THETA CHARACTERISTICS AND PROJECTIVELY CONGRUENCE MODULAR FORMS OF WEIGHT ONE (IN PROGRESS)

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Abstract. The Hodge bundle ω over a modular curve satisfies the Kodaira–Spencer isomorphism, which implies that ω is a square-root of the canonical bundle, or a *theta characteristic*. We prove that, in most cases, any section of a theta characteristic ν different from ω is a *noncongruence modular form*. We investigate their relations to the arithmetic of modular curves, and their possible relations to the theory of automorphic forms.

Contents

1. Introduction

Motivation: Modular curves are not Brill–Noether general.

1.1. **Acknowledgements.** To be added.

1.2. **Notations.** Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup that satisfies the following condition[1].

$$(*) \qquad \text{There exist integers } N_1, N_2 \text{ such that } (N_1, N_2) \text{ is odd, } \mathrm{lcm}(N_1, N_2) \geq 5, \text{ and } \Gamma = \Gamma_1(N_1) \cap \Gamma(N_2).$$

For example, the standard congruence subgroups $\Gamma_1(N)$ and $\Gamma(N)$ for any $N \geq 5$ satisfy $(*)$. Note that $(*)$ implies that Γ is torsion-free.

Let $Y(\Gamma) = \Gamma\backslash\mathbb{H}$ be the (open) modular curve, regarded as a Riemann surface, and let $X(\Gamma)$ be the compactification of $Y(\Gamma)$. Thanks to $(*)$, there is a universal elliptic curve $f : \mathcal{E} \to Y(\Gamma)$. Let $D = X(\Gamma) - Y(\Gamma)$ be the cuspidal divisor. We will add subscripts to these geometric objects (e.g.

[1] We impose this condition just for simplicity, and we expect our results to be extended to more general torsion-free congruence subgroups. On the other hand, the torsion-free-ness seems to be a more crucial assumption.

$Y(\Gamma)_{\mathbb{Q}}$) if we need to specify the base ring. We denote the genus of $X(\Gamma)$ as g_Γ and the number of cusps as n_Γ, and we will omit the subscripts when there is no confusion. As Γ is torsion-free, we have

$$g_\Gamma = 1 + \frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]}{24} - \frac{n_\Gamma}{2}.$$

The space of weight k modular forms (cusp forms, respectively) of level Γ is denoted as $M_k(\Gamma)$ ($S_k(\Gamma)$, respectively).

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g, regarded as a Deligne–Mumford stack over $\mathbb{Q}$. More generally, for a level structure Γ, let $\mathcal{A}_{g,\Gamma}$ be the corresponding moduli space with the Γ-level structure. Let A_g, $A_{g,\Gamma}$ be the associated coarse moduli schemes.

2. The Hodge bundle ω

Definition 2.1. The *Hodge bundle* ω is a line bundle over $Y(\Gamma)$ that is defined as

$$\omega := f_*\Omega^1_{\mathcal{E}/Y(\Gamma)}.$$

The Hodge bundle extends canonically (in the sense of Deligne) over $X(\Gamma)$, and we will denote the canonical extension as ω^{can}, although we will oftentimes omit the superscript when there is no confusion. One can, for example, define ω^{can} as the algebraization of the analytic sheaf of sections of logarithmic growth of ω at infinity over $Y(\Gamma)$.

The following is well-known.

Theorem 2.2 (Kodaira–Spencer isomorphism)**.** *Over $Y(\Gamma)$, one has a natural isomorphism*

$$\omega^{\otimes 2} \xrightarrow{\sim} \Omega^1_{Y(\Gamma)/\mathbb{C}}.$$

Over $X(\Gamma)$, one has a natural isomorphism

$$\omega^{\otimes 2} \xrightarrow{\sim} \Omega^1_{X(\Gamma)/\mathbb{C}}(D).$$

Proof. There is a natural morphism $\omega \to \omega^{-1} \otimes \Omega^1_{Y(\Gamma)/\mathbb{C}}$ which is the Higgs field corresponding to the variation of Hodge structures coming from $\mathcal{E}/Y(\Gamma)$. Since the Gauss–Manin connection has no singularities on $Y(\Gamma)$, the natural morphism is nonvanishing everywhere, thus an isomorphism. The Kodaira–Spencer isomorphism over $X(\Gamma)$ follows by taking the canonical extension of both sides of the said isomorphism over $Y(\Gamma)$. □

Corollary 2.3. *The degree of ω is $g - 1 + \frac{n}{2}$.*

Because of the condition on the level, we have

$$M_k(\Gamma) = H^0(X(\Gamma), \omega^{\otimes k}), \quad S_k(\Gamma) = H^0(X(\Gamma), \omega^{\otimes k}(-D)).$$

A simple application of Riemann–Roch yields the following

Proposition 2.4.

(1) If $k \geq 2$, we have $\dim M_k(\Gamma) = (k-1)(g-1) + \frac{nk}{2}$.
(2) If $k \geq 3$, we have $\dim S_k(\Gamma) = (k-1)(g-1) + \frac{n(k-2)}{2}$. We also have $\dim S_2(\Gamma) = g$.
(3) We have $\dim M_1(\Gamma) - \dim S_1(\Gamma) = \frac{n}{2}$.
(4) If $n > 2g-2$, we have $\dim M_1(\Gamma) = \frac{n}{2}$ and $\dim S_1(\Gamma) = 0$.

Proof. Only $\dim M_1(\Gamma) - \dim S_1(\Gamma) = \frac{n}{2}$ requires an explanation. From the short exact sequence $0 \to \omega(-D) \to \omega \to \omega|_D \to 0$, we have the long exact sequence

$$0 \to S_1(\Gamma) \to M_1(\Gamma) \to H^0(\omega|_D) \to H^1(\omega(-D)) \to H^1(\omega) \to 0,$$

as $\omega|_D$ is a skyscraper sheaf. On the other hand, by Serre duality,

$$\begin{aligned}\ker(H^1(\omega(-D)) \to H^1(\omega)) &= \ker(H^0(\omega)^* \to H^0(\omega(-D))^*) \\ &= \big(\operatorname{coker}(H^0(\omega(-D)) \to H^0(\omega))\big)^* = \left(\frac{M_1(\Gamma)}{S_1(\Gamma)}\right)^*.\end{aligned}$$

Thus, we have a short exact sequence

$$0 \to \frac{M_1(\Gamma)}{S_1(\Gamma)} \to H^0(\omega|_D) \to \left(\frac{M_1(\Gamma)}{S_1(\Gamma)}\right)^* \to 0.$$

Therefore, $\dim M_1(\Gamma) - \dim S_1(\Gamma)$ is the half of $\dim H^0(\omega|_D) = n$. □

Remark 2.5. For a cusp form of weight 1 and level Γ to exist, the inequality $n \leq 2g - 2$, or equivalently $24n \leq [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ must be satisfied, which is true when Γ is sufficiently small. For example, if $\Gamma = \Gamma(N)$, the inequality is satisfied if $N \geq 12$.

Remark 2.6. It is expected that there is no simple formula that expresses $\dim S_1(\Gamma)$. It is however conjectured that $S_1(\Gamma)$ is mostly exhausted by dihedral forms. See for example the discussion in [Duk95, §1].

3. Theta characteristics as uniformizing logarithmic Higgs bundles

In the previous section, the computation of $\deg \omega$ and the dimension of the space of modular forms only used the Kodaira–Spencer isomorphism. Thus, the same dimension formulae will hold true for any line bundle ν such that $\nu^{\otimes 2} \cong \Omega^1_{X(\Gamma)/\mathbb{C}}(D)$.

Definition 3.1. A line bundle ν over $X(\Gamma)$ which satisfies

$$\nu^{\otimes 2} \cong \Omega^1_{X(\Gamma)/\mathbb{C}}(D),$$

is called a *(stable) theta characteristic.*[2]

If ν is a theta characteristic, $\nu \otimes \omega^{-1}$ is a square-root of $\mathcal{O}_{X(\Gamma)}$. Thus, there are in total $2^{2g} = \#\operatorname{Jac}(X(\Gamma))[2](\mathbb{C})$ many theta characteristics. For a theta characteristic ν, the isomorphism $\nu^{\otimes 2} \cong \Omega^1_{X(\Gamma)/\mathbb{C}}(D)$ induces an isomorphism $\nu \cong \nu^{-1} \otimes \Omega^1_{X(\Gamma)/\mathbb{C}}(D)$. This in turn deduces a *logarithmic Higgs field* $\theta_\nu : E \to E \otimes \Omega^1_{X(\Gamma)/\mathbb{C}}(D)$ on the vector bundle $E_\nu := \nu \oplus \nu^{-1}$,

$$\theta_\nu : \nu \oplus \nu^{-1} \to \nu \xrightarrow{\sim} \nu^{-1} \otimes \Omega^1_{X(\Gamma)/\mathbb{C}}(D) \to (\nu \oplus \nu^{-1}) \otimes \Omega^1_{X(\Gamma)/\mathbb{C}}(D),$$

making (E_ν, θ_ν) a logarithmic Higgs bundle on $X(\Gamma)$.

In view of the nonabelian Hodge correspondence, one may ask which local systems correspond to the Higgs fields constructed using theta characteristics. In the non-logarithmic setting, Simpson showed in [Sim88] that the Higgs field formed by a theta characteristic of a hyperbolic curve

[2] A line bundle is called a theta characteristic if it is a square root of the canonical bundle Ω^1. A stable theta characteristic is when a line bundle is a square root of the canonical bundle twisted by a specific divisor. As we will only care about the square-roots of $\Omega^1(D)$ in this article, most of the time we will just refer to such line bundles as theta characteristics.

is precisely a lift of the projective representation of the topological π_1 of the curve given by the complex uniformization.

Definition 3.2. For the rest of the paper, we fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$ compatible with the embeddings. We also fix a $\overline{\mathbb{Q}}$-point $* \in Y(\Gamma)(\overline{\mathbb{Q}})$ throughout the paper. The points induced from $*$ by the embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ are again denoted $*$ by abuse of notation.

Using a tame regular analogue of the nonabelian Hodge correspondence over a noncompact Riemann surface, we are able to show that the theta characterstics in our sense are also characterized by the projective lifts of $\pi_1(Y(\Gamma), *) = P\Gamma$. Before formulating the theorem, we introduce some terminologies.

Definition 3.3. Let $P\Gamma$ be the projective image of Γ. Namely,

$$P\Gamma = \mathrm{im}(\Gamma \hookrightarrow \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})).$$

A *projective lift* of $P\Gamma$ is a subgroup $\Gamma' \leq \mathrm{SL}_2(\mathbb{R})$ such that $P\Gamma' = P\Gamma$. A projective lift is *honest* if the natural map $\Gamma' \to P\Gamma' = P\Gamma$ is injective (thus bijective).

As $Y(\Gamma) \cong S_{g,n}$, we could choose a set of generators $a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_n \in P\Gamma$ such that the only relation between the generators is

$$[a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_n = 1.$$

Let $A_1, B_1, \cdots, A_g, B_g, C_1, \cdots, C_n \in \Gamma$ be the corresponding elements in Γ. A *hyperbolic projective lift* is an honest projective lift of the form

$$\langle \epsilon_{11} A_1, \epsilon_{12} B_1, \cdots, \epsilon_{g1} A_g, \epsilon_{g2} B_g, C_1, \cdots, C_n \rangle \subset \mathrm{SL}_2(\mathbb{R}),$$

where $\epsilon_{ij} \in \{\pm 1\}$ for $1 \leq i \leq g$, $1 \leq j \leq 2$.

Lemma 3.4.

(1) The notion of the hyperbolic projective lifts does not depend on the choice of a presentation of $P\Gamma$ as a topological fundamental group of a surface.

(2) Given a hyperbolic projective lift Γ' of $P\Gamma$, let $\rho_{\Gamma'}$ be the two-dimensional real representation of $P\Gamma$ given by $P\Gamma \xleftarrow{\sim} \Gamma' \hookrightarrow \mathrm{SL}_2(\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{R})$. Then, for two different hyperbolic projective lifts $\Gamma'_1 \neq \Gamma'_2$, $\rho_{\Gamma'_1} \not\cong \rho_{\Gamma'_2}$.

Proof.

(1) As $n \geq 1$, $\Gamma = P\Gamma$ is a free group with $2g + n - 1$ generators. Thus, given a presentation of $P\Gamma$ as above, choosing an honest projective lift of $P\Gamma$ is the same as choosing a sign for each of $A_1, B_1, \cdots, A_g, B_g, C_1, \cdots, C_{n-1}$, or equivalently, choosing a homomorphism $P\Gamma \to (\mathbb{Z}/2\mathbb{Z})^{2g+n-1}$.

Note that $X(\Gamma)$ has the fundamental group, denoted $\overline{P\Gamma}$, (with the same choice of basepoint as $Y(\Gamma)$ via the inclusion $Y(\Gamma) \hookrightarrow X(\Gamma)$) whose presentation can be given by

$$\overline{P\Gamma} \cong \langle \overline{a_1}, \overline{b_1}, \cdots, \overline{a_g}, \overline{b_g} \mid [\overline{a_1}, \overline{b_1}] \cdots [\overline{a_g}, \overline{b_g}] = 1 \rangle,$$

and the natural homomorphism $\pi_1(Y(\Gamma), *) \to \pi_1(X(\Gamma), *)$ is given by $a_i \mapsto \overline{a_i}$, $b_i \mapsto \overline{b_i}$, $c_i \mapsto 1$. Thus, an honest projective lift is a hyperbolic projective lift if and only if the corresponding homomorphism $P\Gamma \to (\mathbb{Z}/2\mathbb{Z})^{2g+n-1}$ factors through the morphism $P\Gamma \to \overline{P\Gamma}$. Since the latter condition does not refer to a specific presentation at all and only uses the natural map $P\Gamma \to \overline{P\Gamma}$, the notion of hyperbolic projective lifts is independent of the choice of a presentation of $P\Gamma$.

(2) Choose a presentation of $P\Gamma$ as above. Given $\Gamma'_1 \neq \Gamma'_2$, there is some $1 \leq i \leq g$ such that either a_i or b_i is lifted to matrices with the opposite signs. Let $d \in \{a_i, b_i\}$ be such element. Then, $\operatorname{tr} \rho_{\Gamma'_1}(d) = -\operatorname{tr} \rho_{\Gamma'_2}(d)$. Since A_i and B_i are hyperbolic matrices, $\operatorname{tr} A_i$ and $\operatorname{tr} B_i$ are both nonzero. Thus, $\operatorname{tr} \rho_{\Gamma'_1}(d) \neq \operatorname{tr} \rho_{\Gamma'_2}(d)$, which means that as abstract representations $\rho_{\Gamma'_1}$ and $\rho_{\Gamma'_2}$ are non-isomorphic.

□

The above lemma shows that we can refer to the hyperbolic projective lifts of $P\Gamma$ as being certain two-dimensional real representations of $P\Gamma$, or, after conjugation, two-dimensional representations of $P\Gamma$ valued in $\mathrm{SU}(1,1)$.

We are now able to state the main theorem of this section.

Theorem 3.5 (Theta characteristics are hyperbolic projective lifts)**.** *There is a unique one-to-one correspondence between the theta characteristics and the hyperbolic projective lifts of $P\Gamma$, characterized as follows.*

- *For a theta characteristic ν, ρ_{Γ_ν} is the 2-dimensional local system on $Y(\Gamma)$ corresponding to the logarithmic Higgs bundle (E_ν, θ_ν) via the tame nonabelian Hodge correspondence. Furthermore, there is a natural isomorphism $H^0(X(\Gamma), \nu) \cong M_1(\Gamma_\nu)$.*
- $\Gamma_\omega = \Gamma$.

Proof. We would like to use the tame regular version of nonabelian Hodge correspondence over a noncompact curve as in [Sim90]: for the definitions of the terms, see [Sim90, Synopsis].

Theorem 3.6 (Tame nonabelian Hodge correspondence over non-compact curves, [Sim90, p.718])**.** *Over a smooth algebraic noncompact curve, there is a natural one-to-one correspondence between stable filtered regular Higgs bundles of degree zero, and stable filtered local systems of degree zero. The correspondence preserves the rank on both sides.*

On the other hand, a special case of this correspondence is proved in [Sim88, Theorem 4]: taking the graded piece gives an equivalence of categories from the category of complex variations of Hodge structures on $Y(\Gamma)$ to the category of Hodge bundles on $Y(\Gamma)$. Here, geometric objects on $Y(\Gamma)$ are extended to $X(\Gamma)$ as "canonical extensions" (namely, the filtration is given by the growth behavior at the punctures).

We equip the Higgs bundle (E_ν, θ_ν) with a left-continuous decreasing filtration

$$E_{\nu,\alpha} = E_\nu(-\lceil\alpha\rceil D), \quad \alpha \in \mathbb{R}.$$

This is by definition a filtered regular Higgs bundle of degree zero. Moreover, it is stable as the only proper nonzero θ-stable subbundle of E_ν is ν^{-1}, whose filtered degree is negative. This is the same as the "canonical extension" of $(E_\nu, \theta_\nu)|_{Y(\Gamma)}$.

By the tame nonabelian Hodge correspondence, from $(E_\nu, \{E_{\nu,\alpha}\}, \theta_\nu)$, we obtain a 2-dimensional stable filtered local system L_ν of degree zero. The correspondence of the statement of the Theorem is then

$$\nu \mapsto \text{the underlying local system of } L_\nu.$$

The inverse of the correspondence can be given as follows. Let Γ' be a hyperbolic projective lift of $P\Gamma$. Then, the universal variation of Hodge structures on $\mathbb{H}$ descend to a variation of Hodge structure on $Y(\Gamma') = Y(\Gamma)$ whose underlying local system is the same as the local system corresponding to Γ'. Since the local system has unipotent local monodromies around the punctures, the Hodge filtration extends canonically (in the sense of Deligne) to $X(\Gamma)$ as a filtration of vector bundles. Let $\overline{F^1}$ be the canonical extension of F^1; namely, it is the sheaf of sections of F^1 with at worst logarithmic growth at the punctures. Then the inverse correspondence is

$$\Gamma' \mapsto \overline{F^1}.$$

This is certainly a restriction of the inverse of the tame nonabelian Hodge correspondence as above by [Sim88, Theorem 4]. It sends hyperbolic projective lifts of $P\Gamma$ to theta characteristics. Since the two sets, the set of hyperbolic projective lifts of $P\Gamma$ and the set of theta characteristics, are finite sets with the same cardinality 2^g, it gives rise to a one-to-one correspondence. From the description of the inverse correspondence, the rest of the Theorem follows immediately. $\square$

Note that, for a theta characteristic ν, there is a 2-torsion line bundle L on $X(\Gamma)$ such that $\nu = \omega \otimes L$. It is well-known that (e.g. [DP22, 1.1.1]) there is a one-to-one correspondence between 2-torsion line bundles and étale double covers. Thus, this gives another geometric way to compute Γ_ν.

Proposition 3.7. *For a* 2*-torsion line bundle* L *on* $X(\Gamma)$*, there is a unique étale double cover*

$$\alpha : \widetilde{X}_L \to X(\Gamma),$$

such that $\alpha_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{X(\Gamma)} \oplus L$*. If we define the representation* ρ_L *of* $P\Gamma = \pi_1(Y(\Gamma), *)$ *to be the composition*

$$\pi_1(Y(\Gamma), *) \to \pi_1(X(\Gamma), *) \twoheadrightarrow \mathrm{Gal}(\widetilde{X}_L / X(\Gamma)) = \{\pm 1\},$$

the local system ρ_{Γ_ν} *satisfies* $\rho_{\Gamma_\nu} = \rho_{\Gamma_\omega} \otimes \rho_L$*. In particular,*

$$\Gamma_\nu = \langle \epsilon_{11} A_1, \epsilon_{12} B_1, \cdots, \epsilon_{g1} A_g, \epsilon_{g2} B_g, C_1, \cdots, C_n \rangle,$$

where $\epsilon_{i1} = \rho_L(a_i)$ *and* $\epsilon_{i2} = \rho_L(b_i)$*.*

Proof. The desired étale double cover $\widetilde{X}_L$ can be constructed as a relative Spec over $X(\Gamma)$,

$$\widetilde{X}_L = \mathcal{S}pec_{\mathcal{O}_{X(\Gamma)}} \left(\mathcal{O}_{X(\Gamma)} \oplus L \right),$$

where $\mathcal{O}_{X(\Gamma)} \oplus L$ is the $\mathcal{O}_{X(\Gamma)}$-algebra where the only nontrivial multiplication structure is given by the morphism $L \otimes_{\mathcal{O}_{X(\Gamma)}} L \xrightarrow{\sim} \mathcal{O}_{X(\Gamma)}$.

As first noted by Deligne, the nonabelian Hodge correspondence is compatible with tensor products (see [Sim92, p. 8]). Thus, we only need to show that $\rho_L : \pi_1(X(\Gamma), *) \to \{\pm 1\}$ and the Higgs bundle $(L, 0)$ on $X(\Gamma)$ correspond to each other via the nonabelian Hodge correspondence on $X(\Gamma)$.

Let $c \in \mathrm{Gal}(\widetilde{X}_L / X(\Gamma))$ be the nontrivial Galois element, which gives rise to an automorphism $c \in \mathrm{Aut}_{X(\Gamma)}(\widetilde{X}_L)$. Consider $\mathscr{H}^0_{\mathrm{dR}}(\widetilde{X}_L / X(\Gamma))$, which is a variation of Hodge structures on $X(\Gamma)$ of rank 2 and weight 0. It is isomorphic to

$$\mathscr{H}^0_{\mathrm{dR}}(\widetilde{X}_L / X(\Gamma)) \cong (\mathcal{O}_{X(\Gamma)}, d) \oplus (L, d),$$

where $(\mathcal{O}_{X(\Gamma)}, d)$ denotes the canonical differential $d : \mathcal{O}_{X(\Gamma)} \to \Omega^1_{X(\Gamma)/\mathbb{C}}$, and $(L, d) = L \otimes (\mathcal{O}_{X(\Gamma)}, d)$ (this is still a vector bundle with an integrable connection as $L^{\otimes 2} = \mathcal{O}$, so the transition functions for a sufficiently fine atlas can be taken to be constant functions, namely ± 1).

Furthermore, c gives rise to an endomorphism of $\mathscr{H}^0_{\mathrm{dR}}(\widetilde{X}_L/X(\Gamma))$, where

$$\left(\mathscr{H}^0_{\mathrm{dR}}(\widetilde{X}_L/X(\Gamma))\right)^{c=1} = (\mathcal{O}_{X(\Gamma)}, d), \quad \left(\mathscr{H}^0_{\mathrm{dR}}(\widetilde{X}_L/X(\Gamma))\right)^{c=-1} = (L, d).$$

Thus, ρ_L (considered as a character) is a local system that underlies a variation of Hodge structure, and its associated Hodge bundle is $(L, 0)$ which implies that ρ_L and $(L, 0)$ correspond to each other via the nonabelian Hodge correspondence. $\square$

We will see in §5 that without much difficulty the same construction works motivically.

4. ω is the unique congruence theta characteristic

By Theorem 3.5, for each theta characteristic ν, $H^0(X(\Gamma), \nu)$ is a space of weight one modular forms of level Γ_ν. The main result of this section is the following.

Theorem 4.1. *For a theta characteristic ν, $\Gamma_\nu \leq \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup if and only if $\nu = \omega$.*

For simplicity, we will call ν a *congruence theta characteristic* if Γ_ν is a congruence subgroup. Therefore, the above Theorem is that ω is the only congruence theta characteristic. A quick corollary is that the Hecke operators are zero on $H^0(X(\Gamma), \nu)$ when $\nu \neq \omega$.

Corollary 4.2. *For $(p, N) = 1$, define the Hecke operator T_p on $H^0(X(\Gamma), \nu) = M_1(\Gamma_\nu)$ as follows: let*

$$\alpha = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix},$$

and let $\Gamma_\nu \alpha \Gamma_\nu = \cup_i \Gamma_\nu \alpha \alpha_i$. Then, for $f \in M_1(\Gamma_\nu)$,

$$T_p f = \sum_i f|_{\alpha\alpha_i}.$$

If $\nu \neq \omega$, we always have $T_p f = 0$.

Proof. By [Ber94], we know that T_p factors through the trace map to the congruence closure. In our case, if $\nu \neq \omega$, by Theorem 4.1, the congruence closure of Γ_ν is $\langle \pm 1, \Gamma \rangle$. Since there is no nonzero odd-weight modular form of level $\langle \pm 1, \Gamma \rangle$, the desired statement follows. $\square$

Remark 4.3. The above Hecke operator can be geometrically interpreted as the correspondence

$$\begin{array}{ccc} & X(\Gamma_\nu \cap \alpha^{-1}\Gamma_\nu\alpha \cap \Gamma_0(p)) \xrightarrow{\sim} X(\alpha\Gamma_\nu\alpha^{-1} \cap \Gamma_\nu \cap \Gamma^0(p)) & \\ \swarrow & & \searrow \\ X(\Gamma_\nu) & & X(\Gamma_\nu) \end{array}$$

where $\Gamma^0(p) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right) \pmod p\}$. As Γ_ν is an index 2 subgroup of a congruence subgroup of level prime to p, it is not in general true that α normalizes Γ_ν, but rather sends Γ_ν to a possibly different index 2 subgroup of $\langle \pm 1, \Gamma \rangle$.

The proof of Theorem 4.1 will be a slight generalization of the proofs in [Kim14, §2], and we will use the technical condition $(*)$. As in *loc. cit.*, we will use $V_2(G) := G^{\mathrm{ab}}/(G^{\mathrm{ab}})^2 = G^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{F}_2$ for a group G. Note that V_2 is a functor that sends finitely generated groups to finite-dimensional $\mathbb{F}_2$-vector spaces. We record the following easy

Lemma 4.4.

(1) V_2 *is a right-exact functor.*[3]

(2) $V_2(G_1 \times G_2) \cong V_2(G_1) \times V_2(G_2)$.

Proof. The functor V_2 is the composition of the abelianization functor with $(-) \otimes_{\mathbb{Z}} \mathbb{F}_2$, and both are right exact. □

Proof of Theorem 4.1. As $\Gamma_\omega = \Gamma$, Γ_ω is a congruence subgroup, which proves one direction. Conversely, suppose that Γ_ν is a congruence subgroup. As per Theorem 3.5, we need to prove that there is no hyperbolic projective lift of $P\Gamma$ that is different from Γ. Suppose that Γ is of level N; namely, N is the minimal number such that $\Gamma(N) \leq \Gamma$. By the result of Wohlfahrt and Kiming–Schütt–Verrill (see [Woh64, Theorem 2], [KSV11, Proposition 3]), Γ is of general level either N or $\frac{N}{2}$. Recall that the *general level* of a Fuchsian group is the least common multiple of the widths of the cusps. The general level only depends on the projective image of the Fuchsian group, so Γ_ν is of general level N. By *loc. cit.*, $\Gamma_\nu \geq \Gamma(2N)$. Thus, $\langle \pm 1, \Gamma\rangle \geq \Gamma_\nu \geq \Gamma(2N)$. Thus, Γ_ν corresponds to a subgroup of $\langle \pm 1, \Gamma\rangle/\Gamma(2N) \cong \langle \pm 1\rangle \times \Gamma/\Gamma(2N)$ such that $\{\pm 1\}$ and Γ_ν together generate the whole subgroup $\langle \pm 1, \Gamma\rangle/\Gamma(2N)$. As in [Kim14, Proposition 1], projective lifts of $P\Gamma$ that are also congruence subgroups are in one-to-one corresopndence with a sub-$\mathbb{F}_2$-vector space U of $V_2(\langle \pm 1, \Gamma\rangle/\Gamma(2N)) \cong \langle \pm 1\rangle \times V_2(\Gamma/\Gamma(2N))$ such that U and -1 together span the whole vector space. Such projective lift is honest if U is a proper subspace, and $-1 \notin U$. Thus, the composition $U \hookrightarrow V_2(\langle \pm 1, \Gamma\rangle/\Gamma(2N)) = \langle \pm 1\rangle \times V_2(\Gamma/\Gamma(2N)) \twoheadrightarrow V_2(\Gamma/\Gamma(2N))$ is injective, thus bijective (as the target and the source have the same $\mathbb{F}_2$-dimensions). Thus, choosing an honest congruence projective lift is the same as choosing the signs for the lifts of basis elements of $V_2(\Gamma/\Gamma(2N))$.

By the assumption $(*)$, $N = \mathrm{lcm}(N_1, N_2)$, and $\Gamma(N) \leq \Gamma \leq \Gamma_1(N)$. Let $N = 2^s p_1^{t_1} \cdots p_r^{t_r}$, where p_i's are odd primes. Note also that

$$\mathrm{SL}_2(\mathbb{Z})/\Gamma(2N) \cong \mathrm{SL}_2(\mathbb{Z}/2N\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z}/2^{s+1}\mathbb{Z}) \times \prod_{i=1}^{r} \mathrm{SL}_2(\mathbb{Z}/p_i^{t_i}\mathbb{Z}),$$

so $\Gamma_1(N)/\Gamma(2N)$ injects into $\Gamma_1(2^s)/\Gamma(2^{s+1}) \times \prod_{i=1}^{r} \Gamma_1(p_i^{t_i})/\Gamma(p_i^{t_i})$, which is a bijection as the two groups are finite groups of the same order; for $1 \leq a \leq b$, $\#\Gamma_1(p^a)/\Gamma(p^b) = p^{3b-2a}$. Under this isomorphism, we have

$$\Gamma/\Gamma(2N) \cong A \times \prod_{i=1}^{r} B_i,$$

where $B_i \leq \Gamma_1(p_i^{t_i})/\Gamma(p_i^{t_i})$ is a subgroup, and

$$A = \begin{cases} \mathrm{SL}_2(\mathbb{Z})/\Gamma(2) & \text{if } s = 0, \\ \Gamma_1(2^s)/\Gamma(2^{s+1}) & \text{if } 2|N_1, \\ \Gamma(2^s)/\Gamma(2^{s+1}) & \text{if } 2|N_2. \end{cases}$$

Note that $\Gamma_1(p_i^{t_i})/\Gamma(p_i^{t_i})$ is of odd order, so B is of odd order as well. Thus, the natural projection map $\Gamma/\Gamma(2N) \twoheadrightarrow A$ induces an isomorphism $V_2(\Gamma/\Gamma(2N)) \xrightarrow{\sim} V_2(A)$.

By the right-exactness of V_2, we have a surjective natural map $V_2(\Gamma) \twoheadrightarrow V_2(\Gamma/\Gamma(2N))$. Since a hyperbolic projective lift fixes the signs of the lifts of the loops around the cusps, to prove Theorem 4.1, it suffices to prove that $V_2(\Gamma/\Gamma(2N))$ is spanned by the images of shearing transformations along the cusps. We prove that this is true by dividing into cases.

[3] Even though the category of groups is not an abelian category, the notion of exact sequences makes sense.

(Case 1) If $s = 0$, then $A = \mathrm{SL}_2(\mathbb{F}_2) \cong S_3$, and $V_2(A) \cong (\mathbb{Z}/2\mathbb{Z})$ is generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. As $\Gamma = \Gamma_1(N_1) \cap \Gamma(N_2)$ with N_1, N_2 odd, $\left(\begin{smallmatrix} 1 & N_2 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$ is sent to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in A$ via the natural projection $\Gamma \twoheadrightarrow A$. Since $\left(\begin{smallmatrix} 1 & N_2 \\ 0 & 1 \end{smallmatrix}\right)$ is a shearing transformation along the cusp $\infty \in \mathbb{P}^1(\mathbb{Q})$, there is no hyperbolic projective lift different from Γ.

(Case 2) If $s > 0$ and $2|N_2$, then $A = \Gamma(2^s)/\Gamma(2^{s+1})$. As in the proof of [Kim14, Proposition 2], one notes that $A = V_2(A) \cong (\mathbb{Z}/2\mathbb{Z})^3$ with a generator given by

$$\alpha = \begin{pmatrix} 1 & 2^s \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1+2^s & -2^s \\ 2^s & 1-2^s \end{pmatrix} \quad \gamma = \begin{pmatrix} 1 & 0 \\ 2^s & 1 \end{pmatrix}.$$

Note that we took a slightly different set of generators. Note that

$$\alpha \equiv \begin{pmatrix} 1 & N_2 \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\, 2^{s+1}), \quad \beta \equiv \begin{pmatrix} 1+N_1N_2 & -N_1N_2 \\ N_1N_2 & 1-N_1N_2 \end{pmatrix} (\mathrm{mod}\, 2^{s+1}), \quad \gamma \equiv \begin{pmatrix} 1 & 0 \\ N_1N_2 & 1 \end{pmatrix} (\mathrm{mod}\, 2^{s+1}),$$

and these matrices are genuine elements of $\Gamma = \Gamma_1(N_1) \cap \Gamma(N_2)$. Also note that $\left(\begin{smallmatrix} 1 & N_2 \\ 0 & 1 \end{smallmatrix}\right)$ is a shearing transformation along the cusp $\infty \in \mathbb{P}^1(\mathbb{Q})$, $\left(\begin{smallmatrix} 1 & 0 \\ N_1N_2 & 1 \end{smallmatrix}\right)$ is a shearing transformation along the cusp $0 \in \mathbb{P}^1(\mathbb{Q})$, and $\left(\begin{smallmatrix} 1+N_1N_2 & -N_1N_2 \\ N_1N_2 & 1-N_1N_2 \end{smallmatrix}\right)$ is a shearing transformation along the cusp $1 \in \mathbb{P}^1(\mathbb{Q})$, since

$$\begin{pmatrix} 1+N_1N_2 & -N_1N_2 \\ N_1N_2 & 1-N_1N_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ N_1N_2 & \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Therefore, there is no hyperbolic projective lift different from Γ.

(Case 3) If $s > 0$ and $2|N_1$, then $A = \Gamma_1(2^s)/\Gamma(2^{s+1})$. As per *loc. cit.*, $V_2(A) \cong (\mathbb{Z}/2\mathbb{Z})^2$ with a basis given by

$$\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 2^s & 1 \end{pmatrix}.$$

Since N_2 is odd, 2^{s+1} is invertible modulo N_2, which implies that there exists $k \in \mathbb{Z}$ such that $k2^{s+1} \equiv -1 (\mathrm{mod}\, N_2)$. Now note that

$$\tau \equiv \begin{pmatrix} 1 & 1+k2^{s+1} \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\, 2^{s+1}), \quad \gamma \equiv \begin{pmatrix} 1 & 0 \\ N_1N_2 & 1 \end{pmatrix} (\mathrm{mod}\, 2^{s+1}),$$

and these matrices are genuine elements of $\Gamma = \Gamma_1(N_1) \cap \Gamma(N_2)$. It is clear that these matrices are also shearing transformations along the cusps $\infty, 0 \in \mathbb{P}^1(\mathbb{Q})$, respectively, so there is no hyperbolic projective lift different from Γ.

□

5. Hecke algebra of Γ_ν

6. Twisted Kuga–Sato varieties

We now aim to show that the construction of Proposition 3.7 yields, for each ν, a *geometric étale local system* over $Y(\Gamma)_K$, over an appropriate number field K, that comes from geometry. We will construct this from a geometric object that we will call the *twisted Kuga–Sato variety.*

Definition 6.1. Let $K/\mathbb{Q}$ be a number field, and let $\alpha : \widetilde{X} \to X(\Gamma)_K$ be a finite étale Galois cover of degree r. The *twisted Kuga–Sato variety* $u : \overline{W}_{\widetilde{X}} \to X(\Gamma)_K$ associated with $\widetilde{X}$ is defined as the Weil restriction of the pullback $\alpha^*\overline{\mathcal{E}}_K$,

$$\overline{W}_{\widetilde{X}} := \mathrm{R}_{\widetilde{X}/X(\Gamma)_K}(\alpha^*\overline{\mathcal{E}}_K).$$

The open twisted Kuga–Sato variety $u : W_{\widetilde{X}} \to Y(\Gamma)_K$ is defined as the open subscheme of $\overline{W}_{\widetilde{X}}$ lying over $Y(\Gamma)_K \subset X(\Gamma)_K$.

For the definition of scheme-theoretic Weil restriction of scalars, see [BLR90, §7.6].

Example 6.2. For a trivial étale r-cover $X(\Gamma) \coprod \cdots \coprod X(\Gamma) \to X(\Gamma)$ of r copies of $X(\Gamma)$, the corresponding twisted Kuga–Sato variety is the usual Kuga–Sato variety (before the canonical desingularization), namely the r-fold fiber product of $\overline{\mathcal{E}}$ over $X(\Gamma)$.

Example 6.3. As in the case of the usual Kuga–Sato variety, the twisted Kuga–Sato variety $\overline{W}_{\widetilde{X}}$ is in general singular, even though $\alpha^*\overline{\mathcal{E}}_K$ itself is a smooth K-scheme. Also, the open twisted Kuga–Sato variety $W_{\widetilde{X}}$ is smooth.

The open twisted Kuga–Sato variety $W_{\widetilde{X}}$ still turns out to be a family of principally polarized abelian r-folds over $Y(\Gamma)_K$.

Proposition 6.4. *The open twisted Kuga–Sato variety $u : W_{\widetilde{X}} \to Y(\Gamma)_K$ is a family of principally polarized abelian varieties of dimension r.*

Proof. By [DN03, Proposition 2], the Weil restriction of a principal polarization is a principal polarization. As an elliptic curve is principally polarized, $W_{\widetilde{X}}$ is a family of principally polarized abelian varieties. □

We can thus think of classifying map to the moduli space of principally polarized abelian varieties of dimension r,

$$\pi_{\widetilde{X}} : Y(\Gamma)_K \to \mathcal{A}_{r,K},$$

and this induces a classifying map to the corresponding coarse moduli scheme,

$$p_{\widetilde{X}} : Y(\Gamma)_K \to A_{r,K}.$$

Proposition 6.5. *Let $\pi_{\mathrm{diag}} : Y(\Gamma)_K \to \mathcal{A}_{r,K}$ be the classifying map which corresponds to the r-th self-product $\mathcal{E}_K \times_{Y(\Gamma)_K} \cdots \times_{Y(\Gamma)_K} \mathcal{E}_K$ over $Y(\Gamma)_K$, and let $p_{\mathrm{diag}} : Y(\Gamma)_K \to A_{r,K}$ be the corresponding map to the coarse moduli scheme. Then, $\pi_{\mathrm{diag}} \neq \pi_{\widetilde{X}}$, but $p_{\mathrm{diag}} = p_{\widetilde{X}}$.*

Proof. By [CGP15, Proposition A.5.2], we have

$$W_{\widetilde{X}} \times_{Y(\Gamma)_K} \widetilde{Y} = \mathrm{R}_{\widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y}/\widetilde{Y}}(\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} (\widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y})),$$

where $\widetilde{Y} = \alpha^{-1}(Y(\Gamma)_K)$. Since $\widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y}$ is isomorphic to the disjoint union of r copies of $\widetilde{Y}$, we have

$$W_{\widetilde{X}} \times_{Y(\Gamma)_K} \widetilde{Y} \cong \mathrm{R}_{\widetilde{Y} \coprod \cdots \coprod \widetilde{Y}/\widetilde{Y}} \left(\alpha^*\mathcal{E}_K \coprod \cdots \coprod \alpha^*\mathcal{E}_K \right) \cong \alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \cdots \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K.$$

This implies that

$$W_{\widetilde{X}} \times_{Y(\Gamma)_K} \widetilde{Y} \cong (\mathcal{E}_K \times_{Y(\Gamma)_K} \cdots \times_{Y(\Gamma)_K} \mathcal{E}_K) \times_{Y(\Gamma)_K} \widetilde{Y}.$$

Furthermore, the two abelian schemes are isomorphic as $\widetilde{Y}$-schemes; even though $\widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y} \cong \widetilde{Y} \coprod \cdots \coprod \widetilde{Y}$ is most naturally thought as being indexed by the elements in $\mathrm{Gal}(\widetilde{Y}/Y(\Gamma)_K)$, for any $\sigma \in \mathrm{Gal}(\widetilde{Y}/Y(\Gamma)_K)$, the $\widetilde{Y}$-scheme $\widetilde{Y} \xrightarrow{\sigma} \widetilde{Y}$ is isomorphic as a $\widetilde{Y}$-scheme to $\widetilde{Y} \xrightarrow{\mathrm{id}} \widetilde{Y}$. Thus, $\pi_{\mathrm{diag}} \circ \alpha = \pi_{\widetilde{X}} \circ \alpha$, and $p_{\mathrm{diag}} \circ \alpha = p_{\widetilde{X}} \circ \alpha$. Since α is flat, surjective, and locally of finite presentation, by [Sta18, Tag 05VM], α is an epimorphism (i.e. surjective as a map of sheaves), which implies that $p_{\mathrm{diag}} = p_{\widetilde{X}}$.

The fact that $\pi_{\mathrm{diag}} \neq \pi_{\widetilde{X}}$ is equivalent to $W_{\widetilde{X}} \neq \mathcal{E}_K \times_{Y(\Gamma)_K} \cdots \times_{Y(\Gamma)_K} \mathcal{E}_K$. There are many ways of seeing this – we will soon see that the monodromy representations of their relative $H^1_{\text{ét}}$ are different. A more elementary way of seeing the difference is to observe that the descent data for the two schemes are different for the étale covering $\alpha : \widetilde{Y} \to Y(\Gamma)_K$. Let us fix an isomorphism

$$\iota : \widetilde{Y} \coprod \cdots \coprod \widetilde{Y} \cong \widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y},$$

such that the two projection maps $p_1, p_2 : \widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y} \to \widetilde{Y}$ are identified with

$$p_1 \circ \iota : \widetilde{Y} \coprod \cdots \coprod \widetilde{Y} \xrightarrow{(\mathrm{id}, \cdots, \mathrm{id})} \widetilde{Y},$$

$$p_2 \circ \iota : \widetilde{Y} \coprod \cdots \coprod \widetilde{Y} \xrightarrow{(\sigma_1, \cdots, \sigma_r)} \widetilde{Y},$$

where $\mathrm{Gal}(\widetilde{Y}/Y(\Gamma)_K) = \{\sigma_1 = \mathrm{id}, \sigma_2, \cdots, \sigma_r\}$. Note that $\mathrm{Gal}(\widetilde{Y}/Y(\Gamma)_K) \subset S_r$ where one can identify $\sigma \in \mathrm{Gal}(\widetilde{Y}/Y(\Gamma)_K)$ with the permutation of the components

$$\widetilde{Y} \coprod \cdots \coprod \widetilde{Y} \xrightarrow{\sim} \widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y} \xrightarrow{(\sigma, \mathrm{id})} \widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y} \xleftarrow{\sim} \widetilde{Y} \coprod \cdots \coprod \widetilde{Y}.$$

The descent datum for $\mathcal{E}_K \times_{Y(\Gamma)_K} \cdots \times_{Y(\Gamma)_K} \mathcal{E}_K$ for the covering $\widetilde{Y} \to Y(\Gamma)_K$ is given by

$$\left(\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \cdots \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K\right) \coprod \cdots \coprod \left(\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \cdots \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K\right)$$

$$\xrightarrow{(\mathrm{id}, \cdots, \mathrm{id})} \left(\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \cdots \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K\right) \coprod \cdots \coprod \left(\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \cdots \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K\right),$$

whereas the descent datum for $W_{\widetilde{X}}$ for the covering $\widetilde{Y} \to Y(\Gamma)_K$ is given by

$$\left(\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \cdots \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K\right) \coprod \cdots \coprod \left(\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \cdots \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K\right)$$

$$\xrightarrow{(f, \cdots, f)} \left(\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \cdots \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K\right) \coprod \cdots \coprod \left(\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \cdots \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K\right),$$

where

$$f : \alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \cdots \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K \to \left(\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \cdots \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K\right) \coprod \cdots \coprod \left(\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \cdots \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K\right),$$

is $f = (\sigma_1, \cdots \sigma_r)$, where σ_i is the natural map corresponding to the permutation that it represents. As there is no $\widetilde{Y}$-automorphism that intertwines the two descent data, the two descend to two non-isomorphism $Y(\Gamma)_K$-schemes, as desired. $\square$

Proposition 6.6. *The family of abelian varieties $W_{\widetilde{X}}$ on $Y(\Gamma)_K$ has the following properties.*

1. *Given a field L/K and a point $x \in Y(\Gamma)(L)$, the fiber $(W_{\widetilde{X}})_x$ is the abelian variety over L given by*

$$(W_{\widetilde{X}})_x = \mathrm{R}_{\widetilde{X}_x/L}((E_x)_{\widetilde{X}_x}),$$

   where E_x is the elliptic curve corresponding to x, and $\widetilde{X}_x$ is the étale L-algebra of degree r given by the fiber of $\alpha : \widetilde{X} \to X(\Gamma)_K$ over x.
2. *The family $u : W_{\widetilde{X}} \to Y(\Gamma)_K$ is $\mathrm{Gal}(\widetilde{Y}/Y(\Gamma)_K)$-equivariant, where $\mathrm{Gal}(\widetilde{Y}/Y(\Gamma)_K)$ acts trivially on $Y(\Gamma)_K$.*

Proof. (1) follows directly from the fact that the Weil restriction of schemes is compatible with base-change. The action of $\mathrm{Gal}(\widetilde{Y}/Y(\Gamma)_K)$ on $\widetilde{Y}$ and on $\alpha^*\mathcal{E}_K$ gives, by functoriality of the Weil restriction, the action of $\mathrm{Gal}(\widetilde{Y}/Y(\Gamma)_K)$ on $W_{\widetilde{X}}$, fixing $Y(\Gamma)_K$ on the base, from which (2) follows. □

If the étale cover $\widetilde{X}/X(\Gamma)_K$ is abelian, then $\widetilde{Y}$, when considered as a "relative motive over $Y(\Gamma)_K$", can be further split into pieces. Here, a "relative motive" means merely a collection of various local systems ("realizations") that are compatible with each other.

Definition 6.7. Suppose that $\alpha : \widetilde{X} \to X(\Gamma)_K$ is an abelian étale Galois cover, and that the exponent of the abelian group $\mathrm{Gal}(\widetilde{X}/X(\Gamma)_K)$ is n. For a character $\chi : \mathrm{Gal}(\widetilde{X}/X(\Gamma)_K) \to \mathbb{Z}[\zeta_n]^\times$, the *$\chi$-isotypic part of the open twisted Kuga–Sato variety* $W_{\widetilde{X}}[\chi]$ is the collection of local systems

$$W_{\widetilde{X}}[\chi] := (\{\rho_{\widetilde{X},\chi,\sigma,H}\}_{\sigma:K\hookrightarrow\mathbb{C}}, \{\rho_{\widetilde{X},\chi,p}\}_{(p,Nr)=1}),$$

where

- for a complex embedding $\sigma : K \hookrightarrow \mathbb{C}$, $\rho_{\widetilde{X},\chi,\sigma}$ is a variation of polarized pure $\mathbb{Z}[\zeta_n]$-Hodge structures of weight 1 and rank 2 on $Y(\Gamma)$ (as a Riemann surface) defined as
$$\rho_{\widetilde{X},\chi,H} := \mathscr{H}^1_B(W_{\widetilde{X}} \times_{K,\sigma} \mathbb{C}/Y(\Gamma), \mathbb{Z}[\zeta_n])[\chi],$$
where $\mathscr{H}^1_B(W_{\widetilde{X}} \times_{K,\sigma} \mathbb{C}/Y(\Gamma), \mathbb{Z}[\zeta_n])$ is the relative first Betti cohomology, a $\mathbb{Z}[\zeta_n]$-local system which gives rise to a variation of polarized pure $\mathbb{Z}[\zeta_n]$-Hodge structures of weight 1 and rank $2r$ via the Hodge-de Rham comparison isomorphism, and the χ-isotypic part $[\chi]$ is the χ-isotypic part of the action of $\mathrm{Gal}(\widetilde{X}/X(\Gamma)_K)$ on $\mathscr{H}^1_B(W_{\widetilde{X}} \times_{K,\sigma} \mathbb{C}/Y(\Gamma))$, and
- for $(p, Nr) = 1$, $\rho_{\widetilde{X},\chi,p} : \pi_{1,\text{ét}}(Y(\Gamma)_K, *) \to \mathrm{GL}_2(\mathbb{Z}_p[\zeta_n])$ is an étale $\mathbb{Z}_p[\zeta_n]$-local system of rank 2 over $Y(\Gamma)_K$ defined as
$$\rho_{\widetilde{X},\chi,p} := R^1u_{\text{ét},*}\mathbb{Z}_p[\zeta_n][\chi],$$
where $u : W_{\widetilde{X}} \to Y(\Gamma)_K$, and the χ-isotypic part $[\chi]$ is the $\iota\circ\chi$-isotypic part of the action of $\mathrm{Gal}(\widetilde{X}/X(\Gamma)_K)$ on $R^1u_{\text{ét},*}\mathbb{Z}_p[\zeta_n]$, where $\iota : \mathbb{Z}[\zeta_n] \hookrightarrow \mathbb{Z}_p[\zeta_n]$ is an embedding,

such that the Betti-étale comparison isomorphism holds: namely,

$$\rho_{\widetilde{X},\chi,p}|_{\pi_{1,\text{ét}}(Y(\Gamma)_K\times_{K,\sigma}\mathbb{C},*)} \cong \widehat{\iota \circ \rho_{\widetilde{X},\chi,H}} : \pi_1(\widehat{Y(\Gamma)}, *) \to \mathrm{GL}_2(\mathbb{Z}_p[\zeta_n]).$$

Here, the left hand side is the restriction of the $\mathbb{Z}_p[\zeta_n]$-étale local system $\rho_{\widetilde{X},\chi,p}$ to the geometric fundamental group $\pi_{1,\text{ét}}(Y(\Gamma)_K \times_{K,\sigma} \mathbb{C}, *)$, which is naturally isomorphic to the profinite completion of the topological fundamental group $\pi_1(Y(\Gamma), *)$, and the right hand side is the profinite completion of the topological local system $\iota \circ \rho_{\widetilde{X},\chi,H} : \pi_1(Y(\Gamma), *) \to \mathrm{GL}_2(\mathbb{Z}_p[\zeta_n])$.

Remark 6.8. Even though we have not tried, it is probably not very difficult to define the χ-isotypic part as a Chow motive over K (with an appropriate coefficient, say $\mathbb{Q}(\zeta_r)$) after naturally desingularizing the boundary divisor of $\overline{W}_{\widetilde{X}}$, as in [Sch90].

As these étale local systems come from geometry, the following are also immediate.

Proposition 6.9. *For $(p, Nr) = 1$, the étale local system $\rho_{\widetilde{X},\chi,p}$ extends to an étale local system over an integral model $\mathfrak{Y}$ over $\mathcal{O}_{K,S}$ of $Y(\Gamma)_K$ for a finite set of primes S of K including the primes above p. Furthermore, at every place $\mathfrak{p}$ of K above p, $\rho_{\widetilde{X},\chi,p}|_{\pi_{1,\text{ét}}(Y(\Gamma)_{K_\mathfrak{p}},*)}$ is a de Rham local system in the sense of* [Sch13].

7. Geometric local systems for theta characteristics

We apply the general theory of twisted Kuga–Sato varieties of §6 to the case of theta characteristics. Let ν be a stable theta characteristic of $X(\Gamma)$ that is different from ω, so that $\nu = \omega \otimes L$ for a 2-torsion line bundle $L \in \mathrm{Jac}(X(\Gamma))[2](\overline{\mathbb{Q}})$. Let $K \subset \overline{\mathbb{Q}}$ be a number field over which L can be defined. Let $\alpha : \widetilde{X}_L \to X(\Gamma)_K$ be the étale double cover such that $\alpha_* \mathcal{O}_{\widetilde{X}_L} = \mathcal{O}_{X(\Gamma)_K} \oplus L$, as constructed in Proposition 3.7.

Lemma 7.1. *The étale double cover $\alpha : \widetilde{X}_L \to X(\Gamma)_K$ is Galois.*

Proof. Using the explicit construction

$$\widetilde{X}_L = \mathcal{S}pec_{\mathcal{O}_{X(\Gamma)_K}} \left(\mathcal{O}_{X(\Gamma)_K} \oplus L \right),$$

we can construct a morphism $c : \widetilde{X}_L \to \widetilde{X}_L$ induced by the morphism of $\mathcal{O}_{X(\Gamma)_K}$-algebras,

$$\mathcal{O}_{X(\Gamma)_K} \oplus L \xrightarrow{(x,y)\mapsto(x,-y)} \mathcal{O}_{X(\Gamma)_K} \oplus L.$$

This is a nontrivial element of $\mathrm{Aut}_{X(\Gamma)_K}(\widetilde{X}_L)$, which implies that $\# \mathrm{Aut}_{X(\Gamma)_K}(\widetilde{X}_L) \geq 2$. Since $\deg \alpha = 2$, this implies that $\deg \alpha = \# \mathrm{Aut}_{X(\Gamma)_K}(\widetilde{X}_L) = 2$, so α is a Galois covering. $\square$

Therefore, we can consider the -1-isotypic part of the open twisted Kuga–Sato variety,

$$W_{\widetilde{X}_L}[-1] = (\{\rho_{\widetilde{X}_L,-1,\sigma,H}\}_{\sigma:K\hookrightarrow\mathbb{C}}, \{\rho_{\widetilde{X}_L,-1,p}\}_{(p,2N)=1}),$$

which is a collection of variations of polarized pure $\mathbb{Z}$-Hodge structures and étale $\mathbb{Z}_p$-local systems. For notational simplicity, we will use the notation

$$\rho_{\nu,\sigma} := \rho_{\widetilde{X}_L,-1,\sigma,H}, \quad \rho_{\nu,p} := \rho_{\widetilde{X}_L,-1,p}.$$

We would like to show that this is the "relative motive" that is uniquely attached to the uniformizing logarithmic Higgs bundle (E_ν, θ_ν). This is analogous to the case of $\nu = \omega$, where the associated "relative motive" is just (the various realizations of) the universal elliptic curve $\mathcal{E}$. More precisely, we will show the following.

Theorem 7.2. *Let $E_\nu := \nu \oplus \nu^{-1}$ and $\theta_\nu : E_\nu \to E_\nu \otimes \Omega^1_{X(\Gamma)_K/K}(D_K)$ be the Higgs field induced by the Kodaira–Spencer isomorphism.*

(1) For a complex embedding $\sigma : K \hookrightarrow \mathbb{C}$, $\rho_{\nu,\sigma}$ is the unique, up to the shift of indices, variation of Hodge structures over $Y(\Gamma)$ where the associated graded of its canonical extension is isomorphic to $(E_\nu, \theta_\nu) \times_{K,\sigma} \mathbb{C}$.

(2) For $(p, 2N \,\mathrm{disc}(K/\mathbb{Q})) = 1$ and a prime $v|p$ of K, define $\rho_{\nu,v}$ to be the restriction of $\rho_{\nu,p}$ to $\pi_{1,\text{ét}}(Y(\Gamma)_{K_v})$. Then, $\rho_{\nu,v}$ is the crystalline $\mathbb{Z}_p$-local system associated to a unique filtered convergent F-isocrystal on $Y(\Gamma)_{K_v}$ whose associated graded is isomorhpic to $(E_\nu, \theta_\nu) \times_K K_v$.

In other words, we will show that the uniformizing logarithmic Higgs bundle (E_ν, θ_ν) corresponds to various types of local systems via complex analytic/p-adic nonabelian Hodge correspondence, and that the local systems underlie variations of Hodge structures/F-isocrystals in an essentially unique way.

Remark 7.3. We assume p to be coprime to $\mathrm{disc}(K/\mathbb{Q})$ in Theorem 7.2(2) as we use the formalism of [TT19], which works over an absolutely unramified base field. We believe that this assumption is unnecessary, so that p only needs to be coprime to $2N$, if we use more modern treatment of crystalline local systems, such as [DLMS22, Appendix A].

7.1. **Varitations of Hodge structures attached to** (E_ν, θ_ν).

Proof of Theorem 7.2(1). Let us use the notation $(-)_\sigma$ for the shorthand of $(-)\times_{K,\sigma}\mathbb{C}$. In the proof of Theorem 3.5, we have already seen that $(E_{\nu,\sigma}, \theta_{\nu,\sigma})$ is a stable Higgs bundle. Furthermore, it is clear that $(E_{\nu,\sigma}, \theta_{\nu,\sigma}) \cong (E_{\nu,\sigma}, t\theta_{\nu,\sigma})$ for any $t \in \mathbb{C}^\times$. Therefore, by [Sim92, Lemma 4.1], the local system corresponding to $(E_{\nu,\sigma}, \theta_{\nu,\sigma})$ comes from a complex variation of Hodge structures, which is unique up to the shift of indices.

On the other hand, the variation of polarized pure $\mathbb{Z}$-Hodge structures $\rho_{\nu,\sigma}$ has the underlying vector bundle $\mathscr{H}_{\nu,\sigma} := \rho_{\nu,\sigma} \otimes \mathcal{O}_{Y(\Gamma)}$ isomorphic to the relative de Rham cohomology

$$\mathscr{H}_{\nu,\sigma} \cong \mathscr{H}^1_{\mathrm{dR}}(W_{\widetilde{X},\sigma}/Y(\Gamma))[\chi_{\widetilde{X}}],$$

where $\chi_{\widetilde{X}} : \mathrm{Gal}(\widetilde{X}/X(\Gamma)_K) \to \{\pm 1\} \hookrightarrow \mathbb{Z}^\times$ is the nontrivial character. Note that the local system underlying the variation of Hodge structures $\mathscr{H}^1_B(W_{\widetilde{X},\sigma}/Y(\Gamma), \mathbb{Z})$ is, as the representation of $\pi_1(Y(\Gamma), *)$, isomorphic to $\mathrm{Ind}_{\pi_1(\widetilde{Y},*)}^{\pi_1(Y(\Gamma),*)} \mathrm{Res}_{\pi_1(Y(\Gamma),*)}^{\pi_1(\widetilde{Y},*)} \rho_E$, where ρ_E is the local system underlying the variation of Hodge structures $\mathscr{H}^1_B(\mathcal{E}/Y(\Gamma), \mathbb{Z})$. The fact that $\mathbb{H}$ is the classifying space of pure polarized $\mathbb{Z}$-Hodge structures of weight 1 and rank 2 implies that, as $Y(\Gamma) \cong \mathbb{H}/\Gamma$, ρ_E is isomorphic to the representation $\pi_1(Y(\Gamma), *) \cong \Gamma \hookrightarrow \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z})$.

It is a general fact that, if $H \leq G$ is a finite index subgroup, given a representation ρ of G,

$$\mathrm{Ind}_H^G \mathrm{Res}_G^H \rho \cong \rho \otimes S_{G/H},$$

where $S_{G/H} : G \to \mathrm{GL}_{[G:H]}(\mathbb{Z})$ is the left regular representation. Therefore, the local system underlying the VHS $\mathscr{H}^1_B(W_{\widetilde{X},\sigma}/Y(\Gamma), \mathbb{Z})$ is isomorphic to $\rho \oplus \rho \otimes \chi_{\widetilde{X}}$. Thus, the local system underlying $\rho_{\nu,\sigma}$ is isomorphic to $\rho\otimes\chi_{\widetilde{X}}$. By Proposition 3.7, it follows that the canonical extension of $\rho_{\nu,\sigma}$ has the associated Hodge bundle equal to $(E_{\nu,\sigma}, \theta_{\nu,\sigma})$. This proves Theorem 7.2(1). $\square$

7.2. **Crystalline local systems attached to** (E_ν, θ_ν). We first introduce several concepts regarding the notion of crystalline local systems, following [TT19]. We will then use the notion of periodic Higgs-de Rham flow as in [LSZ19] and [LSYZ19] to prove Theorem 7.2(2). In the following exposition, we will freely use the concepts appearing in [Sch13].

We first recall what it means for an étale $\mathbb{Z}_p$-local system to be crystalline. Let k be a finite field of characteristic p, and let X be a smooth $W(k)[1/p]$-scheme. Let $\mathcal{X}$ be a smooth $W(k)$-scheme whose generic fiber is isomorphic to X. Let $\mathcal{X}_k$ be the mod p fiber of $\mathcal{X}$. There is a crystalline period sheaf $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$, as constructed in [TT19, §2], which is a sheaf on $X_{\text{proét}}$ equipped with a decreasing filtration and a $\mathbb{B}_{\mathrm{cris}}$-linear connection

$$\nabla : \mathcal{O}\mathbb{B}_{\mathrm{cris}} \to \mathcal{O}\mathbb{B}_{\mathrm{cris}} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \Omega^{1,\mathrm{ur}}_{X/W(k)[1/p]},$$

that is integrable and satisfies the Griffiths transversality. Here, $\mathcal{O}_X^{\mathrm{ur}}$ and $\Omega^{1,\mathrm{ur}}_{X/W(k)[1/p]}$ are defined as $w^{-1}\mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p]$ and $w^{-1}\Omega^1_{\mathcal{X}_{\text{ét}}/W(k)}[1/p]$, respecitvely, where $w : \widetilde{X}_{\text{proét}} \to \widetilde{\mathcal{X}}_{\text{ét}}$ is the natural morphism of topoi.

On the other hand, there is a well-accepted notion of *convergent isocrystals* on $\mathcal{X}_k/W(k)$ by Ogus in [Ogu84]. It is classical that various types (e.g. crystalline, convergent, overconvergent) of crystals (isocrystals, respectively) on a characteristic p scheme can be interpreted as coherent sheaves with flat connection on an integral model (the generic fiber of an integral model, respectively), where the connection satisfies certain conditions regarding its “radius of convergence”. A *convergent F-isocrystal* is a convergent isocrystal equipped with a local horizontal isomorphism of itself with its Frobenius pullback (a lift of the Frobenius can be defined locally by Elkik's theorem, and the definition is well-defined by the crystallinity). A *filtered convergent F-isocrystal* is a convergent F-isocrystal equipped with a filtration that satisfies the Griffiths transversality.

Definition 7.4 (Crystalline local systems)**.** An étale $\mathbb{Z}_p$-local system $\mathbb{L}$ on X, regarded as a lisse $\mathbb{Z}_p$-sheaf on $X_{\text{ét}}$, is *crystalline* if there is a filtered convergent F-isocrystal $\mathscr{E}$ on $\mathcal{X}_k/W(k)$ such that there is an isomorphism of $\mathcal{O}\mathbb{B}_{\text{cris}}$-modules

$$w^{-1}\mathscr{E} \otimes_{\mathcal{O}_X^{\text{ur}}} \mathcal{O}\mathbb{B}_{\text{cris}} \cong \nu^{-1}\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\text{cris}},$$

compatible with connection, Frobenius and filtration. Here, $\nu^{-1} : \widetilde{X}_{\text{ét}} \to \widetilde{X}_{\text{proét}}$ is the natural morphism of topoi, and the filtered convergent F-isocrystal $\mathscr{E}$ is regarded as a coherent $\mathcal{O}_{\mathcal{X}}$-module equipped with a flat connection, a horizontal Frobenius structure, and a filtration satisfying the Griffiths transversality. In this case, we say that $\mathbb{L}$ and $\mathscr{E}$ are associated.

There is an equivalent definition resembling the analogous notion for Galois representations. Namely, let

$$\mathbb{D}_{\text{cris}}(\mathbb{L}) := w_*(\nu^{-1}\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\text{cris}}),$$

which is naturally equipped with a flat connection, Frobenius and filtration. Then $\mathbb{L}$ is crystalline if and only if $\mathbb{D}_{\text{cris}}(\mathbb{L})$ is a filtered convergent F-isocrystal, and $\mathbb{L}$ and $\mathbb{D}_{\text{cris}}(\mathbb{L})$ are associated [TT19, Proposition 3.13, Lemma 3.17].

Proof of Theorem 7.2(2). Note that, by [DR73], the universal generalized elliptic curve $\overline{\mathcal{E}}_K \to X(\Gamma)_K$ has a natural model over $\mathcal{O}_{K,v}$, denoted as $\overline{f} : \overline{\mathcal{E}}_{\mathcal{O}_{K,v}} \to X(\Gamma)_{\mathcal{O}_{K,v}}$, which is smooth over $Y(\Gamma)_{\mathcal{O}_{K,v}}$. Furthermore, if we denote k_v by the residue field of v, then $X(\Gamma)_{k_v}$ is a smooth curve.

The same definition of ω,

$$\omega := f_*\Omega^1_{\mathcal{E}/Y(\Gamma)},$$

gives rise to a theta characteristic on $Y(\Gamma)_{\mathcal{O}_{K,v}}$, as the Kodaira–Spencer isomorphism holds on the integral level ([Kat73, A.1.3.17]). There is the unique *canonical extension* on the integral level,

$$\omega^{\text{can}} = \overline{f}_*\Omega^1_{\overline{\mathcal{E}}_{\mathcal{O}_{K,v}}/X(\Gamma)_{\mathcal{O}_{K,v}}}(\log \infty_f) := \overline{f}_*\left(\Omega^1_{\overline{\mathcal{E}}_{\mathcal{O}_{K,v}}/\mathcal{O}_{K,v}}(\overline{f}^{-1}(D))/\overline{f}^*(\Omega^1_{X(\Gamma)_{\mathcal{O}_{K,v}}/\mathcal{O}_{K,v}}(D))\right),$$

which satisfies the Kodaira–Spencer isomorphism[4]

$$(\omega^{\text{can}})^{\otimes 2} \xrightarrow{\sim} \Omega^1_{X(\Gamma)/\mathcal{O}_{K,v}}(D).$$

Moreover, the canonical extension ω^{can} arises as the first Hodge filtration of the log-de Rham cohomology bundle $R^1\overline{f}_{\log \text{dR},*}(\overline{\mathcal{E}}_{\mathcal{O}_{K,v}}/X(\Gamma)_{\mathcal{O}_{K,v}})$, where the log-structures for $\overline{\mathcal{E}}_{\mathcal{O}_{K,v}}$ and $X(\Gamma)_{\mathcal{O}_{K,v}}$ are given by $\overline{f}^{-1}(D)$ and D, respectively. As before, we will omit the superscript $^{\text{can}}$ most of the time as there is no source of confusion regarding this.

Over $\mathcal{O}_{K,v}$, we can formulate a logarithmic Higgs sheaf $(E_\omega, \theta_\omega)$ over $X(\Gamma)_{\mathcal{O}_{K,v}}$ by $E_\omega := \omega \oplus \omega^{-1}$ and $\theta_\omega : E_\omega \to E_\omega \otimes \Omega^1_{X(\Gamma)_{\mathcal{O}_{K,v}}/\mathcal{O}_{K,v}}(D)$ defined as

$$\theta_\omega : E_\omega \twoheadrightarrow \omega \xrightarrow{\text{Kodaira–Spencer}} \omega^{-1} \otimes \Omega^1_{X(\Gamma)_{\mathcal{O}_{K,v}}/\mathcal{O}_{K,v}}(D) \hookrightarrow E_\omega \otimes \Omega^1_{X(\Gamma)_{\mathcal{O}_{K,v}/\mathcal{O}_{K,v}}}(D).$$

[4] We were unable to locate a literature that states the log version of the Kodaira–Spencer isomorphism on the integral modular curve. A much more general version of the log Kodaira–Spencer isomorphism on the integral level is proved in [Lan12, Proposition 6.9], which contains the statements that we would like for the modular curves.

Since $\mathrm{Jac}(X(\Gamma)_{\mathcal{O}_{K,v}})[2]$ is a finite flat group scheme over $\mathcal{O}_{K,v}$, $L \in \mathrm{Jac}(X(\Gamma)_{\mathcal{O}_{K,v}})[2](K_v)$, the base-change of $L \in \mathrm{Jac}(X(\Gamma))[2](K)$ to K_v, extends uniquely to $\mathcal{L} \in \mathrm{Jac}(X(\Gamma)_{\mathcal{O}_{K,v}})[2](\mathcal{O}_{K,v})$. By applying the same construction, we obtain the Galois cover $\alpha : \widetilde{X}_{\mathcal{O}_{K,v}} \to X(\Gamma)_{\mathcal{O}_{K,v}}$ of degree 2. Accordingly, we obtain $u : \overline{W}_{\widetilde{X}_{\mathcal{O}_{K,v}}} \to X(\Gamma)_{\mathcal{O}_{K,v}}$. Using this integral model, one can also extend a logarithmic Higgs sheaf $(E_{\nu,v}, \theta_{\nu,v})$ to $X(\Gamma)_{\mathcal{O}_{K,v}}$ by $(E_{\nu,v}, \theta_{\nu,v}) := (E_\omega, \theta_\omega) \otimes (\mathcal{L}, 0)$.

Since $u : W_{\widetilde{X}_{\mathcal{O}_{K,v}}} \to Y(\Gamma)_{\mathcal{O}_{K,v}}$ is smooth and proper, [TT19, Proposition 5.4] implies that the relative crystalline cohomology $\mathscr{E} := R^1 u_{\mathrm{cris},*}\mathcal{O}$ gives rise to a convergent F-isocrystal on $Y(\Gamma)_{k_v}$ (which is in fact overconvergent). From the relative crystalline comparison theorem, [TT19, Theorem 5.5], it follows that $\rho_{\nu,v}$ is a crystalline $\mathbb{Z}_p$-local system, and is associated to $\mathscr{E}$. The fact that the associated graded of $\mathscr{E}$ is $(E_{\nu,v}, \theta_{\nu,v})$ follows from the de Rham-crystalline comparison and from the fact that ω^{can} is the first Hodge filtration of $R^1\overline{f}_{\log \mathrm{dR},*}(\mathscr{E}_{\mathcal{O}_{K,v}}/X(\Gamma)_{\mathcal{O}_{K,v}})$. □

Remark 7.5. Recall that Proposition 3.7 proved that the topological local systems underlying $\rho_{\nu,\sigma}$ is the local system corresponding to the Higgs sheaf $(E_{\nu,\sigma}, \theta_{\nu,\sigma})$ via the nonabelian Hodge correspondence. Under a reasonable theory of p-adic nonabelian Hodge correspondence (e.g. [LSZ19], [LSYZ19]), one could also prove that $\rho_{\nu,v}$ is the étale local system corresponding to the Higgs sheaf $(E_{\nu,v}, \theta_{\nu,v})$ via the p-adic nonabelian Hodge correspondence.

8. Twisted period map to a Siegel modular threefold

We retain the notation of the previous section. For this section only, we assume for simplicity[5] that Γ is either $\Gamma_1(N)$ or $\Gamma(N)$. Recall that $W_{\widetilde{X}} \to Y(\Gamma)_K$ is a family of abelian surfaces that is different from the square of the universal elliptic curve, but they become isomorphic after an étale base-change to $\widetilde{Y}$:

$$W_{\widetilde{X}} \ncong \mathcal{E}_K \times_{Y(\Gamma)_K} \mathcal{E}_K, \quad \alpha^* W_{\widetilde{X}} \cong \alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K.$$

In this case, it turns out that, from the universal level structure on $\mathcal{E}/Y(\Gamma)$, one can construct a certain natural level structure on $W_{\widetilde{X}}/Y(\Gamma)_K$, which is a twisted version of the natural level structure on $\mathcal{E}_K \times_{Y(\Gamma)_K} \mathcal{E}_K$. This implies that $Y(\Gamma)_K$ admits a twisted period map into the moduli space of abelian surfaces with a level structure that is different from the usual "diagonal embedding." We see this as a hint of possibly automorphic interpretations of the sections of a noncongruence theta characteristic ν, which are a priori noncongruence modular forms.

To construct the twisted level structure, we first describe $W_{\widetilde{X}}$ as a variety over $\widetilde{Y}$ with a descent datum.

Proposition 8.1. *Let $\sigma : \widetilde{Y} \to \widetilde{Y}$ be the nontrivial element of $\mathrm{Gal}(\widetilde{Y}/Y(\Gamma)_K)$. Let $\lambda : \widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y} \xrightarrow{\sim} \widetilde{Y} \coprod \widetilde{Y}$ be an isomorphism of $\widetilde{Y}$-schemes such that the following diagram commutes.*

$$\begin{tikzcd}
\widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y} \arrow[rr, "\lambda"] \arrow[dr, "\mathrm{pr}_2"'] & & \widetilde{Y} \coprod \widetilde{Y} \arrow[dl, "\mathrm{id} \coprod \sigma"] \\
& \widetilde{Y} &
\end{tikzcd}$$

[5]This is merely for the simplicity of the moduli problem that the corresponding level structure represents.

Let $\lambda : W_{\widetilde{X}} \times_{Y(\Gamma)_K} \widetilde{Y} \xrightarrow{\sim} \alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K$ be the natural isomorphism obtained from $\lambda : \widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y} \xrightarrow{\sim} \widetilde{Y} \coprod \widetilde{Y}$. Then, the following diagram commutes.

$$\begin{tikzcd}
W_{\widetilde{X}} \times_{Y(\Gamma)_K} \widetilde{Y} \arrow[r, "(\mathrm{id},\sigma)"] \arrow[d, "\lambda"'] & W_{\widetilde{X}} \times_{Y(\Gamma)_K} \widetilde{Y} \arrow[d, "\lambda"] \\
\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K \arrow[r, "(x,y)\mapsto(\sigma(y),\sigma(x))"'] & \alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K
\end{tikzcd}$$

Proof. Note that

$$\widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y} \xrightarrow{(\mathrm{id},\sigma)} \widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y},$$

after conjugating by λ, is identified with

$$\widetilde{Y} \coprod \widetilde{Y} \xrightarrow{x\coprod y \mapsto y \coprod x} \widetilde{Y} \coprod \widetilde{Y}.$$

By [CGP15, Proposition A.5.2], the Weil restriction of schemes has a natural isomorphism

$$\mathrm{R}_{S'/S}(X') \times_S T \cong \mathrm{R}_{T'/T}(X' \times_{S'} T'),$$

where S' is a finite locally free S-scheme, X' is an S'-scheme and $T' = S' \times_S T$. Therefore, the isomorphism

$$W_{\widetilde{X}} \times_{Y(\Gamma)_K} \widetilde{Y} = \left(\mathrm{R}_{\widetilde{Y}/Y(\Gamma)_K}(\alpha^*\mathcal{E}_K)\right) \times_{Y(\Gamma)_K} \widetilde{Y} \cong \mathrm{R}_{\widetilde{Y}\times_{Y(\Gamma)_K}\widetilde{Y}/\widetilde{Y}}(\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} (\widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y})),$$

is natural, where in the rightmost expression, the morphism $\widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y} \to \widetilde{Y}$ used in the subscript is the second projection, while the morphism $\widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y} \to \widetilde{Y}$ used in the expression in the parenthesis is the first projection. Thus, after conjugating by $\lambda : \widetilde{Y} \times_{Y(\Gamma)_K} \widetilde{Y} \xrightarrow{\sim} \widetilde{Y} \coprod \widetilde{Y}$, this is identified with

$$\mathrm{R}_{\widetilde{Y}\coprod\widetilde{Y}/\widetilde{Y}}(\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} (\widetilde{Y} \coprod \widetilde{Y})),$$

where $\mathrm{id} \coprod \sigma : \widetilde{Y} \coprod \widetilde{Y} \to \widetilde{Y}$ ($\mathrm{id} \coprod \mathrm{id} : \widetilde{Y} \coprod \widetilde{Y} \to \widetilde{Y}$, respectively) is used in the subscript (the expression in the parenthesis, repsectively). Therefore, under this identification, the morphism

$$(\mathrm{id}, \sigma) : W_{\widetilde{X}} \times_{Y(\Gamma)_K} \widetilde{Y} \to W_{\widetilde{X}} \times_{Y(\Gamma)_K} \widetilde{Y},$$

is identified with the morphism

$$\mathrm{R}_{\widetilde{Y}\coprod\widetilde{Y}/\widetilde{Y}}(\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} (\widetilde{Y} \coprod \widetilde{Y})) \to \mathrm{R}_{\widetilde{Y}\coprod\widetilde{Y}/\widetilde{Y}}(\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} (\widetilde{Y} \coprod \widetilde{Y})),$$

where the subscripts are related by the diagram

$$\begin{tikzcd}
\widetilde{Y} \coprod \widetilde{Y} \arrow[r, "x\coprod y\mapsto y\coprod x"] \arrow[d, "\mathrm{id}\coprod\sigma"'] & \widetilde{Y} \coprod \widetilde{Y} \arrow[d, "\mathrm{id}\coprod\sigma"] \\
\widetilde{Y} \arrow[r, "\sigma"'] & \widetilde{Y}
\end{tikzcd}$$

and the expressions in the parentheses are related by the diagram

$$\begin{tikzcd}
\widetilde{Y} \coprod \widetilde{Y} \arrow[rr, "x \coprod y \mapsto y \coprod x"] \arrow[dr, "\mathrm{id} \coprod \mathrm{id}"'] & & \widetilde{Y} \coprod \widetilde{Y} \arrow[dl, "\mathrm{id} \coprod \mathrm{id}"] \\
& \widetilde{Y} &
\end{tikzcd}$$

From this, the statement easily follows. □

We consider the Γ-level structure on an elliptic scheme E/S. In the case of $\Gamma = \Gamma(N)$, it is a pair of sections $P_1, P_2 : S \to E$ that fiberwise generates $E[N]$, and in the case of $\Gamma = \Gamma_1(N)$, it is a section $P : S \to E[N]$ that has exact order N. We take the Γ-level structure on the universal elliptic curve $\mathcal{E}_K/Y(\Gamma)_K$ as either $\mathcal{P}, \mathcal{Q} : Y(\Gamma)_K \to \mathcal{E}_K[N]$ (in the case of $\Gamma(N)$) or $\mathcal{P} : Y(\Gamma)_K \to \mathcal{E}_K[N]$ (in the case of $\Gamma_1(N)$). Using the level structure on $\mathcal{E}_K$, we may define a twisted level structure on $W_{\widetilde{X}}$ as follows.

Definition 8.2 ($\Gamma(N)^+$- and $\Gamma_1(N)^+$-structures on an abelian surface)**.** For a principally polarized abelian surface $(A/S, \lambda)$, a (naive) $\Gamma(N)^+$-structure is a collection of étale-local sections P_1, P_2, P_3, P_4 of $A[N]$ such that they generate $A[N]$ fiberwise, and two such collections

$$P_1, P_2, P_3, P_4, \quad P_1', P_2', P_3', P_4',$$

are equivalent if $\{P_1, P_2\} = \{P_1', P_2'\}$ and $\{P_3, P_4\} = \{P_3', P_4'\}$ (as unordered sets).

A (naive) $\Gamma_1(N)^+$-structure is a collection of étale-local sections P_1, P_2 of $A[N]$ such that they generate a totally isotropic subspace of $A[N]$, with respect to the Weil pairing induced by λ, and two such collections

$$P_1, P_2, \quad P_1', P_2',$$

are equivalent if $\{P_1, P_2\} = \{P_1', P_2'\}$ (as unordered sets).

Remark 8.3. The moduli space of principally polarized abelian surfaces with $\Gamma(N)^+$- or $\Gamma_1(N)^+$-structures are identified with the arithmetic quotient of the Siegel upper half space by a subgroup of $\mathrm{Sp}_4(\mathbb{Z})$. More precisely, if the symplectic form corresponds to the matrix $\left(\begin{smallmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{smallmatrix}\right)$, then

$$\Gamma(N)^+ = \Gamma(N) \cdot \left\langle \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle, \qquad \Gamma_1(N)^+ = \Gamma_1(N) \cdot \left\langle \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle,$$

where $\Gamma(N)$ and $\Gamma_1(N)$ are the standard congruence subgroups of $\mathrm{Sp}_4(\mathbb{Z})$,

$$\Gamma(N) = \{M \in \mathrm{Sp}_4(\mathbb{Z}) \mid M \equiv I_4 \pmod{N}\},$$
$$\Gamma_1(N) = \{M \in \mathrm{Sp}_4(\mathbb{Z}) \mid M \pmod{N} \text{ is upper triangular unipotent}\}.$$

Definition 8.4 (Twisted level structure on $W_{\widetilde{X}}$)**.** Let Γ^+ be $\Gamma(N)^+$ ($\Gamma_1(N)^+$, respectively) if $\Gamma = \Gamma(N)$ ($\Gamma = \Gamma_1(N)$, respectively). We define the twisted level structure, a Γ^+-structure on $\alpha^* W_{\widetilde{X}} \cong \alpha^* \mathcal{E}_K \times_{\widetilde{Y}} \alpha^* \mathcal{E}_K$, as follows.

- If $\Gamma = \Gamma(N)$, we consider the sections $\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2, \widetilde{\mathcal{Q}}_1, \widetilde{\mathcal{Q}}_2 : \widetilde{Y} \to (\alpha^* \mathcal{E}_K \times_{\widetilde{Y}} \alpha^* \mathcal{E}_K)[N]$, where

  $$\widetilde{\mathcal{P}}_1 := (\alpha^* \mathcal{P}, \alpha^* e), \quad \widetilde{\mathcal{P}}_2 := (\alpha^* e, \alpha^* \mathcal{P}), \quad \widetilde{\mathcal{Q}}_1 := (\alpha^* \mathcal{Q}, \alpha^* e), \quad \widetilde{\mathcal{Q}}_2 := (\alpha^* e, \alpha^* \mathcal{Q}).$$

  The above étale-local sections define a $\Gamma(N)^+$-structure on $\alpha^* W_{\widetilde{X}}$.

- If $\Gamma = \Gamma_1(N)$, we consider the sections $\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2 : \widetilde{Y} \to (\alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K)[N]$, where
$$\widetilde{\mathcal{P}}_1 := (\alpha^*\mathcal{P}, \alpha^* e), \quad \widetilde{\mathcal{P}}_2 := (\alpha^* e, \alpha^*\mathcal{P}).$$
The above étale-local sections define a $\Gamma_1(N)^+$-structure on $\alpha^* W_{\widetilde{X}}$.

Lemma 8.5. *The twisted level structure on $\alpha^* W_{\widetilde{X}}$, as a Γ^+-level structure, descends into a twisted level structure, again as a Γ^+-level structure, on $W_{\widetilde{X}}$. Namely, the twisted level structure on $\alpha^* W_{\widetilde{X}}$ is invariant under the automorphism induced by $\sigma : \widetilde{Y} \to \widetilde{Y}$.*

Proof. We already know what descent datum $\alpha^* W_{\widetilde{X}} \cong \alpha^*\mathcal{E}_K \times_{\widetilde{Y}} \alpha^*\mathcal{E}_K$ has, thanks to Proposition 8.1. We only need to check that that the Γ^+-level structure is compatible with the descent datum, which is clear as the level structure is indifferent to the switch between $\widetilde{\mathcal{P}}_1$ and $\widetilde{\mathcal{P}}_2$ (and also the switch between $\widetilde{\mathcal{Q}}_1$ and $\widetilde{\mathcal{Q}}_2$, if $\Gamma = \Gamma(N)$). ☐

Remark 8.6. The twisted period map $\pi_{\widetilde{X}} : Y(\Gamma)_K \to \mathcal{A}_{2,\Gamma^+}$ is different from the usual "diagonal" period map $\pi_{\mathrm{diag}} : Y(\Gamma)_K \to \mathcal{A}_{2,\Gamma^+}$, given by the diagonal morphism between the moduli functors, $E \mapsto E^2$. This is simply because the pullbacks of the universal abelian surface over $\mathcal{A}_{2,\Gamma^+}$ by the two period maps are different.

It may first look strange to have a classifying map into a congruence quotient of a Shimura variety even though the starting object is "noncongruence". This phenomenon happens because the double cover of congruence quotients $\mathcal{A}_{2,\Gamma} \to \mathcal{A}_{2,\Gamma^+}$ of a larger group somehow "absorbs" the double cover $\widetilde{Y} \to Y(\Gamma)_K$. To be more precise, for the diagonal period map, there is a map $Y(\Gamma)_K \to \mathcal{A}_{2,\Gamma}$ that fills in the diagram

$$\begin{tikzcd}
Y(\Gamma)_K \coprod Y(\Gamma)_K \arrow[r] \arrow[d] & \mathcal{A}_{2,\Gamma} \arrow[d] \\
Y(\Gamma)_K \arrow[r] \arrow[ur] & \mathcal{A}_{2,\Gamma^+}
\end{tikzcd}$$

On the other hand, for the twisted period map $\pi_{\widetilde{X}}$, the diagonal arrow cannot be filled:

$$\begin{tikzcd}
\widetilde{Y} \arrow[r] \arrow[d] & \mathcal{A}_{2,\Gamma} \arrow[d] \\
Y(\Gamma)_K \arrow[r] \arrow[ur, dashed, "\times\times\times" description] & \mathcal{A}_{2,\Gamma^+}
\end{tikzcd}$$

It is interesting to note that we had to use the stacky double cover $\mathcal{A}_{2,\Gamma} \to \mathcal{A}_{2,\Gamma^+}$, which seems necessary.

Appendix A. Examples of cusp forms in $H^0(X, \nu(-D))$

In this subsection, we specialize to the case of $\Gamma = \Gamma_1(N)$, where $N = p_1 \cdots p_r$ is a square-free odd integer. In this case, there are $n = 2^{r-1}(p_1 - 1)\cdots(p_r - 1)$ cusps, and for each decomposition $N = M_1 M_2$, there are $\frac{n}{2^r}$ cusps whose widths are M_1. Let us index the cusps into
$$c_{M_1,a,b} := \frac{b}{M_2 a},$$

where $1 \leq a < M_1$, $(a, M_1) = 1$, and $(b, M_2a) = 1$. Note that $c_{M_1,a,b}$ is of width M_1 and $c_{1,1,1} = \infty$. The two cusps $c_{M_1,a,b}, c_{M_1',a',b'}$ are equivalent if and only if $M_1 = M_1'$, and there exists $\epsilon \in \{\pm 1\}$ such that $a \equiv \epsilon a' \pmod{M_1}$ and $b \equiv \epsilon b' \pmod{M_2}$, and these subsume the cusps of Γ.

Definition A.1. Let $\sigma_{M_1,a,b} \in \mathrm{SL}_2(\mathbb{Z})$ be a matrix such that

$$\sigma_{M_1,a,b}^{-1} Z_{c_{M_1,a,b}}(\Gamma)\sigma_{M_1,a,b} \subset \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix},$$

where $Z_{c_{M_1,a,b}}(\Gamma) \subset \Gamma$ is the stabilizer of $c_{M_1,a,b}$ in Γ.

By the definition of width, $\sigma_{M_1,a,b}^{-1} Z_{c_{M_1,a,b}}(\Gamma)\sigma_{M_1,a,b} = \left(\begin{smallmatrix} 1 & M_1\mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)$.

Definition A.2. For a theta characteristic ν and a cusp form $f \in S_k(\Gamma_\nu)$, let

$$e_{M_1,a,b}(f)(q) = \sum_{j=1}^{\infty} a_{M_1,a,b,j}(f) q^{j/M_1} \in \mathbb{C}[[q^{1/M_1}]],$$

be the Fourier expansion of $f_{M_1,a,b}(z) := f(\sigma_{M_1,a,b}(z))$. We also define

$$\mathrm{ord}_{M_1,a,b}(f) := \min\{j \mid a_{M_1,a,b,j} \neq 0\}.$$

Lemma A.3. *The modular discriminant Δ, when seen as an element of $S_{12}(\Gamma) = S_{12}(\Gamma_\nu)$, has*

$$\mathrm{ord}_{M_1,a,b}(\Delta) = M_1.$$

Furthermore, Δ does not vanish on $Y(\Gamma)$.

Proof. The first statement follows from that Δ is of level 1 and $\sigma_{M_1,a,b} \in \mathrm{SL}_2(\mathbb{Z})$. The second statement is also well-known. $\square$

Corollary A.4. *Consider the multiplication-by-Δ map $\times\Delta : S_1(\Gamma_\nu) \hookrightarrow S_{13}(\Gamma_\nu)$. Then,*

$$\begin{aligned} \mathrm{im}(\times\Delta) &= \{f \in S_{13}(\Gamma_\nu) \mid \mathrm{ord}_{M_1,a,b}(f) \geq M_1 + 1 \textit{ for all cusps } c_{M_1,a,b}\} \\ &= \bigcap_{c_{M_1,a,b} \textit{ cusp},\ 1 \leq j \leq M_1} \ker a_{M_1,a,b,j}. \end{aligned}$$

Thus, by knowing all the Fourier expansions of weight 13 modular forms, one can compute $\dim_{\mathbb{C}} S_1(\Gamma_\nu)$. This is beneficial, since $\dim_{\mathbb{C}} S_{13}(\Gamma_\nu) = \dim_{\mathbb{C}} S_{13}(\Gamma)$ by Riemann–Roch.

Lemma A.5. *Let $f \in S_k(N, \chi)$. Then, $\mathrm{ord}_{M_1,a,b}(f)$ only depends on M_1. In partcular,*

$$\bigcap_{c_{M_1,a,b} \textit{ cusp},\ 1 \leq j \leq M_1} K^{\omega}_{M_1,a,b,j} = \bigcap_{N = M_1M_2,\ 1 \leq j \leq M_1} K^{\omega}_{M_1,1,1,j}.$$

Proof. This follows from that every width M_1 cusp of $X_1(N)$ is an orbit under the diamond action, and that f has nebentype. $\square$

This already gives rise to many coincidences, which are not enjoyed by most other Γ_ν's with $\nu \neq \omega$. On the other hand, there are a few Γ_ν's which have the similar property for $\nu \neq \omega$, especially when ν is the pullback of a logarithmic theta characteristic of $X_0(N)$.

We show that there are a few more coincidences, using the results of [Asa76], which showed how to compute the Fourier expansion at a cusp by using the Fourier expansion at ∞.

Proposition A.6 ([Asa76, Theorem 2]). *Let $f \in S_k^{\text{new}}(N,\chi)$ be a Hecke eigenform such that, at $c_{1,1,1} = \infty$, f has the q-expansion*

$$e_{1,1,1}(f)(q) = \sum_{n=1}^{\infty} a_n q^n, \qquad a_1 = 1.$$

Let $\chi = \chi_{p_1} \cdots \chi_{p_r}$ be the product such that χ_{p_i} is a Dirichlet character mod p_i (may or may not be primitive). Then, for all cusps $c_{M_1,a,b}$,

$$e_{M_1,a,b}(f)(q) = \chi\left(bcM_1 + M_2^2 ad\right) \prod_{p|M_1} \left(p^{-\frac{k}{2}} \chi_p\left(\frac{M_1}{p}\right) \overline{a_p} C(\chi_p) \right) \sum_{n=1}^{\infty} a_n^{(M_1)} q^{n/M_1},$$

where $c, d \in \mathbb{Z}$ such that $cM_1 + dM_2 = 1$,

$$C(\chi_p) = \begin{cases} \sum_{1 \le h < p} \chi_p(h) e^{2\pi i h/p} & \text{if } \chi_p \text{ is primitive} \\ -q & \text{if } \chi_p \text{ is trivial,} \end{cases}$$

and $a_n^{(M_1)}$ is

$$a_n^{(M_1)} = \begin{cases} \overline{\chi(dnM_2 + cM_1)} a_n & \text{if } (n, M_1) = 1 \\ \chi(cnM_1 + dM_2) \overline{a}_n & \text{if } (n, M_2) = 1 \\ a_x^{(M_1)} a_y^{(M_1)} & \text{if } n = xy, \, (x,y) = 1. \end{cases}$$

This can be packaged more simply as follows.

Corollary A.7. *Fix χ and a cusp $c_{M_1,a,b} \in X_1(N)$ of width M_1. Then, there exist a constant $\lambda \in \mathbb{C}$ that depends only on M_1, a, b, and, for each $n \ge 1$, a constant $\epsilon_n \in \mathbb{C}$ that depends only on M_1, a, b, n, such that, for a normalized Hecke eigenform $f \in S_k^{\text{new}}(N,\chi)$ that has the q-expansion $e_{1,1,1}(f)(q) = \sum_{n=1}^{\infty} a_n q^n$ at ∞,*

$$a_{M_1,a,b,n}(f) = \begin{cases} \left(\lambda \epsilon_n \prod_{p|M_1} \overline{a_p}\right) a_n & \text{if } (n, M_1) = 1 \\ \left(\lambda \epsilon_n \prod_{p|M_1} \overline{a_p}\right) \overline{a_n} & \text{if } (n, M_2) = 1 \\ \frac{a_{M_1,a,b,x}(f) a_{M_1,a,b,y}(f)}{\lambda \epsilon_1 \prod_{p|M_1} \overline{a_p}} & \text{if } n = xy, \, (x,y) = 1. \end{cases}$$

We also record the oldform analogue of the above result.

Proposition A.8. *Let $f \in S_k(N,\chi)$ be of the form $f(z) = \widetilde{f}(Bz)$ for a Hecke eigenform $\widetilde{f} \in S_k^{\text{new}}(A,\chi)$ with $A|N$ and $B|\frac{N}{A}$. Suppose that $\widetilde{f}$ has the q-expansion*

$$e_{1,1,1}(\widetilde{f})(q) = \sum_{n=1}^{\infty} a_n q^n,$$

at $\infty \in X_1(A)$. Let $\chi = \chi_{p_1} \cdots \chi_{p_s}$ be the product such that χ_{p_i} is a Dirichlet character mod p_i (may or may not be primitive). Then, for all cusps $c_{M_1,a,b}$ of $X_1(N)$,

$$e_{M_1,a,b}(f)(q) = e_{M_1',a',b'}(\widetilde{f})(q^B),$$

where

$$c_{M_1',a',b'} \sim (M_1, B) c_{M_1,a,b},$$

as the cusps of $X_1(A)$. More explcitly,

$$M_1' = \frac{A}{(M_2, A)}, \quad a' \equiv \frac{M_2}{(M_2, A)} a \pmod{M_1'}, \quad b' \equiv (M_1, B) b \left(\operatorname{mod} \frac{A}{M_1'}\right).$$

Proof. Note that $e_{M_1,a,b}(f)(q)$ is the Fourier expansion of

$$f\left(\sigma_{M_1,a,b}(z)\right) = \widetilde{f}\left(\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \sigma_{M_1,a,b}(z)\right).$$

It is easy to see that

$$\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \sigma_{M_1,a,b} \begin{pmatrix} B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \sigma_{M_1,a,(B,M_1)b}.$$

The statement follows. $\square$

Similarly to Corollary A.7, we can package the above result as follows.

Corollary A.9. *Fix χ, a cusp $c_{M_1,a,b} \in X_1(N)$ of width M_1, $A|N$, and $B|\frac{N}{A}$. Then, there exist a constant $\lambda \in \mathbb{C}$ that depends only on M_1, a, b, A, B, and, for each $n \geq 1$, a constant $\epsilon_n \in \mathbb{C}$ that depends only on M_1, a, b, A, B, n, such that, for $f(z) = \widetilde{f}(Bz)$ for a normalized Hecke eigenform $\widetilde{f} \in S_k^{\mathrm{new}}(A, \chi)$, with the q-expansion $e_{1,1,1}(f)(q) = \sum_{n \geq 1,\ B|n} a_n q^n$ at ∞,*

$$a_{M_1,a,b,n}(f) = \begin{cases} 0 & \text{if } B \not| \, n \\ \left(\lambda \epsilon_{\frac{n}{B}} \prod_{p|M_1'} \overline{a_{pB}}\right) a_n & \text{if } B|n,\ (n, M_1') = 1 \\ \left(\lambda \epsilon_{\frac{n}{B}} \prod_{p|M_1'} \overline{a_{pB}}\right) \overline{a_n} & \text{if } B|n,\ (n, M_2') = 1 \\ \frac{a_{M_1,a,b,Bx}(f) a_{M_1,a,b,By}(f)}{\lambda \epsilon_1 \prod_{p|M_1'} \overline{a_{pB}}} & \text{if } B|n,\ n = Bxy,\ (x,y) = 1, \end{cases}$$

where $M_1' = \frac{A}{(M_2,A)}$ and $M_2' = \frac{A}{M_1'}$.

Namely, if there is a section f to $\nu(-D)$ for a logarithmic theta characteristic ν, then f^2 will give rise to a weight 2 cusp form of level $\Gamma_1(N)$ that vanishes at each cusp of $X_1(N)$ up to order ≥ 2. We first prove the following useful lemma.

Lemma A.10. *Let $N \geq 5$ be a square-free integer, and let $f \in S_2(\Gamma_1(N))$ be a cusp form such that it vanishes up to order ≥ 2 at each cusp of $X_1(N)$. Let $f = \sum_\chi f_\chi$ be the decomposition according to $S_2(\Gamma_1(N)) = \bigoplus_\chi S_2(N, \chi)$, where χ runs over all mod N Dirichlet characters. Then, each f_χ vanishes up to order ≥ 2 at each cusp of $X_1(N)$.*

Proof. Let $N = p_1 \cdots p_r$. Then, there are $2^{r-1}(p_1 - 1) \cdots (p_r - 1)$ cusps of $X_1(N)$, and for each $M|N$, there are $\frac{(p_1-1)\cdots(p_r-1)}{2}$ cusps of width M. The diamond operators $\langle d \rangle$, for $1 \leq d \leq \lfloor \frac{N}{2} \rfloor$, $(d, N) = 1$, act faithfully and transitively on the cusps of the same width. For each $M|N$, choose a cusp $C_{M,1}$ of width M, and let $C_{M,d}$ be where the $C_{M,1}$ is sent via the action of $\langle d \rangle$.

Let f_χ have the q-expansion at $C_{M,1}$ given by $e_M(f_\chi)$. Then, the vanishing at $C_{M,1}$ implies that the q-expansion $\sum_\chi e_M(f_\chi)$ vanishes up to order ≥ 2. For $1 \leq d \leq \lfloor \frac{N}{2} \rfloor$, $(d, N) = 1$, since f_χ has the q-expansion at $C_{M,d}$ given by $\chi(d) e_M(f_\chi)$, the vanishing at $C_{M,d}$ implies that the q-expasnion $\sum_\chi \chi(d) e_M(f_\chi)$ vanishes up to order ≥ 2. Since $\chi(-1) = 1$ for all χ, it follows that $\sum_\chi \chi(d) e_M(f_\chi)$ vanishes up to order ≥ 2 for all $(d, N) = 1$. Since $\widehat{(\mathbb{Z}/N\mathbb{Z})}^\times$ is (noncanonically) isomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times$, it follows that each $e_M(f_\chi)$ vanishes up to order ≥ 2 for all χ. Combining this for all M, we obtain the desired result. $\square$

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