Existence of canonical models

The canonical models are still smooth quasi-projective varieties.

Descent of the base field

Let Ω alg. closed field of char 0, $k \subset \Omega$ subfield. $A = \mathrm{Aut}(\Omega/k)$.

Def. An action of A on $V(\Omega)$ of variety V/Ω is regular if for every $\sigma \in A$, $\exists$ isomorphism $f_\sigma : \sigma V \longrightarrow V$ s.t. on Ω points the map $(\sigma V)(\Omega) \longrightarrow V(\Omega)$ sends σP to $\sigma \cdot P$.

RMK. If V smooth then $f_\sigma(\Omega)$ isom. $\Rightarrow$ f_σ isom.

$\{f_\sigma\}$ satisfy $f_\sigma \circ \sigma f_\tau = f_{\sigma\tau}$. Such families are called descent system.

Def. An action of A on $V(\Omega)$ is continuous if $\exists\ L \subset \Omega$ finitely generated over k and a model V_0 of V on L s.t. the action of $\mathrm{Aut}(\Omega/L)$ on $V(\Omega)$ defined by V_0 is the same as A. V_0 splits the descent system

descent datum: continuous + regular action.

Prop. A regular action of A on $V(\Omega)$ is continuous if $\exists\ P_1, \dots, P_n \in V(\Omega)$ s.t.

$$* \begin{cases} \{\sigma \in \mathrm{Aut}(V),\ \sigma P_i = P_i\} = id \\ \exists\ L \subset \Omega \text{ finitely generated over } k,\ \forall \sigma \text{ fixes } L,\ \sigma \text{ fix } P_i. \end{cases}$$

Pf. Choose (V_0, φ) any model over L, $\varphi^{-1} P_i \in V_0(L)$.

$\forall \sigma$ fixes L, $f_\sigma = \varphi \circ (\sigma\varphi)^{-1} \Rightarrow \varphi(\sigma P) = \sigma \cdot \varphi(P)$. ///

Descent effective $\Leftarrow$ the functor $Sch/k \longrightarrow Sets$ is rep'l.

$S \longmapsto (W/S \text{ variety},\ \varphi: W\times_k \Omega \xrightarrow{\sim} V\times_k S)$

Thm. If V/Ω is quasi-projective with regular continuous $Aut(\Omega/k)$ action $\cdot$ on $V(\Omega)$ then $(V,\cdot)$ arises from a model over k.

Cor. $(V,\cdot)$ has a model over k if V is quasi-projective, $Aut(\Omega/k)$ acts on $V(\Omega)$ regular and $\exists P_1,\dots,P_n \in V(\Omega)$ satisfying $(*)$.

$\exists$ model V_0 of V over $K\subset\Omega$ finitely generated over $\bar{k}$ splitting $(V,\cdot)$.

(Ω/k of infinite transcendental degree) Pick $\sigma\in Aut(\Omega/\bar{k})$, K, σK linearly disjoint over $\bar{k}$, σV_0 is a splitting model of $(V,\cdot)$ over σK

$\Rightarrow (V,\cdot)$ has splitting model over $\bar{k}$

$\varphi: V_{0,\Omega} \xrightarrow{\sim} V$, $\forall Q\in V_0(\Omega)$, $\alpha\in Aut(\Omega/K)$, $\varphi(\alpha Q) = \alpha\cdot\varphi(Q)$.

$$\begin{tikzcd} K \arrow[r, "\sigma"] \arrow[d, "i_K"'] & \sigma K \arrow[d, "i_{\sigma K}"] \\ \Omega \arrow[r, "\sigma"] & \Omega \end{tikzcd}$$

$\Rightarrow (\sigma V_0)_\Omega = \sigma V_{0,\Omega}$.

$\beta\in Aut(\Omega/\sigma K) \Rightarrow \sigma^{-1}\beta\sigma \in Aut(\Omega/K)$.

$P\in(\sigma V_0)(\Omega)$, $P=\sigma Q$, $Q\in V_0(\Omega)$

$\sigma V_{0,\Omega} \xrightarrow{\sigma\varphi} \sigma V \xrightarrow{f_\sigma} V$,

$$\begin{aligned} (f_\sigma\circ\sigma\varphi)(\beta P) &\qquad \beta\cdot(f_\sigma\circ\sigma\varphi)(P) \\ \| &\qquad\qquad \| \\ f_\sigma(\sigma\varphi(\sigma\sigma^{-1}\beta\sigma Q)) &\qquad \beta\cdot f_\sigma(\sigma\varphi(\sigma Q)) \\ \| &\qquad\qquad \| \\ \sigma\cdot\varphi(\sigma^{-1}\beta\sigma Q) &= (\beta\sigma)\cdot\varphi(Q) \end{aligned}$$

Quasi-projective + descent datum $\Rightarrow (V,\cdot)$ split over k

Descending schemes is not always effective.

Review of local systems and families of abelian varieties

S connected topological manifold, a local system of $\mathbb{Z}$-modules on S is a sheaf on S locally isom. to constant sheaf $\underline{\mathbb{Z}}^n$.

Prop. The functor is an equivalence of categories

$$\{\text{local system of } \mathbb{Z}\text{-modules on } S\} \longrightarrow \{\text{finite free } \mathbb{Z}\text{-modules with } \pi_1(S,s) \text{ action}\}$$

$$F \longmapsto (F_s,\ \gamma \text{ acts by } \varphi_{\gamma,1} \text{ where}$$

$$\varphi_\gamma : \underline{F_s} \xrightarrow{\sim} \gamma^* F \text{ unique trivialization})$$

S complex manifold, F local system of $\mathbb{Z}$-modules on S

$\forall s \in S$, $\exists$ Hodge structure h_s on $F_s \otimes_{\mathbb{Z}} \mathbb{R}$

$(F, \{h_s\})$ is called a variation of integral Hodge structure on S if on every $U \subset S$ open trivialization of F, $(F \otimes \mathbb{R}, \{h_s\})$ is a VHS.

A polarization of integral VHS $(F, \{h_s\})$ is a pairing $\psi : F \times F \to \mathbb{Z}$ s.t. ψ_s is polarization of (F_s, h_s) for every s.

Thm. Let $V/\mathbb{C}$ smooth variety. The functor is an equivalence of categories

$$\{AV/V\} \longrightarrow \{\text{polarizable integral VHS of type } (-1,0), (0,-1) \text{ on } V^{an}\}$$

$$(f : A \to V) \longmapsto (R^1 f_* \mathbb{Z})^\vee$$

The Siegel modular variety.

(V, ψ) symplectic space $/\mathbb{Q}$, $(G, X) = (GSp(\psi), X(\psi))$ SD.

$K \subset G(\mathbb{A}_f)$ compact open, $\mathcal{M}_K$ set of triples $(A, s, \eta K)$ where

- A AV$/\mathbb{C}$
- s alternating form on $H_1(A, \mathbb{Q})$ s.t. $\pm s$ is a polarization
- η isom. $V(\mathbb{A}_f) \xrightarrow{\sim} V_f A$ sending ψ to a multiple of s

$\mathcal{M}_K \longrightarrow Sh_K(\mathbb{C})$ whose fibres are isom. classes.

$E(G, X) = \mathbb{Q}$ as V has a symplectic base over $\mathbb{Q}$.

Def. A CM algebra is a finite product of CM fields. A AV$/\mathbb{C}$ is CM if $\exists$ CM alg. E and $E \longrightarrow \mathrm{End}^\circ A$ s.t. $H_1(A, \mathbb{Q})$ is free E-mod of rank 1.

Write $E = E_1 \times \cdots \times E_m$, E_i CM field, then A is isogenous to $A_1 \times \cdots \times A_m$, A_i AV$/\mathbb{C}$ of CM type (E_i, Φ_i).

Prop. $A/\mathbb{C}$ AV is CM iff $\exists$ torus $T \subset GL(H_1(A, \mathbb{Q}))$, $h_A(\mathbb{C}^\times) \subset T(\mathbb{R})$.

Sketch of proof.

May assume A simple.

TFAE.

(a) A is of CM type

$\dim_{\mathbb{Q}} \mathrm{End}^\circ A \le \dim_{\mathbb{Q}} H_1(A, \mathbb{Q})$: (a) $\Leftrightarrow$ (b)

(b) $\mathrm{End}^\circ A$ is a CM field of degree $2\dim A$ over $\mathbb{Q}$

(c) $\exists$ torus $T \subset GL(H_1(A,\mathbb{Q}))$ s.t. $h_A(\mathbb{C}^\times) \subset T(\mathbb{R})$

(c) $\Rightarrow$ (d)

(d) $\exists$ torus $T \subset GL(H_1(A,\mathbb{Q}))$ s.t. $\mu_A(\mathbb{C}^\times) \subset T(\mathbb{C})$.

(b) $\Rightarrow$ (c) : $E \otimes \mathbb{R}$ action on $H_1(A, \mathbb{R})$ preserves Hodge structure $\Rightarrow h_A(\mathbb{C}^\times)$ commute with $E \otimes \mathbb{R}$ in $\mathrm{End}(H_1(A,\mathbb{R})) \Rightarrow h_A(\mathbb{C}^\times) \subset (E\otimes\mathbb{R})^\times = (\mathbb{G}_m)_{E/\mathbb{Q}}(\mathbb{R})$.

maximal torus

(d) $\Rightarrow$ (b) : $\mathrm{End}^\circ A$ is subalg. of $\mathrm{End}(H_1)$ preserving Hodge structure

$\Rightarrow$ commuting with $\mu_A(\mathbb{C}^\times) \subset T(\mathbb{C})$

$\Rightarrow \mathrm{Cent}_T \mathrm{End}(H_1) \subset \mathrm{End}^\circ A$

$\Rightarrow \mathrm{Cent}_{T_{\mathbb{C}}} \mathrm{End}(H_1 \otimes \mathbb{C}) \subset \mathrm{End}^\circ A \otimes \mathbb{C}$

projections to eigen base

$T_{\mathbb{C}}$ split torus $\Rightarrow T_{\mathbb{C}}$ is diagonalizable in $\mathrm{End}(H_1 \otimes \mathbb{C}) \Rightarrow \mathrm{End}^\circ A \otimes \mathbb{C}$ contains a $\mathbb{C}$-alg. of deg $2\dim A$ $\Rightarrow \dim_{\mathbb{Q}} \mathrm{End}^\circ A = 2\dim A$.

Then define the involution on $\mathrm{End}^\circ A$ by Riemann form ψ and Rosati involution. Check it has a totally real fixed field F and $\mathrm{End}^\circ A$ is totally imaginary over F. ///

Cor. If $(A, s, \eta K)$ maps to $[x, a]_K$, then A CM $\Leftrightarrow$ x special.

Define $\mathrm{Aut}(\mathbb{C})$ action on $\mathcal{M}_K$ as follows: $\sigma \in \mathrm{Aut}(\mathbb{C})$, $(A, s, \eta K) \in \mathcal{M}_K$

$\sigma \cdot (A, s, \eta K) = (\sigma A, {}^{\sigma}s, {}^{\sigma}\eta K)$ where

- as $s \in H^2(A, \mathbb{Q})$ is Hodge tensor. $s \in H^2(A, \mathbb{Q}) \cap H^{1,1} \overset{\text{Lefschetz}}{=\!=} \mathrm{Im}\,\mathrm{Pic}\,A \otimes \mathbb{Q}$

hence $s = r[D]$ for some $r \in \mathbb{Q}^\times$ and D divisor on A, ${}^{\sigma}s = r[\sigma D]$.

$\pm {}^{\sigma}s$ is still a polarization for $H_1(\sigma A, \mathbb{Q})$.

- ${}^{\sigma}\eta$: $V(\mathbb{A}_f) \xrightarrow{\eta} V_f A \xrightarrow{\sigma} V_f \sigma A$.

Such action is natural in terms of moduli interpretation.

Prop. Suppose Sh_K has a model M_K over $\mathbb{Q}$ s.t. the map $\mathcal{M}_K \to M_K(\mathbb{C})$ commutes with the actions of $\mathrm{Aut}(\mathbb{C})$, then M_K is canonical.

If (T, x) special, $a \in G(\mathbb{A}_f)$, $(A, s, \eta K) \longmapsto [x, a]_K$, A is of CM type (E, Φ), $E(x) = E^*$, $r_x = N_{E^*}$. $\sigma \in \mathrm{Aut}(\mathbb{C}/E(x))$, $\sigma[x, a]_K = [x, r_x(s) a]_K$.

Define the action of $\mathrm{Aut}(\mathbb{C})$ on $\mathcal{M}_K/\simeq \;\simeq Sh_K(\mathbb{C})$ by the action on $\mathcal{M}_K$ defined above, to show existence of canonical model, enough to show such an action satisfies the descent conditions.

- Sh_K is quasi-projective.
- The action is regular.

To show the map $\sigma Sh_K(\mathbb{C}) \xrightarrow{f_\sigma} Sh_K(\mathbb{C})$, $\sigma P \longmapsto \sigma \cdot P$ is regular, by Borel's thm, enough to show f_σ is holomorphic.

For $\forall\, \varepsilon \in \pi_0(Sh_K)$, let Sh_K^ε be the corresponding connected component

then $Sh_K^\varepsilon = \Gamma_\varepsilon \backslash X^+$.

Consider

$$\begin{tikzcd} U \arrow[r, "\tilde{\sigma}"] \arrow[d, "\pi"'] & X^+ \arrow[d] \\ \sigma(\Gamma_\varepsilon \backslash X^+) \arrow[r, "f_\sigma"] & \Gamma_{\sigma\varepsilon} \backslash X^+ \\ \sigma P \arrow[r, mapsto] & \sigma \cdot P \end{tikzcd}$$

where π is universal covering.

Fix a lattice Λ in V stable under Γ_ε, then get a local system of $\mathbb{Z}$-mods M on $\Gamma_\varepsilon \backslash X^+$ and actually is a polarized integral VHS, hence coming from a polarized AV $f: \mathcal{A} \to X^+(\Gamma_\varepsilon)$. Apply σ, we get a polarized AV $\sigma f: \sigma\mathcal{A} \to \sigma X^+(\Gamma_\varepsilon)$ and $(R^1(\sigma f)_* \mathbb{Z})^\vee$ is a polarized integral VHS on $\sigma(\Gamma_\varepsilon \backslash X^+)$. Pull it back to U and tensor with $\mathbb{Q}$. We get a polarized rational VHS over U, whose underlying local system can be identified with $\underline{V}$ naturally so that each $u \in U$ defines a complex structure on V positive for ψ, i.e. a point $x \in X^+$. The map $\tilde{\sigma}: u \longmapsto x$ makes the diagram commute and is holomorphic as it comes from VHS. Hence f_σ is holomorphic.

- As $\mathrm{Aut}(Sh_K)$ is finite and $\{[x, g], g \in G(\mathbb{A}_f)\}$ is dense, we may choose x special and $g_1, \dots, g_n \in G(\mathbb{A}_f)$ s.t. the only automorphism of Sh_K fixing $[x, g_i]$ is identity. The main theorem of CM tells that $\sigma \cdot [x, g_i] = [x, g_i]$ for all σ fixing some fixed finite extension of $E(x)$.

Simple PEL SV of type (A) or (C).

The proof is similar. $Sh_K(G, X)(\mathbb{C})$ classifies $(A, i, s, \eta K)$.

- σ fix $E(G, X) \Rightarrow \sigma(A, i, s, \eta K) \in Sh_K(\mathbb{C})$.
- special points $\leftrightsquigarrow$ CM AV
- Any model M_K over $E(G, X)$ for which the action of $\mathrm{Aut}(\mathbb{C}/E(G,X))$ on $M_K(\mathbb{C}) = Sh_K(\mathbb{C})$ agrees with the action on the quadruples is canonical.

SV of Hodge type

The main problem is to define ${}^{\sigma}s$ for s Hodge tensor, which may need Hodge conjecture. Deligne showed existence (and uniqueness) by passing to $\mathbb{A}_f$-coefficient.

Alternatively, if $(G, X) \hookrightarrow (G', X')$ is an inclusion of SD, then existence of canonical model for $Sh(G', X')$ will imply the existence for $Sh(G, X)$.

SV of abelian type

Deligne defined the notion of a canonical model of a connected SV and proved that $Sh(G, X)$ has canonical model $\Leftarrow$ $Sh^\circ(G^{der}, X^+)$ has canonical model.

- (G_1, X_1), (G_2, X_2) SD, $(G_1^{der}, X_1^+) = (G_2^{der}, X_2^+)$ then one of $Sh(G_i, X_i)$ has canonical model, so will both.
- (G_i, X_i) conn. SD, $Sh^\circ(G_i, X_i)$ has canonical model M_i then the conn. SD $(\prod G_i, \prod X_i)$, $Sh^\circ(\prod G_i, \prod X_i)$ has canonical model $\prod M_i$.
- $(G_1, X_1) \to (G_2, X_2)$ isogeny of conn. SD, $Sh^\circ(G_1, X_1)$ has canonical model, so does $Sh^\circ(G_2, X_2)$.

Thus enough to show conn. SD (H, X^+) of primitive abelian type, $Sh^\circ(H, X^+)$ has a canonical model. But then $\exists$ SD (G, X) of Hodge type s.t. $(G^{der}, X^+) = (H, X^+)$ and the existence for Hodge type implies the result.

General SV.

Milne proved Langland's conjugacy conjecture $\Rightarrow$ existence of canonical models.

Langlands formulated the conjecture when studying zeta functions of SV.

Milne's proof relies on results for SV defined by groups of type A_1 over totally real fields.

RMK. This definition for canonical model is rigid, i.e. ours is the only condition for which the canonical model can exist.