LEARNING SEMINAR ON THETA CORRESPONDENCES

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Started off as a representation-theoretic interpretation of classical invariant theory, theta correspondence has been an extremely useful tool in constructing automorphic forms. We would like to first review classical theory of theta series, motivate the archimedean local correspondence and then study the global correspondence. We will then study arithmeticity properties. Note that nonarchimedean local aspect is pretty much ignored as I do not know much about it; it would be great if someone could cover it during the course.

The following schedule is very much tentative and is subject to change.

Talk 1. Classical theta series and the Weil representation.

Classical theta series and their relation to abelian varieties. Heisenberg group, Stone-von Neumann theorem. Weil representation and how theta series fit into the picture. One should consult to more classical references like [MNN], [Ig], or the very paper of Weil, [We]. For a shorter treatment of Weil representation, one could take a look at some introductory references suggested for Talk 3.

Talk 2. Classical invariant theory and the archimedean local theta correspondence.

Review Howe's [Ho1] (and maybe [Ho2]) illustrating how Howe ended up with theta correspondence from considerations of classical invariant theory.

Talk 3. Statements of theta correspondences.

Review the usually-known forms of local and global theta correspondences using introductory references, e.g. [Ho3], [Ku], [PrD2], [PrD3]. [Ga4] is helpful for understanding what's known.

Talk 4. The Shimura correspondence I.

Study the Shimura correspondence in detail, which says that for a modular form f of half-integral weight $k + \frac{1}{2}$, there is a modular form g of weight $2k$ such that the eigenvalue of T_{n^2} for f is equal to the eigenvalue of T_n for g. We can use a wonderful course note of Wee Teck Gan [Ga2], which will guide us to learn most aspects of theta correspondences. For the first part, cover [Ga2, §1-6], introducing the general problem and its context.

Talk 5. The Shimura correspondence II.

Cover [Ga2, §7-10], introducing more precise period relations, including a certain case of Siegel-Weil formula and Rallis inner product formula, the usefulness of doubling and see-saw. Prove the Waldspurger's formula for torus periods and the global Shimura correspondence.

Talk 6. The role of root numbers I.

Review the notion of local root numbers. Discuss several qualitative phenomena of local theta correspondences, including the "first occurrence" and the "epsilon dichotomy", using [KR] and

[HKS].

Talk 7. Gan-Gross-Prasad conjectures I.

Discuss the work of Gross-Prasad (and inevitably the formalism of Vogan's local Langlands conjectures) which gave interpretations on "symplectic local root numbers", following Gross-Prasad [GP]. It might worth to briefly review some precursors: Deligne's elegant topological interpretation of orthogonal root numbers [De]; Prasad's PhD thesis [PrD1] which is a special case of local Gross-Prasad conjecture for "triple product" case (a more streamlined proof can be found in [Ga1]). If time permits one could try to relate a work of Harris-Kudla [HK] to the global Gross-Prasad conjecture.

Talk 8. The role of root numbers II.

Discuss and state the conjectured relation between nonvanishing of global theta lifting and nonvanishing of certain L-values. [Fu] has symplectic-orthogonal case and [PrD4] has unitary case.

Talk 9. Gan-Gross-Prasad conjectures II.

Motivate and state the Gan-Gross-Prasad conjecture and its refinement on period relations, using [GGP], [Ga3], [HaN] and [Li3], focusing on the case of unitary groups. One might want to exhibit why this is in accordance with Waldspurger's formula.

Talk 10-11. Algebraicity/arithmeticity of periods and theta lifts.

Discuss the problem of algebraicity and arithmeticity of periods and L-values using explicit period relations coming from theta correspondences. Focus on either Shimura correspondence or Shimizu correspondence (i.e. Jacquet-Langlands for GL(2)). For example, start from [HK], [PrK3], [PrK2], and understand the calculations of [Hi], [PrK1].

We can then proceed to study e.g. arithmetic theta lifting (cf. [Li1], [Li2]) or algebraicity/arithmeticity of theta correspondence for other groups (cf. [HaM1], [HaM2], [IP]).

References

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