEAN LOCAL LANGLANDS

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1. Kazhdan–Lusztig inversion formula as Koszul duality

Somehow the categorification of local Langlands correspodence started off as interpreting the Kazhdan–Lusztig inversion formula. We recall the setup. Namely, consider the block $\mathcal{O}_0$ of the category of representations of a complex semisimple Lie algebra $\mathfrak{g}$ which contains the trivial representation. The simple objects are the simple highest weight modules M_w of highest weight $-w(\rho)-\rho$, for the half sum of positive roots ρ and $w \in W$. This is the irreducible quotient of the Verma module M_w of highest weight $-w(\rho)-\rho$ which is universal among highest weight modules. The **Kazhdan–Lusztig polynomial** $P_{y,w}(q) \in \mathbb{Z}[q]$ for $y, w \in W$ encodes the multiplicity of Verma module, in that

$$[M_w] = \sum_{y \leq w} P_{w_0 w, w_0 y}(1)[L_y].$$

The Kazhdan–Lusztig inversion formula says that the inversion is also expressible in terms of K-L polynomials, namely

$$[L_w] = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1)[M_y].$$

[BGS] found that the categorification of this is that

$$D^b(\mathcal{O}_0) \cong D^b_{(B^\vee)}(G^\vee/B^\vee) \qquad \text{(Categorification of local Langlands over } \mathbb{C}\text{)},$$

where the right hand side is the bounded derived category of $B^\vee$-equivariant constructible sheaves on $G^\vee/B^\vee$, and this is an instance of **Koszul duality**. We will see what Koszul duality means shortly. Soergel then generalized this picture to real group representations, which we will also see. To at least superficially understand what this $\cong$ is, we translate the left-hand side to a geometric picture using Beilinson–Bernstein localization;

$$D^b_{(B)}(G/B) \cong D^b(\mathcal{O}_0) \cong D^b_{(B^\vee)}(G^\vee/B^\vee).$$

Then L_w corresponds to the **IC sheaf**, which is a simple perverse sheaf corresponding to the Schubert cell associated with w, and M_w corresponds to the **standard object**. In terms of $i_w : BwB/B \hookrightarrow G/B$,

$$L_w = i_{w!*}k[\ell(w)], \quad M_w = i_{w!}k[\ell(w)].$$

Similar definition $N_w = i_{w*}k[\ell(w)]$ corresponds to the **dual Verma module** $N_w = M_w^\vee$.

Now the "Koszul duality" $D^b_{(B)}(G/B) \cong D^b_{(B^\vee)}(G^\vee/B^\vee)$ is such that $L_w \mapsto N_{w^{-1}w_0}$, and $M_w \mapsto L_{w^{-1}w_0}$. Somehow you can now imagine that the roles of L and M are "flipped."

2. Koszul duality basics

So why is this called Koszul duality? Koszul duality is basically a generalization of

$$D^b(\mathrm{Mod}_{S(V)}) \cong D^b(\mathrm{Mod}_{\wedge^*(V^\vee)}).$$

This is usually called the BGG correspondence. Let us briefly recall what this is. Let $M = \oplus M_d$ be a graded $S(V)$-module. Then, define $R(M)$ to be the complex

$$\cdots \xrightarrow{\phi} \mathrm{Hom}_K(\wedge^*(V^\vee), M_d) \xrightarrow{\phi} \mathrm{Hom}_K(\wedge^*(V^\vee), M_{d+1}) \xrightarrow{\phi} \cdots,$$

$$\phi(\alpha)(e) = \sum_i v_i \alpha(\widetilde{v}_i \wedge e),$$

where v_i and $\widetilde{v}_i$ are dual bases of V and $V^\vee$. This being a complex is basically equivalent to that the induced morphisms $V \otimes M_d \to M_{d+1}$ satisfy commutativity and associativity so that it gives an $S(V)$-module structure on M. This gives an equivalence of categories between graded left $S(V)$-modules and certain category of **complexes of $\wedge^* V^\vee$-modules**. This can be upgraded to an equivalence between derived categories by taking the total complex of R.

So, Koszul duality is really a "derived" phenomenon that involves two different t-structures encoded in terms of two different "gradings." Let us develop a more general theory of Koszul duality.

Definition 2.1 ([BGS, Def. 1.1.2])**.** A Koszul ring is a positively graded ring $A = \oplus_{j \geq 0} A_j$ such that

(1) A_0 is semisimple (the category of A-modules is semisimple, i.e. every short exact sequence splits),
(2) A_0 admits a graded projective resolution of graded left A-modules
$$\cdots \to P^2 \to P^1 \to P^0 \twoheadrightarrow A_0,$$
such that P^i is generated by its degree i component, $P^i = AP^i_i$.

Example 2.2. The name of Koszul comes from the fact that the Koszul complex of a regular ring shows that a regular ring is Koszul. For example, for $A = k[x, y]$ with usual grading,

$$0 \to xyA \xrightarrow{f \mapsto (f,-f)} (xA) \oplus (yA) \xrightarrow{(a,b) \mapsto a+b} A \twoheadrightarrow k \to 0,$$

shows that A is Koszul. More generally, a polynomial ring, or in other words a symmetric algebra is Koszul. Similarly, an exterior algebra is also Koszul.

Koszulity is something really close to semisimplicity.

Proposition 2.3. *Let A be a Koszul k-algebra for semisimple k.*

(1) *For any two pure modules M, N of weights m, n (namely, concentrated in one degree of grading), $\mathrm{Ext}^i_A(M, N) = 0$ unless $i = m - n$.*
(2) *A is quadratic, namely $A = T(V)/R$ for a k-vector space V and $R \subset V \otimes V$.*

Definition 2.4. Let $A = T(V)/R$ be a Koszul k-algebra. Then its **Koszul dual** $A^!$ is $T(V^*)/R^\perp$, where $R^\perp \subset V^* \otimes V^* = (V \otimes V)^*$ consists of those on which the elements of R vanish.

Now here is Koszul duality.

Proposition 2.5. *Let A be Koszul.*
(1) *$A^!$ is Koszul.*
(2) *$A^{!!} = A$.*
(3) *$\mathrm{Ext}^*_A(k, k) = (A^!)^{\mathrm{op}}$.*
(4) *There is a natural equivalence of categories between $D^+(A - \mathrm{mod})$ and $D^-(A^! - \mathrm{mod})$.*

Proof. The way that $A^! = T(V^*)/R^\perp$ is realized gives a graded projective resolution of k as an A-module,

$$\cdots \to \mathrm{Hom}_k(A^!_2, A) \to \mathrm{Hom}_k(A^!_1, A) \to A \to k \to 0,$$

$$(df)(a) = \sum_i f(a\widehat{v}_i)v_i,$$

where $\{v_i\}$ is a basis of V and $\{\widehat{v}_i\}$ is the dual basis of V^*. This complex ("Koszul complex") can be thought as $\mathrm{Hom}_A(A \otimes A^!, A)$ with the differential $a \otimes b \mapsto \sum_i av_i \otimes \widehat{v}_i b$. In general, the equivalence of categories should be constructed as

$$M \in D^+(\mathrm{Mod}_A) \mapsto \mathrm{Hom}_A(A \otimes A^!, A) \in D^-(\mathrm{Mod}_{A^!}),$$

$$N \in D^-(\mathrm{Mod}_{A^!}) \mapsto A \otimes A^! \otimes_{A^!} N \in D^+(\mathrm{Mod}_A).$$

□

Remark 2.6. The duality does not given an equivalence of unbounded derived category. For example, let $A = k[x]$ and $A^! = k[\epsilon]/\epsilon^2$. Then, $A \in \mathrm{Mod}_A$ is sent to $0 \to k[x] \xrightarrow{x} k[x] \to 0 \cong k\langle 1\rangle[-1]$. Note that A is compact in $D(\mathrm{Mod}_A)$, but k is not compact in $D(\mathrm{Mod}_{A^!})$. Indeed, k is resolved in $D(\mathrm{Mod}_{A^!})$ (here we use unboundedness in the wrong direction) as

$$\cdots \to k[\epsilon]\langle 2\rangle \xrightarrow{\epsilon} k[\epsilon]\langle 1\rangle \xrightarrow{\epsilon} k[\epsilon],$$

so $\mathrm{Hom}_{D(\mathrm{Mod}_{k[\epsilon]})}(k, k[\epsilon]\langle n\rangle[n]) = 0$, whereas $\mathrm{Hom}_{D(\mathrm{Mod}_{k[\epsilon]})}(k, \oplus_{n\geq 1} k[\epsilon]\langle n\rangle[n])$ is k.

Now the real reason why Kazhdan–Lusztig inversion formula gives rise to Koszul duality is via the following.

Proposition 2.7 (Numerical Koszulity criterion)**.** *If k is a semisimple F-algebra for a field F such that $k = \bigoplus_{x\in W} F1_x$ for pairwise orthogonal idempotents indexed by a finite set W, then for a positively graded ring A which is an F-algebra with $A_0 = k$, the Hilbert polynomial of A is a $|W| \times |W|$-matrix*

$$P(A, t)_{x,y} = \sum_{i\geq 0} t^i \dim(1_x A_i 1_y).$$

Then, for a noetherian A, A is Koszul iff $P(A, t)P(A^!, -t)^T = 1$.

Proof. We just prove one side. If A is Koszul, because of Koszul complex, for any $n > 0$,

$$\sum_{i=0}^{n} (-1)^i \dim(1_x \operatorname{Hom}_k(A^!_{n-i}, A_i) 1_y) = 0$$

(Euler characteristic). □

We will see that a similar pattern holds for local Langlands for real reductive groups.

3. Archimedean local Langlands

We review what archimedean local Langlands is. This is a consequence of two theories, Langlands classification and Knapp–Zuckerman theory.

Theorem 3.1 (Langlands classification)**.** *Every irreducible $(\mathfrak{g}, K)$-module is the unique irreducible quotient of the parabolic induction of a tempered representation. This gives a parametrization of irreducible $(\mathfrak{g}, K)$-modules.*

Theorem 3.2 (Knapp–Zuckerman)**.** *Every tempered irreducible $(\mathfrak{g}, K)$-module is **basic**, meaning that it is a constituent of the parabolic induction of a (limit of) discrete series. This gives a parametrization; you know exactly how parabolic induction reduces, etc.*

Now limits of discrete series are classified by the weight and a Weyl chamber (with fixed positive system of roots for $\mathfrak{k}$). In contrast to p-adic case, real case is completely classified because

- one has simple classification of discrete series = supercuspidals,
- and one knows how principal series reduces.

4. Vogan duality and Adams–Barbasch–Vogan

5. Koszul duality and Soergel's conjecture

6. Langlands parameter space as loop space

7. Koszul duality: perverse vs. coherent

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