Uniqueness of canonical models.

Prop. Let Ω alg. closed field of char 0, $k \subset \Omega$ subfield. If V, W are varieties over k, then any morphism $V_\Omega \to W_\Omega$ commuting with the action of $\mathrm{Aut}(\Omega/k)$ on $V(\Omega)$ and $W(\Omega)$ arises from a unique morphism $V \to W$.

Cor. A variety V over k is uniquely determined up to unique isom. by V_Ω and the action of $\mathrm{Aut}(\Omega/k)$ on $V(\Omega)$.

(G, X) SD.

Lemma. (For every finite extension $L/E(G,X)$ in $\mathbb{C}$) there exists a special point x_0 (s.t. $E(x_0)$ is linearly disjoint from L).

Sketch of proof.

$\mathfrak{g}$ Lie algebra of a semisimple algebraic group G over char 0 alg. closed field K, $\{x \in \mathfrak{g} \mid x \text{ regular}\}$ connected open dense in $\mathfrak{g}$.

Maximal tori in G $\overset{\text{Lie}}{\Longleftrightarrow}$ Cartan subalg. in $\mathfrak{g}$ $\Longleftrightarrow$ centralizers of regular elements

(G, X) SD, $x \in X$, T maximal torus in $G_{\mathbb{R}}$ containing $h_x(\mathbb{C})$

$\Rightarrow T$ centralizer of regular λ in $\mathrm{Lie}(G_{\mathbb{R}})$

Choose $\lambda_0 \in \mathrm{Lie}\, G$ sufficiently close to λ in $\mathrm{Lie}\, G_{\mathbb{R}}$, λ_0 regular.

its centralizer T_0 in G is a maximal torus, $T_{0,\mathbb{R}} \simeq gTg^{-1}$ for some $g \in G(\mathbb{R}) \Rightarrow h_{gx}(\mathbb{S}) = ghg^{-1}(\mathbb{S}) \subset T_{0,\mathbb{R}} \Rightarrow gx$ is special. ///

Lemma. $\forall\, x \in X$, $\{[x, a]_K \mid a \in G(\mathbb{A}_f)\}$ is dense in $Sh_K(G, X)$.

Pf. $Sh_K(G, X)(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$

real approximation $\Rightarrow$ $G(\mathbb{Q})x$ dense in X

$\Rightarrow$ $G(\mathbb{Q})x \times G(\mathbb{A}_f)$ dense in $X \times G(\mathbb{A}_f)$

$\Rightarrow$ its image dense in $Sh_K(G, X)(\mathbb{C})$ ///

for complex topology hence for Zariski

$g \in G(\mathbb{A}_f)$, $K, K' \subset G(\mathbb{A}_f)$ compact open, $g^{-1}Kg \subset K'$.

Recall $T(g)$: $Sh_K(\mathbb{C}) \to Sh_{K'}(\mathbb{C})$ $\Rightarrow$ $T(g)$ morphism of varieties $/\mathbb{C}$

$[x, a]_K \longmapsto [x, ag]_{K'}$

Thm. If Sh_K, $Sh_{K'}$ have canonical models over $E(G, X)$ then $T(g)$ is defined over $E(G, X)$.

Pf. Let $x_0 \in X$ special, $E(G, X) \subset E(x_0)$, $\sigma \in \mathrm{Aut}(\mathbb{C}/E(x_0))$, $s \in \mathbb{A}^{\times}_{E_0}$ s.t. $\mathrm{art}(s) = \sigma|_{E(x_0)^{ab}}$.

$$\begin{tikzcd}
{[x_0, a]_K} \arrow[r, mapsto, "T(g)"] \arrow[d, mapsto, "\sigma"] & {[x_0, ag]_{K'}} \arrow[d, mapsto] \\
{[x_0, r_{x_0}(s)a]_K} \arrow[r, mapsto, "T(g)"] & {[x_0, r_{x_0}(s)ag]_{K'}}
\end{tikzcd}$$

$\Rightarrow T(g)$ commutes with σ on $\{[x_0, a]_K\}$

$\Rightarrow T(g)$ commutes with $\mathrm{Aut}(\mathbb{C}/E(x_0))$ on $Sh_K(\mathbb{C})$

$\Rightarrow T(g)$ commutes with $\mathrm{Aut}(\mathbb{C}/E(G,x))$ on $Sh_K(\mathbb{C})$ as $\{\mathrm{Aut}(\mathbb{C}/E(x_0))\}_{x_0}$ generates $\mathrm{Aut}(\mathbb{C}/E(G,x))$ ///

Thm.

(a) A canonical model of $Sh_K(G, X)$ is unique up to unique isom.

(b) If for enough K open compact in $G(\mathbb{A}_f)$, $Sh_K(G, X)$ has a canonical model then so does $Sh(G, X)$, and it is unique up to unique isom.

Pf.

(a) $K = K'$, $g = 1$.

(b) all $T(g)$ defined over $E(G, x)$. ///

RMK. Let $a: (G, x) \longrightarrow (G', x')$ morphism of SD, $Sh(G, x)$ and $Sh(G', x')$ have canonical models, then $Sh_a: Sh(G, X) \longrightarrow Sh(G', x')$ is defined over $E(G, X) \cdot E(G', x')$.

$K\backslash G(\mathbb{A}_f)/K$ - action on $\pi_0(Sh_K)$ hence $H^0(Sh_K)$ factors through $T(\mathbb{A}_f)/K$. The action is trans. hence trivial auto. rep. indexed part in H^0 is 1-dim. For Gal, need r surj.

Canonical model of $Sh_K(G, x) \Rightarrow \mathrm{Aut}(\mathbb{C}/E(G,x))$ acts on $\pi_0(Sh_K)$

G^{der} simply connected $\Rightarrow \pi_0(Sh_K) \simeq T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_f)/\nu(K)$, $Y = T(\mathbb{R})^+\backslash T(\mathbb{R})$.

$h = \nu \circ h_x$, μ_h defined over $E(G, x) \Rightarrow r(\mu_h): \mathbb{A}^\times_{E(G,x)} \longrightarrow T(\mathbb{A}_\mathbb{Q})$

$\sigma \in \mathrm{Aut}(\mathbb{C}/E(G,x))$, $s \in \mathbb{A}^\times_{E(G,x)}$, $\mathrm{art}(s) = \sigma|_{E(G,x)^{ab}}$, then

$\sigma[y, a]_K = [r(s)_\infty y, r(s)_f a]_K$ (canonical model of 0-dim SV)