Complex multiplication: the main theorem.

elliptic curves with CM endomorphism $\Rightarrow$ action of Gal on torsion points

$\Rightarrow$ class field theory, L-functions, ...

AV of CM type $\Rightarrow$ action of Gal of reflex field on torsion points

Recall CFT.

E number field, $\exists!$ continuous surjective homomorphism $\phi_E : \mathbb{A}_E^\times \to \mathrm{Gal}(E)^{ab}$

s.t. $\forall$ L/E finite abelian,

$$\begin{array}{ccc} E^\times \backslash \mathbb{A}_E^\times & \xrightarrow{\phi_E} & \mathrm{Gal}(E^{ab}/E) \\ \downarrow & \circlearrowright & \downarrow \; \sigma \mapsto \sigma|_L \\ (E^\times\backslash \mathbb{A}_E^\times)/\mathrm{Nm}_{L/E}(L^\times \backslash \mathbb{A}_L^\times) & \xrightarrow[\sim]{\phi_{L/E}} & \mathrm{Gal}(L/E) \end{array}$$

$\mathrm{art}_E = (\phi_E)^{-1}$

and rigidified by

- $\phi_{L/E}(u) = 1$, $u = (u_v) \in \mathbb{A}_E^\times$ s.t.

  u_v unit if v unramified in L

  u_v close to 1 if v ramified in L

  $u_v > 0$ if v real becomes complex in L

- v unramified in L, $\alpha_v = (\cdots, 1, v, 1, \cdots)$ maps to $\mathrm{Frob}_v \in \mathrm{Gal}(L/E)$.

$\ker \phi_E$ is the identity connected component of $E^\times \backslash \mathbb{A}_E^\times$.

If E totally imaginary, ϕ_E factors through $E^\times \backslash \mathbb{A}_{E,f}^\times$

The reflex field and norm of a CM type.

(E, Φ) CM type.

Def. The reflex field E^* is the subfield of $\overline{\mathbb{Q}}$ characterized by one of the following equivalent conditions:

(a) $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixes E^* $\Leftrightarrow$ $\sigma\Phi = \Phi$

(b) $E^* = \mathbb{Q}\left(\sum_{\varphi \in \Phi} \varphi(a),\ a \in E\right)$

(c) E^* is the smallest subfield k of $\overline{\mathbb{Q}}$ admitting k-v.s. V with action of E s.t. $\mathrm{tr}(a|V) = \sum_{\varphi \in \Phi} \varphi(a)$.

Let V be as in (c), the reflex norm $N_{E^*} : (\mathbb{G}_m)_{E^*/\mathbb{Q}} \to (\mathbb{G}_m)_{E/\mathbb{Q}}$ is $N_{E^*}(a) = \det_{E \otimes_{\mathbb{Q}} \mathbb{R}}(a | V \otimes_{\mathbb{Q}} \mathbb{R})$, $a \in (E^* \otimes_{\mathbb{Q}} \mathbb{R})^\times$. The definition is independent of V.

Statement of the main theorem of CM.

The theorem specifies the action of σ on $V_f A$, i.e. what is ${}^\sigma\eta$.

Thm. (A, i) AV/$\mathbb{C}$ of CM type (E, Φ), $\sigma \in \mathrm{Aut}(\mathbb{C}/E^*)$. For any $s \in \mathbb{A}^\times_{E^*, f}$ with $\mathrm{art}_{E^*}(s) = \sigma|_{E^*, ab}$, $\exists!$ E-isogeny $\alpha : A \to \sigma A$ s.t. $\alpha(N_{E^*}(s) \cdot x) = \sigma x$ for all $x \in V_f A$.

Pf. $T_0(\sigma A) \simeq T_0 A \otimes_{\mathbb{C}, \sigma} \mathbb{C}$ as $E \otimes_{\mathbb{Q}} \mathbb{C}$-module $\Rightarrow$ $(\sigma A, {}^\sigma i)$ is of CM type $\sigma\Phi = \Phi$ as σ fixes E^*. Hence $\exists$ E-isogeny $\alpha : A \to \sigma A$

if $\sigma|_{\bar{E}=\bar{\mathbb{Q}}} = id$ then consider on the canonical $\bar{\mathbb{Q}}$-model of A, both σ and α are id.

and the map $V_f A \xrightarrow{\sigma} V_f(\sigma A) \xrightarrow{V_f(\alpha)^{-1}} V_f A$ is $E \otimes_{\mathbb{Q}} \mathbb{A}_f$ - linear of $V_f A$ as free rank 1 module, hence is a multiplication by $a \in \mathbb{A}_{E,f}^{\times}$.

Different choice of α changes a by $E^{\times}$, so we have a well-defined map $\mathrm{Gal}(\bar{\mathbb{Q}}/E^*) \longrightarrow \mathbb{A}_{E,f}^{\times}/E^{\times}$ which is a homomorphism. Compose with art_{E^*} to get $\eta: \mathbb{A}_{E^*,f}^{\times}/E^{*,\times} \longrightarrow \mathbb{A}_{E,f}^{\times}/E^{\times}$. ETS this is N_{E^*}.

The basic idea is that N_{E^*} is computed by Shimura-Taniyama on primes. Let K/E^* finite Galois, $K \subset \bar{\mathbb{Q}}$ s.t. A has CM by E over K. Let $\mathfrak{P}$ be a prime of K s.t.

- A has good reduction at $\mathfrak{P}$
- $\mathfrak{P}$ unramified over $P = \mathfrak{P} \cap \mathcal{O}_{E^*}$
- $p = \mathfrak{P} \cap \mathbb{Z}$ unramified in E

Let σ be the $\mathrm{Frob}_{\mathfrak{P}}$ of K/E^*, then as $\sigma\Phi = \Phi$, we can find (after suitable modification) an E-isogeny $\beta: A \to \sigma A$ whose reduction is Frobenius map. Then $\sigma^{f-1}\beta \circ \sigma^{f-2}\beta \circ \cdots \circ \sigma\beta \circ \beta = \pi$ where $f = f(\mathfrak{P}/p)$.

Also by Shimura-Taniyama formula

$$\pi \overset{S-T}{=} \prod_{\varphi \in \Phi} \varphi^{-1}(Nm_{K/\varphi E}\mathfrak{P}) = N_{E^*}(Nm_{K/E^*}\mathfrak{P}) = N_{E^*}(P^f) = N_{E^*}(P)^f$$

Thus for such P primes in E^*, $\eta(\pi_P)^f = N_{E^*}(P)^f$ hence the two maps η, N_{E^*} agree. Then conclude by Dirichlet's density theorem.

The uniqueness comes from the faithful functor $A \longrightarrow V_f A$. ///

Both η and N_{E^*} are continuous and agree on a dense subset.