Hermitian Symmetric Domains

$$\mathcal{H} \quad z \longmapsto \frac{z-i}{z+i} \quad \mathcal{D}$$

$$\frac{z+1}{i(z-1)} \longleftarrow z$$

M smooth manifold, $p \in M$, $\mathcal{O}_{M,p}$. $T_p(M) = \{\mathbb{R}$ derivations $X_p : \mathcal{O}_{M,p} \to \mathbb{R}\}$

$U \subset M$ open, X smooth vector field on U if $X = (X_p)$, $X_p \in T_pM$ s.t. $\forall f \in \mathcal{O}_M(U)$, $p \longmapsto X_p f_p$ is smooth. A smooth r-tensor field on U is a family $t = (t_p)$, $t_p : (T_pM)^r \to \mathbb{R}$ multilinear s.t. $\forall$ smooth vector fields $X_1, \dots, X_r$ on U, $p \longmapsto t_p(X_{1p}, \dots, X_{rp})$ is smooth.

A Riemannian manifold is a smooth manifold M with a smooth 2-tensor field g s.t. $\forall p \in M$, $g_p : T_pM \times T_pM \to \mathbb{R}$ is symmetric and positive def. Locally for coordinates $(x^1, \dots, x^n)$ around p, $g_p = \sum g_{ij}(p)\, dx^i \otimes dx^j$ where $g_{ij}(p) = g_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$.

An almost-complex structure on a smooth manifold M is a smooth tensor field $J = (J_p)$, $J_p : T_pM \to T_pM$, $J_p^2 = -1$. Integrable almost-complex structure arises from complex structure. (locally $x^1, \dots, x^n, y^1, \dots, y^n$,

$J \frac{\partial}{\partial x^r} = \frac{\partial}{\partial y^r}$, $J \frac{\partial}{\partial y^r} = -\frac{\partial}{\partial x^r}$)

A Hermitian metric on a complex manifold M is a Riemannian metric g s.t. $g(JX, JY) = g(X, Y)$. (M, g) is called a Hermitian manifold.

A manifold is called homogeneous if its automorphism group acts transitively, i.e. $\forall\, p, q$, $\exists\, \varphi \in \mathrm{Aut}$ s.t. $\varphi(p) = q$. A manifold is called symmetric if it is homogeneous and $\exists\,(\forall)\, p$, $\exists\, s_p \in \mathrm{Aut}$, $s_p^2 = 1$, p is the only fixed point for s_p in a nbhd. The involution s_p is called the symmetry at p.

A connected symmetric Riemannian manifold is called a symmetric space. For example $(\mathbb{R}^n, \sum dx^i \otimes dx^j)$ is a symmetric space, the translations are isometries and $x \to -x$ is a symmetry at 0.

Let (M, g) be a Hermitian manifold, $\mathrm{Aut}(M, g)$ be the automorphism group of holomorphic isometries. A connected symmetric hermitian manifold is called a Hermitian symmetric space.

Ex.

(1) $\mathcal{H}$, $g = \dfrac{dx \otimes dx + dy \otimes dy}{y^2}$. $SL_2(\mathbb{Z})$ acts on $\mathcal{H}$, $SL_2(\mathbb{Z}) \subset \mathrm{Aut}(\mathcal{H})$.

$\forall\; x + iy = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} i$ and $z \longmapsto -\frac{1}{z}$ is a symmetry at i.

(2) $\mathbb{P}^1(\mathbb{C})$ endowed with the standard metric on $\mathbb{R}^4$, the group of rotations is transitive and reflections along great circles through a point is a symmetry (rotate π along axis passing the point and its polar opposite).

(3) $\mathbb{C}/\Lambda$, $\Lambda \subset \mathbb{C}$ discrete additive subgroup, endowed with the standard metric. The group of transitions is transitive, $z \longmapsto -z$ is a symmetry at 0.

Let (M, g) be a symmetric space. The group of isometries is a Lie group. If (M, g) is Hermitian symmetric space then $Is(M, g)$, the group of holomorphic isometries is closed in $Is(M^{\infty}, g)$, hence also a Lie group.

Name	Example		Curvature	$Is(M,g)^+$
noncompact	$\mathcal{H}$	simply connected	negative	adjoint, noncompact
compact	$\mathbb{P}^1(\mathbb{C})$	simply connected	positive	adjoint, compact
Euclidean	$\mathbb{C}/\Lambda$		zero	

adjoint = semisimple, center trivial

G reductive, then

G semisimple $\Leftrightarrow$ $Z(G)$ finite

$G^{ad} = G/Z(G)$ adjoint.

Every Hermitian symmetric space M, when viewed as Hermitian manifold, decomposes into $M = M^0 \times M^- \times M^+$, M^0 Euclidean, M^- noncompact type, M^+ compact type. $M^0 = \mathbb{C}^g/\Lambda$. M is called irreducible if it is not the product of two Hermitian symmetric spaces of lower dimension. Both M^-, M^+ are products

M is called of noncompact type if curvature nonpositive everywhere and no M^0 part.

of irreducible Hermitian symmetric spaces, each having a simple isometry group.

$\Rightarrow Is(M,g)^+$ semisimple

Hermitian symmetric spaces of noncompact type : Hermitian symmetric domains.

simply connected

Ex. Siegel upper half plane $\mathcal{H}_g = \{ Z = X + iY \in M_g(\mathbb{C}) \mid Z \text{ symmetric}, Y > 0 \}$.

$\mathcal{H}_g \subset \mathbb{C}^{g(g+1)/2}$ open. $Sp_{2g}(\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2g}(\mathbb{R}) \,\middle|\, \begin{matrix} A^t \cdot C = C^t \cdot A & A^t \cdot D - C^t \cdot B = I_g \\ D^t \cdot A - B^t \cdot C = I_g & B^t \cdot D = D^t \cdot B \end{matrix} \right\}$

acts transitively on $\mathcal{H}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ+B) \cdot (CZ+D)^{-1}$$

$\begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$ acts as an involution on $\mathcal{H}_g$ and only fixes iI_g.

$\Rightarrow \mathcal{H}_g$ is homogeneous and symmetric as a complex manifold.

A domain D in $\mathbb{C}^n$ is a nonempty open connected subset.

Thm. Every bounded domain has a canonical Hermitian metric with negative curvature.

Pf. $H(D)$ the Hilbert space of holomorphic square-integrable functions $f: D \to \mathbb{C}$.

There is a unique Bergman kernel $K: D \times D \to \mathbb{C}$ s.t.

1) $\forall \zeta$, $z \mapsto K(z,\zeta) \in H(D)$

2) $K(z,\zeta) = \overline{K(\zeta, z)}$

3) $\forall f \in H(D)$, $f(z) = \int_D K(z,\zeta) f(\zeta)\, dv(\zeta)$

Let $\{e_m\}_{m\in\mathbb{N}}$ be a complete orthonormal basis of $H(D)$.

Set $K(z, \zeta) = \sum_m e_m(z) \overline{e_m(\zeta)}$.

Assume D bounded, $K(z, z) > 0$, $\log K(z, z)$ is smooth.

Let $h = \sum h_{ij} \, dz^i \, d\bar{z}^j$ where $h_{ij}(z) = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log K(z, z)$. ///

Ex. $D = D_1 \subset \mathbb{C}$, $K(z, \zeta) = \frac{1}{\pi} \frac{1}{(1 - z\bar{\zeta})^2}$, $h = \frac{\partial^2}{\partial z \partial \bar{z}} \log K(z, z) = \frac{1}{\pi} \frac{2}{(1 - z\bar{z})^2}$

$\Rightarrow$ the metric is $\frac{1}{(1 - z\bar{z})^2} \, dz \, d\bar{z}$

Cor. Bounded symmetric domain is Hermitian symmetric domain.

Fact. Every Hermitian symmetric domain can be embedded into some $\mathbb{C}^n$ as a bounded symmetric domain.

Cor. $\mathcal{H}_g$ is Hermitian symmetric domain.

Pf. $D_g = \{ Z \in M_g(\mathbb{C}) \mid Z \text{ symmetric}, \ I_g - Z^* Z > 0 \} \subset \mathbb{C}^{\frac{g(g+1)}{2}}$ bounded domain.

Consider $\mathcal{H}_g \to D_g$, $Z \to (Z - i I_g)(Z + i I_g)^{-1}$ this is an isomorphism.

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So far we defined Hermitian Symmetric Domain.

Next we consider automorphisms.

Lemma. Let (M, g) be a symmetric space, $p \in M$. Then the subgroup K_p of $Is(M, g)^+$ fixing p is compact and $Is(M, g)^+ / K_p \to M$, $[a] \mapsto a \cdot p$ is isomorphism of smooth manifolds.

Prop. Let (M, g) be a Herm. Sym. Dom. The inclusions $Is(M^\infty, g) \supset Is(M, g) \subset Hol(M)$ induces $Is(M^\infty, g)^+ = Is(M, g)^+ = Hol(M)^+$, i.e. they have the same connected component of identity. In particular $Hol(M)^+$ acts transitively on M, $Stab_p(Hol(M)^+)$ compact and $Hol(M)^+ / Stab_p \simeq M^\infty$.

(*) G is adjoint and $G(\mathbb{R})$ not compact.

Prop. Let (M, g) be a Herm. Sym. Dom. and $\mathfrak{h} = Lie(Hol(M)^+)$. There is a unique connected algebraic subgroup G of $GL(\mathfrak{h})$ s.t. $G(\mathbb{R})^+ = Hol(M)^+ \subset GL(\mathfrak{h})$ and $G(\mathbb{R})^+ = G(\mathbb{R}) \cap Hol(M) \subset GL(\mathfrak{h})$ therefore $G(\mathbb{R})^+ = Stab_M(G(\mathbb{R}))$.

$Hol(M)^+$ adjoint $\Rightarrow Hol(M)^+ \subset GL(\mathfrak{h}) \Rightarrow \exists G \subset GL(\mathfrak{h})$, $Lie\, G = [\mathfrak{h}, \mathfrak{h}] = \mathfrak{h} \Rightarrow G(\mathbb{R})^+ = Hol(M)^+$.

Ex. The map $z \longmapsto \bar{z}^{-1}$ is an antiholomorphic isometry of $\mathcal{H}$, so every isometry of $\mathcal{H}$ is either holo. or differs from $z \longmapsto \bar{z}^{-1}$ by a holo. one.

$G = PGL_2$, $PGL_2(\mathbb{R})$ acts holo. on $\mathbb{C} - \mathbb{R}$ with $PGL_2(\mathbb{R})^+$ stab. of $\mathcal{H}$.

Let $U_1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$.

Thm. Let D be a Herm. Sym. Dom. For each $p \in D$, $\exists!$ homomorphism $u_p : U_1 \to Hol(D)$ s.t. $u_p(z)$ fixes p and acts on T_pD as z. $\Rightarrow u_p : U_1 \longrightarrow Is(M, g)^+ = Hol(D)^+ = G(\mathbb{R})^+$

Ex. $D = \mathcal{H} = \mathcal{H}_1$, $p = i$, $h : \mathbb{C}^\times \longrightarrow SL_2(\mathbb{R})$, $h(a + ib) = \frac{1}{a^2+b^2}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Then $h(z)$ acts on $T_p\mathcal{H}$ via $\frac{z}{\bar{z}}$. For $z \in U_1$, choose $\sqrt{z} \in U_1$ and set $u(z) = h(\sqrt{z})$ mod $\pm I$. Then $u : U_1 \longrightarrow PSL_2(\mathbb{R})$ s.t. $u(z)$ acts on $T_p\mathcal{H}$ by z.

A Riemannian manifold is called (geodesically) complete if every maximal geodesic is defined on $\mathbb{R}$.

Prop. (M, g) symmetric space.

1) $p \in M$, S_p acts as -1 on T_pM, sends $\gamma(t)$ to $\gamma(-t)$ for every geodesic γ with $\gamma(0) = p$

2) (M, g) is complete.

Pf. of Thm.

Fix $p \in D$, each $z \in U_1$ defines an automorphism of T_pD preserving g_p. Let $u_p(z)(e^{x}) = e^{zx}$, this is defined locally but extends to M. ///
as D complete and homogeneous.

Cartan involutions: Study G as in $(*)$

G connected algebraic group $/\mathbb{R}$. $g \mapsto \bar{g}$ on $G(\mathbb{C})$ is the complex conjugate.

An involution θ of G is called Cartan if

$$G^{(\theta)}(\mathbb{R}) = \{ g \in G(\mathbb{C}) \mid g = \theta(\bar{g}) \} \text{ is compact.}$$

Ex. $G = SL_2$, $\theta = \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\theta\left(\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\right) = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}$

$SL_2^{(\theta)}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) \mid d = \bar{a},\ c = -\bar{b} \right\} = SU_2(\mathbb{R})$ compact

Thm. Let G be a connected algebraic group over $\mathbb{R}$, there exists a Cartan involution of G $\Leftrightarrow$ G reductive, any two are conjugate by element of $G(\mathbb{R})$.

In particular existence of Cartan involution implies reductiveness.

Ex. G conneted algebraic group over $\mathbb{R}$.

(a) id_G is a Cartan involution $\Leftrightarrow$ $G(\mathbb{R})$ compact

(b) $G \hookrightarrow GL(V)$, G reductive iff G stable under $g \longmapsto g^t$ for some basis of V, and $g \longmapsto (g^t)^{-1}$ is a Cartan involution on G. All Cartan involutions of G arise in this way from the choice of a basis for V.

(c) Let θ be an involution on G. There is a unique real form $G^{(\theta)}$ of $G_{\mathbb{C}}$ s.t. the complex conjugate on $G^{(\theta)}(\mathbb{C})$ is $g \longmapsto \theta(\bar{g})$. θ is Cartan iff $G^{(\theta)}$ compact. All compact real forms of $G_{\mathbb{C}}$ arise in this way.

Prop. G conneted algebraic group over $\mathbb{R}$. If G compact, every finite dimensional real representation $G \to GL(V)$ carries a G-invariant positive-definite symmetric bilinear form. Conversely if one faithful f.d. $\mathbb{R}$-rep. carries such a form, G cpt.

Let G be a real algebraic group, $C \in G(\mathbb{R})$, C^2 central. A C-polarization on a $\mathbb{R}$-rep V of G is a G-invariant bilinear form φ s.t. $\varphi_C : (u, v) \longmapsto \varphi(u, Cv)$ is symmetric, positive-definite.

Prop. If $Ad(C)$ is a Cartan involution of G, then every f.d. $\mathbb{R}$-rep of G has a C-polarization. Conversely if one faithful f.d. $\mathbb{R}$-rep of G carries a C-polarization, then $Ad(C)$ is Cartan involution.

Pf. Let $\rho: G \to GL(V)$ be a $\mathbb{R}$-rep. of G. For any G-invariant bilinear form φ on V, $\varphi_{\mathbb{C}}$ is $G(\mathbb{C})$-invariant, $\varphi'(u,v) = \varphi_{\mathbb{C}}(u,\bar{v})$. Then

$$\varphi'(gu, \bar{g}v) = \varphi'(u,v), \quad \forall g \in G(\mathbb{C}), \quad u,v \in V(\mathbb{C})$$

$$\Rightarrow \varphi'(gu, C C^{-1}\bar{g}C v) = \varphi'(u, Cv)$$

$$\Rightarrow \varphi_C'(gu, ((Ad\,C)\bar{g})v) = \varphi_C'(u,v)$$

If ρ faithful and φ is a C-polarization, then φ_C' is positive-definite Hermitian form $\Rightarrow G^{(Ad\,C)}(\mathbb{R})$ compact $\Rightarrow Ad\,C$ Cartan involution.

If $G^{(Ad\,C)}(\mathbb{R})$ compact $\Rightarrow \rho$ carries a $G^{(Ad\,C)}(\mathbb{R})$ invariant positive-definite symmetric bilinear form $\varphi \Rightarrow \varphi_{C^{-1}}$ is a C-polarization. ///

Classification of Hermitian symmetric domains.

Representation theory for split torus $\Rightarrow$ every irreducible $\mathbb{R}$-rep of U_1 is of the form:

1) $V = \mathbb{R}$, U_1 acts trivially

2) $V = \mathbb{R}^2$, $x + iy \in U_1(\mathbb{R})$ acting as $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^n$, $n > 0$.

$u_p : U_1 \to \mathcal{H}ol(D)^+ = G(\mathbb{R})^+$ is a homomorphism of real Lie groups.

The representations of U_1 have the same description whether as Lie group or algebraic group, so every homomorphism $\underset{\substack{\triangle \\ \text{unitary group}}}{U_1} \to GL(V)$ of real Lie groups is algebraic.

Thm. D Herm. Sym. Dom. G associated real adjoint algebraic group as in (*). Then the map $u_p : U_1 \to G$ satisfies:

(a) only characters $z, 1, z^{-1}$ occur in the rep. of U_1 on $(\mathrm{Lie}\, G)_{\mathbb{C}}$ defined by $\mathrm{Ad} \circ u_p$

(b) $\mathrm{Ad}(u_p(-1))$ is a Cartan involution

(c) $u_p(-1)$ does not project to 1 in any simple factor of G.

Conversely, let G be a real adjoint algebraic group, $u : U_1 \to G$ satisfies (a)(b)(c). Then the set D of conjugates of u by elements of $G(\mathbb{R})^+$ has a natural structure of a Herm. Sym. Dom. s.t. $\underline{G(\mathbb{R})^+ = \mathcal{H}ol(D)^+}$ and $u(-1)$ is the symmetry at u.

Pf. $G(\mathbb{R})^+ / K_p \simeq D$ as smooth manifolds. For $z \in U_1$, the action of $u_p(z)$ on $G(\mathbb{R})^+$ by conjugate preserves K_p and corresponds to the action of $u_p(z)$ on D.

$\Rightarrow$ $\mathrm{Lie}\, G / \mathrm{Lie}\, K_p \simeq T_pD$ as $\mathbb{R}$-v.s., note $u_p(z)$ acts on T_pD by z.

$k \in K_p$, $u_p(z) \cdot k \cdot u_p(z)^{-1}$ fixes p and acts as $(dk)_p$ on T_pD $\Rightarrow$ $u_p(z) \cdot k \cdot u_p(z)^{-1} = k$ $\Rightarrow$ $u_p(z)$ acts trivially on $\mathrm{Lie}\, K_p$ $\Rightarrow$ $u_p(z)$ acts on $\mathrm{Lie}\, G$ by 1 or z $\Rightarrow$ (a) holds.

The symmetry S_p at p and $u_p(-1)$ both fix p and act as -1 on T_pD, hence they equal. Let $\mathfrak{h} = \mathrm{Lie}(G(\mathbb{R})^+)$, by construction $G(\mathbb{R})^+ \hookrightarrow GL(\mathfrak{h})$. The Killing form B on $\mathfrak{h}$ is nondegenerate. Let $\mathfrak{k} = \mathrm{Lie}\, K_p \Rightarrow \mathfrak{h} = \mathfrak{k} \oplus \mathfrak{k}^\perp$. The $S_p = u_p(-1)$ action on $G(\mathbb{R})^+$ induces an involution S_p on $\mathfrak{h}$ preserving $\mathfrak{k}$ and acting as -1 on $\mathfrak{k}^\perp$. Now $\varphi = -B$ is G-invariant bilinear symmetric form on $\mathfrak{h}$ hence φ_{S_p} is also symmetric. As K_p compact, $B|_{\mathfrak{k}}$ is negative definite hence $\varphi_{S_p}|_{\mathfrak{k}}$ is positive definite. Claim all sectional curvature ≤ 0 implies $B|_{\mathfrak{k}^\perp}$ positive definite, hence φ_{S_p} positive definite and $\mathrm{Ad}\, S_p$ is a Cartan involution.

Clearly $\mathfrak{k}^\perp \simeq T_pD$. Let $S \subset T_pD$ be a 2-dimensional subspace and X, Y an orthonormal basis of S under g_p. Then $K(S) = g_p([[X,Y],X],Y)$. As g_p symmetric bilinear positive definite on $\mathfrak{k}^\perp$ and B symmetric bilinear, we can find a basis $\{X_i\}$ of $\mathfrak{k}^\perp$, $\{\beta_i\}$ s.t. if $u = \sum \lambda_i X_i$ then

$$g_p(u,u) = \sum \lambda_i^2 \quad \text{and} \quad B(u,u) = \sum \beta_i \lambda_i^2$$

The values $\{\beta_i\}$ are eigenvalues of $b \in \mathrm{End}(\mathfrak{k}^\perp)$, $g_p(bu, v) = B(u,v)$. Write $X = \sum X_i$, $Y = \sum Y_i$ where $X_i, Y_i \in \mathfrak{k}_i^\perp$, the eigenspace of β_i. Then if $i \neq j$, $[X_i, X_j] \in \mathfrak{k}$ and $B(\mathfrak{k}, [X_i, X_j]) = 0 \Rightarrow [X_i, X_j] = 0$ $\Rightarrow K(S) = \sum \frac{1}{\beta_i} B([X_i, Y_i], [X_i, Y_i])$. As $K(S) \leq 0$, $[X_i, Y_i] \in \mathfrak{k}$, by varying S, we see $\beta_i > 0$ hence $B|_{\mathfrak{k}^\perp}$ positive definite.

If the projection of $u(-1)$ into a simple factor of G is trivial then the factor would be compact, contributing to D an irreducible factor of compact type.

Conversely D the set of $G(\mathbb{R})^+$ conjugates of u.

$K_u = C_u(G(\mathbb{R})^+) \subset \{g \in G(\mathbb{C}) \mid g = u(-1)\bar{g}\,u(-1)^{-1}\}$ compact by (b).

K_u closed $\Rightarrow$ K_u compact.

$D = (G(\mathbb{R})^+/K_u)\cdot u$ $\Rightarrow$ D smooth manifold.

$T_uD = \operatorname{Lie} G(\mathbb{R})^+ / \operatorname{Lie} K_u$ is the subspace of $(\operatorname{Lie} G)_{\mathbb{C}}$ on which $u(z)$ acts by z by (a) $\Rightarrow$ T_uD has a $\mathbb{C}$-v.s. structure $\xRightarrow{D \text{ homogeneous}}$ D has an almost complex structure $\xRightarrow{\text{integrable}}$ D is a complex manifold.

K_u acts on D $\Rightarrow$ K_u acts on T_uD $\xRightarrow{K_u \text{ compact}}$ there is K_u-invariant positive definite bilinear form on T_uD which is Hermitian since $J = u(i) \in K_u$ $\xRightarrow{D \text{ homogeneous}}$ D Hermitian manifold $\xRightarrow{u(-1)\text{ symmetry}}$ D Hermitian symmetric space $\xRightarrow{(b)(c)}$ D Herm. Sym. Dom. ///

Cor. There is a bijective correspondence

$$\left\{\simeq \text{ class of pointed Herm. Sym. Dom.}\right\} \longleftrightarrow \left\{\begin{array}{l}(G,u),\ G \text{ real adjoint Lie group} \\ u: U_1 \to G(\mathbb{R}) \text{ homomorphism } (a)(b)(c)\end{array}\right\}$$

and

$$\left\{\simeq \text{ class of Herm. Sym. Dom.}\right\} \longleftrightarrow \left\{(G,[u]) \text{ as above, } [u] \text{ conjugacy class of } u\right\}$$

Ex. $u : U_1 \to PSL_2(\mathbb{R})$, $u(-1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathrm{Ad}\, u(-1)$ is a Cartan involution of SL_2 hence PSL_2, the corresponding Herm. Sym. Dom. is $\mathcal{H}_1$.

as $\mathrm{Ad}(u(-1))$ Cartan involution, G is inner form of its compact form $\Rightarrow G_{\mathbb{C}}$ simple

Let (G, u), G simple adjoint Lie group over $\mathbb{R}$, $u : U_1 \to G$ homomorphism with (a)(b). $G_{\mathbb{C}}$ is also simple adjoint Lie group, $\mu = u_{\mathbb{C}}$ is a cocharacter of $G_{\mathbb{C}}$

(**) in the action of $\mathbb{G}_m$ on Lie$(G_{\mathbb{C}})$ defined by $\mathrm{Ad} \circ \mu$, only the characters $z, 1, z^{-1}$ occur.

Prop. $(G, u) \to (G_{\mathbb{C}}, \mu)$ defines a bijection between the sets of $\simeq$ class of pairs

$$\left\{ \begin{matrix} G \text{ simple adjoint} / \mathbb{R} \\ u \text{ conjugacy class} : U_1 \to G,\ (a)(b) \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} G_{\mathbb{C}} \text{ simple adjoint} / \mathbb{C} \\ \mu \text{ conjugacy class of cocharacter s.t. } ** \end{matrix} \right\}$$

Pf. Given (G, μ), let $g \to \bar{g}$ be the complex conjugate of $G(\mathbb{C})$ relative to a maximal compact subgroup of G containing $\mu(U_1)$. There is a real form H of G s.t. the complex conjugate on $H(\mathbb{C})$ is $g \longmapsto \mu(-1) \cdot \bar{g} \cdot \mu(-1)^{-1}$ and $u = \mu|_{U_1}$ values in $H(\mathbb{R})$. Then (H, u) is s.t. $(H_{\mathbb{C}}, u_{\mathbb{C}}) \simeq (G, \mu)$. ///

Let G be a simple algebraic group $/ \mathbb{C}$, $T \subset G$ a maximal torus, $(\alpha_i)_{i \in I}$ a base for the roots of (G, T) $\Rightarrow$ the nodes of Dynkin diagram of (G, T) is indexed by I. The highest root $\tilde{\alpha} = \sum n_i \alpha_i$. An α_i is called special if $n_i = 1$.

Thm. The Isom. classes of irreducible Herm. Sym. Dom. are classified by special nodes on connected Dynkin diagram.

Pf. Let M be a conjugacy class of nontrivial cocharacters of G satisfying $**$ then M has a representative with image in T. As the Weyl group acts simply transitively on the Weyl chambers there is a unique μ for M s.t. $(\alpha_i, \mu) \geq 0$ for all $i \in I$.

Here the pairing is $\mathrm{Hom}(\mathbb{G}_m, T) \times \mathrm{Hom}(T, \mathbb{G}_m) \to \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$ and the roots α_i actually come from $\mathrm{Hom}(T, \mathbb{G}_m) \to \mathrm{Hom}(\mathfrak{h}, \mathbb{C}) = \mathfrak{h}^\vee$.

The condition $**$ implies $(\alpha_i, \mu) \in \{0, \pm 1\}$, as μ nontrivial, there is exact one α_i s.t. $(\alpha_i, \mu) = 1$ and this α_i is special.

Conversely let α_0 be a special simple root. Then consider μ s.t. $(\alpha_0, \mu) = 1$ and $(\alpha, \mu) = 0$ for all other simple roots α. Then $(\beta, \mu) \in \{0, \pm 1\}$ for all roots β. Then $\mu \in \mathrm{Hom}(\mathrm{Hom}(T, \mathbb{G}_m), \mathbb{Z}) = \mathrm{Hom}(\mathbb{G}_m, T)$. ///

$\alpha \in \mathrm{Hom}(T, \mathbb{G}_m)$, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{Ad}(t)x = \alpha(t)x, \ \forall t \in T\}$.

$\mu \in \mathrm{Hom}(\mathbb{G}_m, T)$, $z \in \mathbb{G}_m$, $x \in \mathfrak{g}_\alpha$, $\mathrm{Ad}(\mu(z))x = \alpha(\mu(z))x = z^{(\alpha, \mu)} x$

$\Rightarrow$ $(**) \Leftrightarrow (\alpha, \mu) \in \{0, \pm 1\}, \ \forall \alpha$

Given $(\alpha, \mu) = 1 \Rightarrow \mathbb{G}_m$ acts on $\mathfrak{g}_\alpha$ by z.

Type	$\tilde{\alpha}$	special roots	#
A_n	$\alpha_1 + \cdots + \alpha_n$	$\alpha_1, \ldots, \alpha_n$	n
B_n	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$	α_1	1
C_n	$2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n$	α_n	1
D_n	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	$\alpha_1, \alpha_{n-1}, \alpha_n$	3
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	α_1, α_6	2
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	α_7	1
E_8, F_4, G_2		—	0