Definition of canonical models

$(G, X)$ SD , $(G', X^+)$ associated conn. SD

$K \subset G(\mathbb{A}_f)$ compact open , $f_K : Sh_K(G, X) \to \pi_K$ of varieties over $\mathbb{C}$ ,

$\pi_K$ variety of dim 0 , fibres of $f_K$ are conn. SV from $(G', X^+)$.

Theory of canonical models :

- reflex field $E = E(G, X)$ , number field contained in $\mathbb{C}$
- canonical model $(f_K)_0 : Sh_K(G, X)_0 \to (\pi_K)_0$ of $f_K$ over $E$ , uniquely

characterized by reciprocity laws at special points

( • action of $\mathrm{Aut}(\mathbb{C}/E)$ on $\pi_K$ corresponding to $(\pi_K)_0$ )

The reflex field.

$G/\mathbb{Q}$ reductive , $k \subset \mathbb{C}$ subfield , $C(k) = G(k) \backslash \mathrm{Hom}(\mathbb{G}_m, G_k)$ be the set

of $G(k)$-conj. classes of cocharacters of $G_k$ .

$k \to k'$ induce $C(k) \to C(k')$ , $\mathrm{Aut}(k'/k)$ acts on $C(k')$

Assume $G$ splits over $k$ , $T$ maximal split torus of $G_k$ , Weyl group

$W = W(G_k, T)$ is the constant etale algebraic group $N/N^\circ$ , $N$ normalizer

of $T$ in $G_k$ . $\forall\, k \subset k'$ , $W(k) = W(k') = N(k')/T(k')$.

Lemma. $T$ maximal split torus in $G_k$, $W\backslash \mathrm{Hom}(\mathbb{G}_m, T_k) \longrightarrow G(k)\backslash \mathrm{Hom}(\mathbb{G}_m, G_k)$ is bijective.

$(G, X)$ SD, $x \in X$, $\mu_x$ cocharacter of $G_{\mathbb{C}}$, $\mu_x(z) = h_{x,\mathbb{C}}(z, 1)$

Different $x$ gives conjugate of $\mu_x$ $\Rightarrow$ $c(X) = [\mu_x] \in C(\mathbb{C})$.

If we change $\mathbb{C}$ to $\bar{\mathbb{Q}}$, $\mathrm{Hom}(\mathbb{G}_m, T_{\bar{\mathbb{Q}}})$ and $W$ do not change

so $c(X)$ contains a $\mu$ defined over $\bar{\mathbb{Q}}$ and $G(\bar{\mathbb{Q}})$-conj. class of $\mu$ is independent of $\mu$. Hence we may think $c(X) \in C(\bar{\mathbb{Q}})$.

Def. The reflex field $E(G, X)$ is the fixed field of the subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ fixing $c(X) \in C(\bar{\mathbb{Q}})$.

RMK.

Any subfield $k$ of $\bar{\mathbb{Q}}$ splitting $G$ contains $E(G, X)$ $\Rightarrow$ $E(G, X)$ number field.

If $c(X)$ contains a $\mu$ defined over $k$ then $E(G, X) \subset k$.

If $G$ quasi-split over $k$, $E(G, X) \subset k$ then $c(X)$ contains a $\mu$ defined over $k$.

$(G, X) \xhookrightarrow{i} (G', X')$ inclusion of SD $\Rightarrow$ $E(G', X') \subset E(G, X)$.

The case (b) corresponds to PEL datum $(E, *, V=E, \psi)$ where $\psi$ is defined by $f \in E$, $f^* = -f$, $\operatorname{Im}\varphi(f) > 0$ $\forall \varphi \in \Phi$. $V^{-1,0} \simeq \mathbb{C}^{\Phi}$ and $\forall b \in E$, the action is given by $(\varphi(b))_{\varphi \in \Phi}$ hence $tr_x b = \sum_{\varphi \in \Phi} \varphi(b)$.

Ex.

(a) $T/\mathbb{Q}$ torus, $h: \mathbb{S} \to T_{\mathbb{R}}$. $\mu_h: \mathbb{G}_{m,\mathbb{C}} \to T_{\mathbb{C}}$ is defined over $\overline{\mathbb{Q}}$.

$E(T, \{h\})$ is the fixed field of the subgrp of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixing $\mu_{h,\overline{\mathbb{Q}}}$.

(b) $(E, \Phi)$ CM type, $T = (\mathbb{G}_m)_{E/\mathbb{Q}}$, $T(\mathbb{R}) = (E \otimes_{\mathbb{Q}} \mathbb{R})^\times \simeq (\mathbb{C}^{\Phi})^\times$ and

$T(\mathbb{C}) = (E \otimes_{\mathbb{Q}} \mathbb{C})^\times \simeq (\mathbb{C}^{\Phi} \otimes_{\mathbb{R}} \mathbb{C})^\times \simeq (\mathbb{C}^{\Phi})^\times \times (\mathbb{C}^{\overline{\Phi}})^\times$.

Let $h_\Phi: \mathbb{S} \to T_{\mathbb{R}}$, $h_\Phi(\mathbb{R}): z \longmapsto (z, \cdots, z) \in (\mathbb{C}^{\Phi})^\times$ and

$h_\Phi(\mathbb{C}): (z_1, z_2) \longmapsto (z_1, \cdots, z_1, z_2, \cdots, z_2) \in (\mathbb{C}^{\Phi})^\times \times (\mathbb{C}^{\overline{\Phi}})^\times$. Then

$\mu_\Phi(z) = (z, \cdots, z, 1, \cdots, 1) \in (\mathbb{C}^{\Phi})^\times \times (\mathbb{C}^{\overline{\Phi}})^\times$. Clearly $\mu_\Phi$ is defined over $\overline{\mathbb{Q}}$ and the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixing it is just the stabilizer of $\Phi$ $\Rightarrow$ $E(T, \{h_\Phi\}) = E^*$ is the reflex field of $(E, \Phi)$.

(c) If $(G, X)$ is simple PEL datum of type (A) or (C), then $E(G,X)$ is the field generated over $\mathbb{Q}$ by $\{tr_x b,\ b \in B\}$. The $\sigma$ fixing the reflex field are exactly those preserve the family of AV parametrized by $Sh(G, X)$.

(d) $(G, X)$ SD coming from quaternion algebra $B$ over totally real $F$.

$c(X)$ contains $\mu: \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$,

$$z \longmapsto (1, \cdots, 1) \times \left( \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \cdots, \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \right) \in GL_{2,\mathbb{C}}^{I_c} \times GL_{2,\mathbb{C}}^{I_{nc}}$$

This is defined over $\overline{\mathbb{Q}}$, $E(G, X)$ is the fixed field of the subgrp of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ stabilizing $I_{nc} \subset I$.

$I_{nc} = \{v\}$, $Sh(G, X)$ curve, $E(G, X) = v(F) \subset \mathbb{C}$.

(e) $(G, X)$ SD, $G$ adjoint, $T$ maximal torus in $G_{\overline{\mathbb{Q}}}$, $\Delta$ base for

roots $\Phi$ of $(G,T)_{\overline{\mathbb{Q}}}$. $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathrm{Hom}(T, \mathbb{G}_m)$, preserving $\Phi$.
For any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $\sigma(\Delta)$ is another base for $\Phi$, hence $\exists\, w_\sigma$ lies in the Weyl group of $(G,T)$ s.t. $w_\sigma(\sigma(\Delta)) = \Delta$. Define an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\Delta$ by $\sigma * \alpha = w_\sigma(\sigma\alpha)$.

Each $c \in C(\overline{\mathbb{Q}})$ contains a unique $\mu_c : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}}$ s.t. $\langle \alpha, \mu_c \rangle \geq 0$ for all $\alpha \in \Delta$ as the Weyl group acts simply transitively on Weyl chambers.
The map $C(\overline{\mathbb{Q}}) \longrightarrow \mathbb{N}^{\Delta}$, $c \longmapsto (\langle \alpha, \mu_c \rangle)_{\alpha \in \Delta}$ is bijective.
$E(G,X)$ is the fixed field of the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixing the element $(\langle \alpha, \mu_{c(x)} \rangle)_{\alpha \in \Delta} \in \mathbb{N}^{\Delta}$.

As $G_{\mathbb{R}}$ contains a compact maximal torus, complex conjugation acts as the opposition involution on $\Delta$ $\Rightarrow$ $E(G,X)$ is totally real if the opposite involution fixes $(\langle \alpha, \mu_{c(x)} \rangle)_{\alpha \in \Delta}$ and otherwise it is CM field.
(f) $(G,X)$ SD, $\{G_i\}$ simple factor of $G^{ad}$, $T = G/G^{der}$.
$(G,X) \Rightarrow \{(G_i, X_i)\}$, $(T,h)$ SD, $E(G,X) = E(G_1, X_1) \cdots E(G_r, X_r) \cdot E(T,h)$
If $T$ split by a CM field (SV 6), $E(G,X)$ is CM or totally real.

CM condition $\sim$ maximal Hodge tensors
$\sim$ minimal Mumford-Tate group
$\sim$ factor through torus

Special points.
Def. $(G,X)$ SD, $x \in X$ is special if $\exists$ torus $T \subset G$ s.t. $\mathrm{Im}\, h_x \subset T(\mathbb{R})$.
Such a pair $(T,x)$ is called a special pair in $(G,X)$.

If SV 4 and SV 6, they are called CM points / pairs.
weight rational — $Z^\circ$ split over CM field

$z \in \mathcal{H}^{\pm}$ special $\Rightarrow \exists\, T \subset GL_2/\mathbb{Q}$, $T(\mathbb{R})$ fixes $z$

$\Rightarrow T(\mathbb{Q})$ fixes $z$

$\Rightarrow \frac{az+b}{cz+d} = z$, $z = \frac{a-d \pm \sqrt{\Delta}}{2c}$

$\Rightarrow \mathbb{Q}(z)/\mathbb{Q}$ quadratic imaginary.

RMK. $(T, x)$ special $\underset{\overset{T \text{ maximal}}{\Longleftarrow}}{\Longrightarrow}$ $T(\mathbb{R})$ fixes $x$.

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Ex. $G = GL_2$, $\mathcal{H}^{\pm} = \mathbb{C} - \mathbb{R}$, $z \in \mathbb{C} - \mathbb{R}$ generates a quadratic imaginary $E/\mathbb{Q}$.

$\{1, -z\}$ $\mathbb{Q}$-basis of $E$, $E \hookrightarrow M_2(\mathbb{Q})$ hence $(\mathbb{G}_m)_{E/\mathbb{Q}} \overset{\text{maximal}}{\subset} G$. Then $((\mathbb{G}_m)_{E/\mathbb{Q}}, z)$ is a special pair.

$z = i$, $h_z(\alpha + \beta i) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in GL_2(\mathbb{R})$, $1 \otimes a + (-i) \otimes b \in E \otimes_{\mathbb{Q}} \mathbb{R}$ acts on the basis $1 \otimes 1, (-i) \otimes 1$ as $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Conversely if $z \in \mathcal{H}^{\pm}$ is special, $\mathbb{Q}[z]/\mathbb{Q}$ is of deg 2. Thus special points in $\mathcal{H}^{\pm}$ are exactly those $\tau$ s.t. $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ are CM elliptic curves.

More generally, SV moduli for AV, special points correspond to AV of CM type $\Rightarrow$ action of open subgroups of $\mathrm{Aut}(\mathbb{C})$ on special points in SV.

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$T/\mathbb{Q}$ torus, $\mu : \mathbb{G}_m \to T_E$, $E/\mathbb{Q}$ finite extension.

$q \in T(E)$, $\prod_{\rho: E \to \overline{\mathbb{Q}}} \rho(q) \in T(\overline{\mathbb{Q}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} = T(\mathbb{Q})$.

$r(T, \mu) : (\mathbb{G}_m)_{E/\mathbb{Q}} \to T$, $r(T,\mu)(e) = \prod_{\rho: E \to \overline{\mathbb{Q}}} \rho(\mu(e))$, $e \in E^{\times}$.

$(T, x) \subset (G, X)$ special pair, $E(x)$ field of definition of $\mu_x$

$r_x : \mathbb{A}_{E(x)}^{\times} \xrightarrow{r(T, \mu_x)} T(\mathbb{A}_{\mathbb{Q}}) \xrightarrow{\text{projection}} T(\mathbb{A}_f)$.

$$a = (a_\infty, a_f) \longmapsto \prod_{\rho: E(x) \to \overline{\mathbb{Q}}} \rho(\mu_x(a_f))$$

If $\mu_x : \mathbb{G}_m \to T_{E(x)}$ factors through $T'_{E(x)}$, then $r_x$ are the same.

① $\forall (T,x)$ special, $a\in G(\mathbb{A}_f)$ $\Rightarrow [x,a]$ coordinate in $E(x)^{ab}$ $\sigma\in Gal(E(x))^{ab}$, $\sigma$ acts by $(*)$

② $\forall (T,x)$ special, $a\in G(\mathbb{A}_f)$ $\Rightarrow [x,a]$ coordinate in $E(x)^{ab}$ $\sigma\in Aut(\mathbb{C}/E(x))$, $\sigma$ acts by $(*)$

③ $\forall (T,x)$ special, $a\in G(\mathbb{A}_f)$ $\Rightarrow$ $\sigma\in Aut(\mathbb{C}/E(x))$, $\sigma$ acts by $(*)$ (i.e. at special $x$, $Aut(\mathbb{C}/E(x))$ action factor through $Gal(E(x))^{ab}$ by $(*)$).

**Definition of a canonical model.** ① $\Leftarrow$ ② $\Leftrightarrow$ ③

$(T,x)\subset(G,X)$ special pair, $art_{E(x)}: \mathbb{A}_{E(x)}^{\times} \twoheadrightarrow Gal(E(x)^{ab}/E(x))$.

Deligne: $(T,x)$ special, $a\in G(\mathbb{A}_f)$, $g\in G(\mathbb{Q})$ $\quad r_x: \mathbb{A}_{E(x)}^{\times} \longrightarrow T(\mathbb{A}_f)$

$gx=x$, $r\in T(\mathbb{A}_f)$ $\Rightarrow [x, g r a]=[x, ra]$.

Def. $(G,X)$ SD, $K$ open compact $\subset G(\mathbb{A}_f)$. A model $M_K(G,X)$ of $Sh_K(G,X)$ over $E(G,X)$ is canonical if for every special pair $(T,x)$, $a\in G(\mathbb{A}_f)$, $[x,a]_K$ has coordinates in $E(x)^{ab}$ and $\sigma[x,a]_K \overset{(*)}{=} [x, r_x(s)a]_K$ for all $\sigma\in Gal(E(x)^{ab}/E(x))$, $s\in\mathbb{A}_{E(x)}^{\times}$, $art_{E(x)}(s)=\sigma$.

$\forall x$ special, ETS to check $(T,x)$ for one $T$.

A model $M(G,X)$ of $Sh(G,X)$ over $k\subset\mathbb{C}$ is an inverse system $\{M_K(G,X)\}_K$ of varieties over $k$ with a right action of $G(\mathbb{A}_f)$ s.t. $M(G,X)_{\mathbb{C}}\simeq Sh(G,X)$ with its $G(\mathbb{A}_f)$-action. It is canonical if each $M_K(G,X)$ is canonical.

Examples: SV defined by tori.

char $k=0$, $\{0\text{-dim varieties}/k\}\longrightarrow\{\text{finite sets with continuous } Gal(\bar{k}/k)\text{-action}\}$

$$V\longmapsto V(\bar{k})$$

Given finite set, viewed as variety over $\bar{\mathbb{Q}}$, a model over $E$ is just a continuous $Gal(\bar{\mathbb{Q}}/E)$ action on it.

$T/\mathbb{Q}$ torus, $h: \mathbb{S} \to T_{\mathbb{R}}$, $(T, h)$ SD, $E = E(T, h)$ field of definition of $\mu_h$. $Sh_K(T, h)$ is finite set, $\sigma[x, a]_K = [x, r_x(s)a]_K$, $\mathrm{art}_E\, s = \sigma$ gives a continuous action of $\mathrm{Gal}(E^{ab}/E)$ on $Sh_K(T, h)$, hence a canonical model of $Sh_K(T, h)$ over $E$. $\mathrm{Aut}(\mathbb{C}/E) \twoheadrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}/E) \twoheadrightarrow \mathrm{Gal}(E^{ab}/E)$.

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$(E, \Phi)$ CM type, $(T, h_\Phi)$ as in the example before. $E(T, h_\Phi) = E^*$,

$r(T, \mu_\Phi): (\mathbb{G}_m)_{E^*/\mathbb{Q}} \to (\mathbb{G}_m)_{E/\mathbb{Q}}$ is just $N_{E^*}$.

For simplicity, assume $E^*$ contains all conjugates of $E$ and fix $\mathrm{Hom}(E, E^*) = \mathrm{Hom}(E, \bar{\mathbb{Q}})$.

$$r(\mathbb{Q}): \quad E^{*,\times} \longrightarrow (E \otimes_{\mathbb{Q}} E^*)^\times \xrightarrow{\sim} (E^*)^{\Phi, \times} \times (E^*)^{\bar{\Phi}, \times}$$

$$z \longmapsto \sum a_i \otimes b_i \longmapsto \left(\sum \varphi(a_i) b_i\right)_\varphi = (z, \dots, z, 1, \dots, 1)$$

then $\sum \varphi(a_i) b_i = \begin{cases} z & \varphi \in \Phi \\ 1 & \varphi \in \bar{\Phi} \end{cases}$

$$Nm_{E^*/\mathbb{Q}}\left(\sum a_i \otimes b_i\right) = \prod_{\rho: E^* \to \bar{\mathbb{Q}}} \left(\sum a_i \otimes \rho(b_i)\right) \longmapsto \prod_{\rho: E^* \to \bar{\mathbb{Q}}} \left(\sum \varphi(a_i) \rho(b_i)\right)_\varphi = (\varphi(e))_\varphi$$

ETS $e = \prod_{\varphi \in \Phi} \varphi^{-1} Nm_{E^*/\varphi E}\, z$, fix any $\psi \in \mathrm{Hom}(E, E^*)$.

For any $\rho: E^* \to \bar{\mathbb{Q}}$, $\exists!\ \psi_\rho \in \mathrm{Hom}(E, E^*)$ s.t. $\rho \circ \psi_\rho = \varphi$.

Then
$$\prod_{\rho: E^* \to \bar{\mathbb{Q}}} \sum \varphi(a_i) \rho(b_i) = \prod_{\rho: E^* \to \bar{\mathbb{Q}}} \rho\left(\sum \psi_\rho(a_i) b_i\right) = \prod_{\psi \in \Phi} \prod_{\substack{\rho: E^* \to \bar{\mathbb{Q}} \\ \psi_\rho = \psi}} \rho(z)$$

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$\prod_{\substack{\rho: E^* \to \bar{\mathbb{Q}} \\ \psi_\rho = \psi}} \rho(z)$ is fixed by $\mathrm{Gal}(\bar{\mathbb{Q}}/\varphi E)$ hence lies in $\varphi E$.

Note $\varphi \psi^{-1} Nm_{E^*/\psi E}\, z = \varphi \psi^{-1} \prod_{\substack{\rho: E^* \to \bar{\mathbb{Q}} \\ \rho\psi = \psi}} \rho(z)$. Choose $\sigma: \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}$ s.t. $\sigma \psi = \varphi$.

Then
$$\varphi \psi^{-1} \prod_{\substack{\rho: E^* \to \bar{\mathbb{Q}} \\ \rho\psi = \psi}} \rho(z)$$

$$= \varphi \psi^{-1} \sigma^{-1} \prod_{\substack{\rho: E^* \to \bar{\mathbb{Q}} \\ \rho\psi = \psi}} \sigma\rho(z) = \prod_{\substack{\rho: E^* \to \bar{\mathbb{Q}} \\ \psi_\rho = \psi}} \rho(z).$$

Alternatively, let $\iota$ be the unique involution on $E$ fixing its maximal totally real subfield. Then $(E, \iota)$ is a semisimple $\mathbb{Q}$-alg. with positive involution, $(V, \psi)$ be symplectic module where $V = E$, $\psi: E \times E \to \mathbb{Q}$ is of the form $\psi(u, v) = tr_{E/\mathbb{Q}}\, f u v^{\iota}$ where $f \in E$ s.t. $f^{\iota} = -f$ and $i\varphi(f) > 0$ for $\forall \varphi \in \Phi$ by weak approximation. $G(\mathbb{Q}) = E^{\times}$, $G = T = (\mathbb{G}_m)_{E/\mathbb{Q}}$.
$V \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{C}^{\Phi}$, the corresponding $h_{\Phi}: \mathbb{S} \to G$ is then just the map $\mathbb{C}^{\times} \longrightarrow \mathbb{C}^{\Phi, \times}$, $z \longmapsto (z, \cdots, z)$ and the corresponding $\mu_{\Phi}$ is just $\mathbb{C}^{\times} \longrightarrow \mathbb{C}^{\Phi, \times} \times \mathbb{C}^{\bar{\Phi}, \times}$, $z \longmapsto (z, \cdots, z, 1, \cdots, 1)$ which is defined over $E^*$. Hence we get a map $E^{*, \times} \to T(E^*)$.
The map $\mu_{\Phi}$ gives the action of $E^*$ on $V^{-1,0}$, and after composing $Nm_{E^*/\mathbb{Q}}$, is exactly $\det_E(\cdot \mid V^{-1,0})$, which is the reflex norm.

Let $K \subset T(\mathbb{A}_f)$ open compact, then $Sh_K(T, h_{\Phi})$ classifies isom. classes of $(A, i, \eta K)$, $(A, i)$ AV/$\mathbb{C}$ of CM type $(E, \Phi)$ and $\eta$ is an $E \otimes \mathbb{A}_f$-linear isom. $V(\mathbb{A}_f) \to V_f A$. An isom. $(A, i, \eta K) \to (A', i', \eta' K)$ is an $E$-linear isom. $A \to A'$ in $AV^{\circ}(\mathbb{C})$ sending $\eta K$ to $\eta' K$.

Let $V = E$, $(A, i)$ of CM type $(E, \Phi)$ $\Rightarrow$ $\exists$ $E$-isomorphism $a: H_1(A, \mathbb{Q}) \to V$ sending $h_A$ to $h_{\Phi}$, $V(\mathbb{A}_f) \xrightarrow{\eta} V_f(A) \xrightarrow{a} V(\mathbb{A}_f)$ is $E \otimes \mathbb{A}_f$ linear, giving rise to $g \in (E \otimes \mathbb{A}_f)^{\times} = T(\mathbb{A}_f)$. Then send $(A, i, \eta K)$ to $[g]$.

To see the moduli interpretation agrees with the PEL moduli interpretation defined by $(E, *, V = E, \psi)$ where $\psi$ $\psi'$ is s.t. the complex structure on

$V \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{C}^{\Phi}$ is given by $z \cdot (z_i) = (\bar{z} z_i)$ $z \cdot (z_i) = (z z_i)$.

Consider the map $\alpha : (A, i, s, \eta K) \longmapsto (A, i, \eta K)$.

As $\exists$ $E$-linear $a : H_1(A, \mathbb{Q}) \to V$ sending $h_A$ to $h_x$, for any $b \in E$,

$\mathrm{tr}_{\mathbb{C}}(b \mid T_0 A) = \mathrm{tr}_{\mathbb{C}}(b \mid V^{-1,0}) = \sum_{\varphi \in \Phi} \varphi(b) \Rightarrow (A, i)$ is of CM type $(E, \Phi)$

$\sum_{\varphi \in \Phi} \overline{\varphi(b)}$ $(E, \bar{\Phi})$

i.e. the CM type of $(A, i)$ depends on the symplectic form on $V$, hence the complex structure on $V \otimes \mathbb{R}$ by the map $a$.

Clearly the map $\alpha$ is well-defined on isom. classes.

Surj: for any $(A, i, \eta K)$ choose $a : H_1(A, \mathbb{Q}) \xrightarrow{\sim} V$ $E$-isom. sending $h_A$ to $h_x$ and let $s$ on $H_1(A, \mathbb{Q})$ be the pullback of $\psi$. Then $V(\mathbb{A}_f) \xrightarrow{\eta} V_f A \xrightarrow{a} V(\mathbb{A}_f)$ is given by $g \in (E \otimes \mathbb{A}_f)^{\times}$, hence $s(\eta x, \eta y) = \psi(a\eta x, a\eta y) = \psi(gx, gy) = \mu(g)\psi(x, y)$ for $\mu(g) \in \mathbb{A}_f^{\times}$. Thus $(A, i, s, \eta K)$ lies in the preimage.

Inj: Suppose $(A, i, s, \eta K)$ and $(A', i', s', \eta' K)$ have same image, then $\exists\, \alpha : A \to A'$ $E$-isogeny sending $\eta K$ to $\eta' K$. Then $\alpha$ induces $\alpha : H_1(A', \mathbb{Q}) \to H_1(A, \mathbb{Q})$ being $E$-isom. Using $a$ and $a'$ to identify $\alpha : V \to V$ $E$-isom. $\Rightarrow \exists\, e \in E^{\times}$, $\alpha v = ev$.

Then $s(\alpha v, \alpha w) = \delta \psi(\alpha v, \alpha w) = \delta \mu(e) \psi(v, w) = \delta' \mu(e) s'(v, w)$, $\delta, \delta', \mu(e) \in \mathbb{Q}^{\times}$

Hence $\alpha : (A, i, s, \eta K) \xrightarrow{\sim} (A', i', s', \eta' K)$.

We can change from $\mathbb{C}$ to $\bar{\mathbb{Q}}$.

$Gal(\bar{\mathbb{Q}}/E^*)$ acts on $(A, i, \eta K)$ by $\sigma(A, i, \eta K) = (\sigma A, {}^{\sigma}i, {}^{\sigma}\eta K)$ where

${}^{\sigma}\eta : V(A_f) \xrightarrow{\eta} V_f A \xrightarrow{\sigma} V_f \sigma A$. As $\sigma$ fixes $E^*$, $(\sigma A, {}^{\sigma}i)$ is again of

CM type $(E, \Phi)$. This action is natural in the sense of moduli.

$Gal(\bar{\mathbb{Q}}/E^*)$ acts on $Sh_K(T, h_\Phi)$ by $\sigma[g] = [r_{h_\Phi}(s)g]$, $art_{E^*}(s) = \sigma|_{E^{*,ab}}$

Prop. The map $(A, i, \eta K) \longmapsto [a \circ \eta]$ is compatible with $Gal(\bar{\mathbb{Q}}/E^*)$-action.

Pf. $(A, i, \eta K) \longmapsto [a \circ \eta]$, $a: H_1(A, \mathbb{Q}) \xrightarrow{\sim} V$, $\sigma \in Gal(\bar{\mathbb{Q}}/E^*)$.

Main theorem of CM $\Rightarrow \exists!$ $E$-linear isogeny $\alpha: A \to \sigma A$, $\alpha(N_{E^*}(s)x) = \sigma x$,

$\forall x \in V_f A$.

Then $\sigma(A, i, \eta K) \longmapsto [a \circ V_f\alpha^{-1} \circ \sigma \circ \eta]$ and $V_f \alpha^{-1} \circ \sigma = N_{E^*}(s) = r_{h_\Phi}(s)$

hence $[a \circ V_f\alpha^{-1} \circ \sigma \circ \eta] = [r_{h_\Phi}(s) \cdot (a \circ \eta)]$. ///