Abelian varieties over finite fields

Goal: understand the zeta function of SV and Galois representation in its cohomology.

$\rightsquigarrow$ understand the points of canonical model reducing primes of $E(G, X)$ in terms of abelian varieties / motives over finite fields with extra structure

Langlands & Rapoport: introduce a category of fake abelian varieties / motives over algebraic closure of a finite field which looks like the true category and can be plugged into the trace formula, hence can be used to compute the local zeta function.

Semisimple categories

An object of an abelian category is simple if it is nonzero and has no proper nontrivial subobjects.

F field, a semisimple F-category is an abelian category, every object is a direct sum of simple objects and $\mathrm{Hom}(x, y)$ are finite dimensional F-v.s. and composition is F-bilinear.

Such a category M is described up to equivalence by

(a) The set $\Sigma(M)$ of isom. classes of simple objects

(b) For each $\sigma \in \Sigma(M)$, the isom. classes $[D_\sigma]$ of the division F-alg. $\mathrm{End}(\sigma)$

We call $(\Sigma(M), ([D_\sigma])_{\sigma\in\Sigma(M)})$ the numerical invariants of M.

F-alg: ring A containing F in center, finite dimensional as F-v.s.

Central F-alg: F-alg. A with center F

Ex.

(a) F alg. closed or finite, every central division alg. is isom. to F

(b) $F=\mathbb{R}$, every central division alg. is isom. to $\mathbb{R}$ or $\mathbb{H}$

(c) F p-adic field, π uniformizer

L/F unramified extension of deg n, σ Frob generator of $Gal(L/F)$

$\forall\ 1\le i\le n$, $D_{i,n}=Le_0\oplus\cdots\oplus Le_{n-1}$, $e_j\cdot c=\sigma^j c\cdot e_j$, $c\in L$

$$e_j\cdot e_l=\begin{cases} e_{j+l} & j+l\le n-1\\ \pi^i e_{j+l-n} & j+l\geq n\end{cases}$$

$\Rightarrow e_0=1$, $e_1=a$, $D_{i,n}=L\oplus La\oplus\cdots\oplus La^{n-1}$, $a^n=\pi^i$, $a\cdot c\cdot a^{-1}=\sigma(c)$

$\Rightarrow D_{i,n}$ is central simple F-alg., and is division $\Leftrightarrow (i,n)=1$

every central division F-alg. is isom. to $D_{i,n}$ for exactly one relatively prime pair (i,n).

The isom. classes of central division / simple F-alg. form a group $Br(F)$.

Ex.

(a) F alg. closed or finite, $Br(F) = 0$

(b) $Br(\mathbb{R}) = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$

(c) F p-adic field, $Br(F) = \mathbb{Q}/\mathbb{Z}$

(d) F number field, v place, $\mathrm{inv}_v : Br(F_v) \to \mathbb{Q}/\mathbb{Z}$

B central simple F-alg., $[B \otimes_F F_v]$ is trivial for all but finitely many v. and the sequence is exact

$$0 \longrightarrow Br(F) \longrightarrow \bigoplus_v Br(F_v) \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

$$B \longmapsto [B \otimes_F F_v]$$

Fact: $Br(F) \simeq H^2(F, \mathbb{G}_m) = H^2(Gal(F^s/F), F^{s,\times})$.

Ex. Let L be the completion of $\mathbb{Q}_p^{ur}$ (fraction field of $W(\overline{\mathbb{F}}_p)$) and σ be the automorphism of L inducing Frob_p on residue field. An isocrystal is a finite dimensional L-v.s. V with a σ-linear isom. $F : V \to V$.

The category Isoc of isocrystals is a semisimple $\mathbb{Q}_p$-category, $\Sigma(\mathrm{Isoc}) = \mathbb{Q}$ and $\mathrm{End}(\lambda)$ is a division alg. E^λ over $\mathbb{Q}_p$ with invariant λ.

If $\lambda \geq 0$, $\lambda = \frac{r}{s}$, $(r,s) = 1$, $s > 0$, $E^\lambda = \mathbb{Q}_p[T]/(T^r - p^s) \otimes_{\mathbb{Q}_p} L$ and if $\lambda < 0$, E^λ is the dual of $E^{-\lambda}$.

Abelian varieties.

Weil: $AV^\circ(k)$ is a semisimple $\mathbb{Q}$-category with simple AV as simple objects.

A Weil q-integer π is an alg. integer s.t. $\forall\, \rho: \mathbb{Q}[\pi] \to \mathbb{C}$, $|\rho(\pi)| = \sqrt{q}$.

Two Weil q-integers π, π' are conjugate if $\exists$ isom. $\mathbb{Q}[\pi] \to \mathbb{Q}[\pi']$.
$$\pi \mapsto \pi'$$

Thm. (Honda-Tate)

The map $A \mapsto \pi_A$ defines a bijection from $\Sigma(AV^\circ(\mathbb{F}_q))$ to the set of conjugacy classes of Weil q-integers. For any simple A, the center of $D = \mathrm{End}^\circ A$ is $F = \mathbb{Q}[\pi_A]$ and for a prime v of F

$$\mathrm{inv}_v(D) = \begin{cases} \frac{1}{2} & v \text{ real} \\ \frac{\mathrm{ord}_v \pi_A}{\mathrm{ord}_v q}[F_v : \mathbb{Q}_p] & v \mid p \\ 0 & \text{otherwise} \end{cases}$$

Moreover $2\dim A = \sqrt{[D:F]} \cdot [F:\mathbb{Q}]$.

Let $W_1(q)$ be the set of Weil q-integers in $\overline{\mathbb{Q}} \subset \mathbb{C}$, then the theorem gives a bijection $\Sigma(AV^\circ(\mathbb{F}_q)) \longrightarrow \Gamma \backslash W_1(q)$, $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

If π is Weil p^n-integer, π^m is Weil p^{nm}-integer, so we have a homomorphism $W_1(p^n) \longrightarrow W_1(p^{nm})$. Let $W_1 = \varinjlim W_1(p^n)$.
$$\pi \longmapsto \pi^m$$

For $\pi \in W_1$, define $\mathbb{Q}\{\pi\}$ to be the field of smallest degree over $\mathbb{Q}$ generated by a representative of π.

Let A be a simple AV / $\overline{\mathbb{F}}_p$, A_0 a model of A over $\mathbb{F}_q$. then $s_A(v) = \dfrac{\mathrm{ord}_v(\pi_{A_0})}{\mathrm{ord}_v\, q}$ is independent of the choice of A_0. Moreover for any $\rho: \mathbb{Q}[\pi_{A_0}] \longrightarrow \overline{\mathbb{Q}}$, the Γ-orbit of $\pi_A \in W_1$ represented by $\rho\pi_{A_0}$ depends only on A.

Thm. The map $A \longmapsto \Gamma\pi_A$ defines a bijection $\Sigma(AV^\circ(\overline{\mathbb{F}}_p)) \to \Gamma \backslash W_1$. For any simple A, the center of $D = \mathrm{End}^\circ A$ is $F = \mathbb{Q}\{\pi_A\}$ and for any v prime of F

$$\mathrm{inv}_v(D) = \begin{cases} \frac{1}{2} & v \text{ real} \\ s_A(v)\cdot [F_v : \mathbb{Q}_p] & v \mid p \\ 0 & \text{otherwise} \end{cases}$$

Tori and their representations.

Let F be a field of char 0, T/F torus, split by L/F Galois. Write $X^*(T)$ the character group of $T_{\overline{F}}$. Then to give a rep. ρ of T on F-v.s. V amounts to give $V \otimes_F L = \bigoplus_{\chi \in X^*(T)} V_\chi$ s.t. $\sigma V_\chi = V_{\sigma\chi}$ for any $\sigma \in \mathrm{Gal}(L/F)$.

Rep(T): semisimple $\mathbb{Q}$-category with $\Sigma = \Gamma \backslash N$, N f.g. free $\mathbb{Z}$-mod with continuous Γ-action, but End are all commutative.

Prop. Let $\Gamma = \mathrm{Gal}(\bar{F}/F)$. The category of rep. of T on F-v.s., $\mathrm{Rep}(T)$ is semisimple and $\underline{\Sigma(\mathrm{Rep}(T)) = \Gamma \backslash X^*(T)}$. If $V_{\Gamma\chi}$ is a simple object then $\dim V_{\Gamma\chi}$ is the order of $\Gamma\chi$ and $\underline{\mathrm{End}(V_{\Gamma\chi}) \simeq F(\chi)}$ where $F(\chi)$ is the fixed field of the subgroup $\Gamma(\chi)$ of Γ fixing χ.

RMK. For any $\chi \in X^*(T)$, $\mathrm{Hom}(F(\chi), \bar{F}) \simeq \Gamma/\Gamma(\chi)$ hence $X^*((\mathbb{G}_m)_{F(\chi)/F}) = \mathbb{Z}^{\Gamma/\Gamma(\chi)}$ and the map $\mathbb{Z}^{\Gamma/\Gamma(\chi)} \longrightarrow X^*(T)$ defines a homomorphism

$$\sum_\sigma n_\sigma \sigma \longmapsto \sum_\sigma n_\sigma \sigma\chi$$

$T \longrightarrow (\mathbb{G}_m)_{F(\chi)/F}$ hence $H^2(F, T) \to H^2(F, (\mathbb{G}_m)_{F(\chi)/F}) \overset{\text{Shapiro}}{\simeq} H^2(F(\chi), \mathbb{G}_m)$

Affine extensions.

Let F be a field of char 0, L/F Galois extension with Galois group Γ, G/F alg. group.

Consider extensions $1 \to G(L) \to E \to \Gamma \to 1$ where for each $\sigma \in \Gamma$, $e_\sigma \in E$ mapping to σ, $\exists\, a \in G(L)$ s.t. $\forall g \in G(L)$, $e_\sigma g e_\sigma^{-1} = a(\sigma g)a^{-1}$.

For example, the split extension $E_G = G(L) \rtimes \Gamma$, $(g,\tau)(h,\sigma) = (g\tau(h), \tau\sigma)$.

equivalently, $\exists \{e_\sigma\}_{\sigma\in\Gamma} \subset E$, e_σ maps to σ s.t. $\forall g \in G(L)$, $e_\sigma g e_\sigma^{-1} = \sigma g$, we have to fix such a lift as G may not be commutative. $\rightsquigarrow$ special section.

An extension is affine if for some open subgrp of Γ, the pullback is split, i.e. for some open subgrp Γ', $\exists$ lifts $\{e_\sigma\}_{\sigma\in\Gamma'} \subset E$ s.t. $e_{\sigma\tau} = e_\sigma e_\tau$.

$\{$ Affine groupoid scheme over L/F $\}$ $\sim$ $\}$ affine extensions $\}$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$\{$ Affine groupoid schemes over L/F with special sections $\}$ $\sim$ $\}$ affine extensions with kernel $/F$ $\}$

Consider affine extension $1 \to T \to E \to \Gamma \to 1$ where T commutative.

Then we may choose lifts $\{e_\sigma\}$ s.t. $\{d_{\sigma,\tau}\}$ defined by $e_{\sigma\tau} = d_{\sigma,\tau} e_\sigma e_\tau$

is a continuous 2-cocycle, hence E gives a class $cl(E) \in H^2(F, T)$.

A homomorphism of affine extensions is a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & G_1(L) \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "\phi"] & \Gamma \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & G_2(L) \arrow[r] & E_2 \arrow[r] & \Gamma \arrow[r] & 1 \end{tikzcd}$$

s.t. $\phi|_{G_1(L)}$ is defined by a homomorphism of alg. groups over L.

We should consider L-representations whose kernel could be defined $/F$ $\overset{\text{descent}}{\Rightarrow}$ F-v.s.

For an F-v.s. V, let E_V be the split affine extension $GL_V(L) \rtimes \Gamma$.

A representation of an affine extension E is a homomorphism $E \to E_V$.

An L-representation of a categorical object M is just a functor P from M to $Vect_L$ and a morphism of representations is just a natural transform of the functors.

$$G(V)(L) = \left\{ \begin{tikzcd} L \arrow[r] & G(V) \arrow[d] \\ & L\times_F L \end{tikzcd} \right\} \overset{(L\times_F L)(L) = \Gamma}{=\!=\!=} \coprod_{\sigma\in\Gamma} \left\{ \begin{tikzcd} L \arrow[r, "\varphi"] \arrow[dr, "{id,\sigma}"'] & G(V) \arrow[d] \\ & L\times_F L \end{tikzcd} \right\} = \coprod_{\sigma\in\Gamma} \mathrm{Isom}(V_L, \sigma^* V_L)$$

$$\|$$

Note that $GL_V(L) \rtimes \Gamma \overset{\text{bijection}}{\longleftrightarrow} \{ \varphi \in GL_F(V\otimes_F L),\ \exists \sigma\in\Gamma,\ \varphi(lv) = \sigma(l)\varphi(v) \}$

$$(g, \sigma) \longmapsto (v \longmapsto g(\sigma(v))).$$

Suppose we have representation $E \xrightarrow{\varphi_V} E_V$, $E \xrightarrow{\varphi_W} E_W$, then a homomorphism of representations $\varphi_V \longrightarrow \varphi_W$ is just an L-linear homomorphism $\rho: V\otimes L \to W \otimes L$ s.t. $\forall e \in E$, the diagram commutes

$$\begin{tikzcd} V\otimes_F L \arrow[r, "\varphi_V(e)"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\circlearrowright"] & V\otimes_F L \arrow[d, "\rho"] \\ W\otimes_F L \arrow[r, "\varphi_W(e)"] & W\otimes_F L \end{tikzcd}$$

RMK. The functor $\mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(E_G)$ is an equivalence of categories.

V F-v.s., $G \xrightarrow{\varphi} GL_V$, $E_G \xrightarrow{\tilde{\varphi}} E_V$, $\tilde{\varphi} = \varphi \times id$.

$\tilde{\varphi}(g,\sigma) = (\varphi(g),\sigma)$, $\tilde{\varphi}(g,\sigma)(h,\tau) = \tilde{\varphi}(g\sigma(h), \sigma\tau) = (\varphi(g)\varphi(\sigma h), \sigma\tau)$

$$\| $$

$(\varphi(g),\sigma)(\varphi(h),\tau) = (\varphi(g)\sigma\varphi(h), \sigma\tau)$

$\varphi(\sigma h) = \sigma\varphi(h)$ as φ is defined over F.

V, W F-v.s., $G \xrightarrow{\varphi_V} GL_V$, $G \xrightarrow{\varphi_W} GL_W$, $E_G \xrightarrow{\tilde{\varphi}_V} E_V$, $E_G \xrightarrow{\tilde{\varphi}_W} E_W$.

$\rho: V \to W$ morphism of G-rep. $\Rightarrow \forall g \in G(L)$,

$$\begin{tikzcd} V\otimes L \arrow[r, "\varphi_V(g)"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\circlearrowright"] & V\otimes L \arrow[d, "\rho"] \\ W\otimes L \arrow[r, "\varphi_W(g)"] & W\otimes L \end{tikzcd}$$

then $\rho(\tilde{\varphi}_V(g,\sigma)v) = \rho(\varphi_V(g)(\sigma v)) = \rho(\sigma(\sigma^{-1}(\varphi_V(g))v)) = \sigma\rho(\varphi_V(\sigma^{-1}g)v)$

$$\|$$

$\tilde{\varphi}_W(g,\sigma)\rho v = \varphi_W(g)(\sigma\rho v) = (\sigma\varphi_W)(g)(\sigma\rho v) = \sigma(\varphi_W(\sigma^{-1}g)\rho(v))$

Hence ρ defines a homomorphism of $\tilde{\varphi}_V \longrightarrow \tilde{\varphi}_W$.

Let V, W F-v.s., $G \xrightarrow{\varphi_v} GL_v$, $G \xrightarrow{\varphi_w} GL_w$, $E_G \xrightarrow{\tilde{\varphi}_v} E_v$, $E_G \xrightarrow{\tilde{\varphi}_w} E_w$.

Let $\rho: V \otimes L \to W \otimes L$ s.t. $\forall e \in E_G$,

$$\begin{tikzcd} V\otimes L \arrow[r, "\tilde{\varphi}_v(e)"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\circlearrowright"] & V \otimes L \arrow[d, "\rho"] \\ W \otimes L \arrow[r, "\tilde{\varphi}_w(e)"] & W \otimes L \end{tikzcd}$$

then $\rho(\tilde{\varphi}_v(g,\sigma)v) = \rho(\varphi_v(g)(\sigma v)) = \rho(\sigma(\varphi_v(\sigma^{-1}g)v))$

$\quad\;\; \shortparallel$

$\tilde{\varphi}_w(g,\sigma)(\rho v) = \varphi_w(g)\sigma(\rho v) = \sigma(\varphi_w(\sigma^{-1}g)(\rho v))$

take $\sigma = 1$ $\Rightarrow$ $\rho(\varphi_v(\sigma^{-1}g)v) = \varphi_w(\sigma^{-1}g)(\rho v)$ $\Rightarrow$ $\rho(\sigma v) = \sigma(\rho v)$ $\Rightarrow$ $\rho\sigma = \sigma\rho$

i.e. ρ is defined over F, which is clearly a morphism of G-rep.

Let V F-v.s.,
$$\begin{tikzcd} 1 \arrow[r] & G(L) \arrow[r] \arrow[d, "\varphi"] & E_G \arrow[r] \arrow[d, "\tilde{\varphi}"] & \Gamma \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & GL_v(L) \arrow[r] & E_v \arrow[r] & \Gamma \arrow[r] & 1 \end{tikzcd}$$
, let $\tilde{\varphi}(1,\sigma) = (g_\sigma, \sigma)$.

$\tilde{\varphi}(1,\sigma\tau) = \tilde{\varphi}((1,\sigma)(1,\tau))$ $\Rightarrow$ $g_{\sigma\tau} = g_\sigma \cdot \sigma g_\tau$ $\Rightarrow$ $\{g_\sigma\}$ is 1-cocycle in $H^1(\Gamma, GL_v)$

Note $H^1(\Gamma, GL_v) = 0$ $\Rightarrow$ $\exists\, \tilde{g} \in GL_v(L)$, $g_\sigma = \tilde{g}^{-1} \cdot \sigma\tilde{g}$.

$\tilde{\varphi}(\sigma g) = \tilde{\varphi}((1,\sigma)(g,1)(1,\sigma^{-1})) = (g_\sigma,\sigma)(\varphi(g),1)(g_\sigma^{-1}, \sigma^{-1}) = (\tilde{g}^{-1}\cdot\sigma\tilde{g}\cdot\sigma\varphi(g)\cdot\sigma\tilde{g}^{-1}\cdot\tilde{g}, 1)$

$\Rightarrow$ $\sigma(\tilde{g}\cdot\varphi(g)\cdot\tilde{g}^{-1}) = \tilde{g}\cdot\varphi(\sigma g)\cdot\tilde{g}^{-1}$ $\Rightarrow$ $\varphi' = \tilde{g}\cdot\varphi\cdot\tilde{g}^{-1}$ is defined over F.

Now $\tilde{g}\cdot\tilde{\varphi}(g,\sigma)\cdot\tilde{g}^{-1} = (\tilde{g},1)(\varphi(g)\cdot g_\sigma, \sigma)(\tilde{g}^{-1},1) = (\tilde{g}\cdot\varphi(g)\cdot\tilde{g}^{-1}, 1) = (\varphi'(g), 1)$

Hence $\tilde{\varphi}$ is isom. to $\tilde{g}\cdot\tilde{\varphi}\cdot\tilde{g}^{-1}$ which comes from rep. of G on V via φ'.

Prop. Let E be an L/F affine extension whose kernel is a torus T split by L. Then the category $\mathrm{Rep}(E)$ is a semisimple F-category, $\Sigma(\mathrm{Rep}(E)) = \Gamma \backslash X^*(T)$.

Let $V_{\Gamma\chi}$ be a simple representation, then $\mathrm{End}(V_{\Gamma\chi})$ has center $F(\chi)$, and its class in $Br(F(\chi))$ is the image of $d(E)$ under the homomorphism

$H^2(F, T) \to H^2(F(\chi), \mathbb{G}_m)$.

Consider affine extensions with kernel being a protorus.

Let $T = \varprojlim T_i$, $X^*(T) = \varinjlim X^*(T_i)$, $T \longmapsto X^*(T)$ defines an equivalence of categories $\{\text{protori}\} \longrightarrow \{\text{torsion free } \mathbb{Z}\text{-mod with continuous } \Gamma\text{-action}\}$

Let $L = \bar{F}$. An affine extension with kernel T is an exact sequence

$1 \to T(\bar{F}) \to E \to \Gamma \to 1$

whose pushout by $T(\bar{F}) \to T_i(\bar{F})$: $1 \to T_i(\bar{F}) \to E_i \to \Gamma \to 1$ is an affine extension.

RMK. Let
$$\begin{array}{ccc} L & \subset & L' \\ \Gamma \big| & & \big| \Gamma' \\ F & \subset & F' \end{array}$$
be a commutative diagram of fields where L/F and L'/F' are Galois. From an L/F-affine extension $1 \to G(L) \to E \to \Gamma \to 1$ with kernel G we obtain an L'/F'-affine extension $1 \to G(L') \to E' \to \Gamma' \to 1$ with kernel $G_{F'}$ by pulling back by $\Gamma' \to \Gamma$, $\sigma \longmapsto \sigma|_L$ and pushing out by $G(L) \to G(L')$

Ex. Let $\mathbb{Q}_p^{un}$ be a maximal unramified extension of $\mathbb{Q}_p$, L_n the subfield of $\mathbb{Q}_p^{un}$ of degree n over $\mathbb{Q}_p$. $\Gamma_n = \mathrm{Gal}(L_n/\mathbb{Q}_p)$, $D_{1,n}$ the division algebra and $1 \to L_n^\times \to N(L_n^\times) \to \Gamma_n \to 1$ the corresponding extension where $N(L_n^\times)$ is the normalizer of $L_n^\times$ in $D_{1,n}$, $N(L_n^\times) = \coprod_{0 \le i \le n-1} L_n^\times a^i$.
This is an $L_n/\mathbb{Q}_p$-affine extension with kernel $\mathbb{G}_m$. Pull back by $\Gamma \to \Gamma_n$ and push out by $L_n^\times \to \mathbb{Q}_p^{un,\times}$ we get a $\mathbb{Q}_p^{un}/\mathbb{Q}_p$-affine extension

$$1 \to \mathbb{Q}_p^{un,\times} \longrightarrow D_n \longrightarrow \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p) \to 1 \quad \text{with kernel } \mathbb{G}_m$$

From a representation $\rho: D_n \to E_V = GL(V(\mathbb{Q}_p^{un})) \rtimes \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ of D_n we get a $\mathbb{Q}_p^{un}$-v.s. V equipped with a σ-linear map F ($\rho(1, a) = (F, \sigma)$).
Tensor with the completion L of $\mathbb{Q}_p^{un}$, we get an isocrystal which is a sum of E^λ, $\lambda \in \frac{1}{n}\mathbb{Z}$.

There is a canonical section to $N(L_n^\times) \to \Gamma_n$, $\sigma^i \longmapsto a^i$, $0 \le i \le n-1$ giving rise to a canonical section to $D_n \to \Gamma$.

There is a homomorphism $D_{nm} \to D_n$ whose restriction to the kernel is multiplication by m. The inverse limit D is a $\mathbb{Q}_p^{un}/\mathbb{Q}_p$-affine extension with kernel $\mathbb{G} = \varprojlim \mathbb{G}_m$, $X^*(\mathbb{G}) = \varinjlim \frac{1}{n}\mathbb{Z}/\mathbb{Z} = \mathbb{Q}$. There is a natural functor from $\mathrm{Rep}(D)$ to the category of isocrystals which is faithful, essentially surjective on objects but not full. D is called the Dieudonne affine extension.

The affine extension $\mathfrak{P}$.

Let $W(p^n)$ be the subgrp of $\overline{\mathbb{Q}}^\times$ generated by $W_1(p^n)$ and $W = \varinjlim W(p^n)$.

Then W contains all roots of unity, and the quotient is a torsion free $\mathbb{Z}$-mod with a continuous action of $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let P be the corresponding pretorus over $\mathbb{Q}$.

For $\pi \in W$, let $\mathbb{Q}\{\pi\}$ be the smallest field generated by a representative of π. If π is represented by $\pi_n \in W(p^n)$ and $|\rho(\pi_n)| = (p^n)^{\frac{m}{2}}$, we say π has weight m and for a place $v|p$ of $\mathbb{Q}\{\pi\}$, let $s_\pi(v) = \dfrac{\mathrm{ord}_v \pi_n}{\mathrm{ord}_v p^n}$.

Thm. There exists an affine extension $1 \to P(\overline{\mathbb{Q}}) \to \mathfrak{P} \to \Gamma \to 1$ s.t.

- $\mathrm{Rep}\,\mathfrak{P}$ is a semisimple $\mathbb{Q}$-category
- $\Sigma(\mathrm{Rep}\,\mathfrak{P}) = \Gamma \backslash W$
- for $\pi \in W$, let $D(\pi) = \mathrm{End}(V_{\Gamma\pi})$. Then $D(\pi)$ is isom. to the division algebra D with center $\mathbb{Q}\{\pi\}$ and invariants

$$\mathrm{inv}_v D = \begin{cases} \dfrac{wt(\pi)}{2} & v \text{ real} \\ s_\pi(v) \cdot [\mathbb{Q}\{\pi\}_v : \mathbb{Q}_p] & v|p \\ 0 & \text{otherwise} \end{cases}$$

Moreover $\mathfrak{P}$ is unique up to isomorphism.

extend simple objects from $\Gamma \backslash W_1$ to $\Gamma \backslash W$ $\rightsquigarrow$ rep. of (pro) torus

keep endomorphism algebra the same form as AV $\rightsquigarrow$ $H^2(\mathbb{Q}, P) \to \prod \mathrm{Br}(\mathbb{Q}\{\pi\})$

extension of groups

We call a representation of $\mathfrak{P}$ a fake motive over $\overline{\mathbb{F}}_p$, and a fake AV if its simple summands correspond to $\pi \in \Gamma \backslash W_1$. Then the category of fake AV is a semisimple $\mathbb{Q}$-category with the same numerical invariants as $AV^\circ(\overline{\mathbb{F}}_p)$.

Local form $\mathfrak{P}(\ell)$ of $\mathfrak{P}$.

Let ℓ be a prime of $\mathbb{Q}$, choose a prime w_ℓ of $\overline{\mathbb{Q}}$ dividing ℓ. Let $\overline{\mathbb{Q}}_\ell$ be the alg. closure of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_{w_\ell}$. Then $\Gamma_\ell = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ is a closed subgrp of Γ and we have a commutative diagram

$$\begin{array}{ccc} \overline{\mathbb{Q}} & - & \overline{\mathbb{Q}}_\ell \\ |\Gamma & & |\Gamma_\ell \\ \mathbb{Q} & - & \mathbb{Q}_\ell \end{array}$$

hence from $\mathfrak{P}$ we get a $\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$-affine extension $\mathfrak{P}(\ell)$.

$\mathbb{Q}_\ell$-space attached to a fake motive.

Let $\ell \neq p, \infty$.

Prop. There exists a continuous homomorphism ζ_ℓ splitting $\mathfrak{P}(\ell) \to \Gamma_\ell$.

Let $\rho: \mathfrak{P} \to E_V$ be a fake motive, pull back by $\Gamma_\ell \to \Gamma$ and push out by

$$\begin{array}{ccc} P(\overline{\mathbb{Q}}) & \longrightarrow & P(\overline{\mathbb{Q}}_\ell) \\ \downarrow & & \downarrow \\ GL(V(\overline{\mathbb{Q}})) & \longrightarrow & GL(V(\overline{\mathbb{Q}}_\ell)) \end{array}$$

we get a representation $\rho(\ell): \mathfrak{P}(\ell) \to GL(V(\overline{\mathbb{Q}}_\ell)) \rtimes \Gamma_\ell$.

Fix a splitting $\zeta_\ell: \Gamma_\ell \to \mathfrak{P}(\ell)$.

For $\sigma \in \Gamma_\ell$, let $(\rho(\ell) \circ \zeta_\ell)(\sigma) = (e_\sigma, \sigma)$, then $e_\sigma \in GL(V(\overline{\mathbb{Q}}_\ell))$ satisfies $e_\sigma \cdot \sigma e_\tau = e_{\sigma\tau}$. Thus $\sigma \cdot v = e_\sigma(\sigma v)$ is an action of Γ_ℓ on $V(\overline{\mathbb{Q}}_\ell)$.

This is a continuous action hence $V_\ell(p) = V(\bar{\mathbb{Q}}_\ell)^{\Gamma_\ell}$ is a $\mathbb{Q}_\ell$-structure on $V(\bar{\mathbb{Q}}_\ell)$.

We get a functor $\{\text{fake motives} / \bar{\mathbb{F}}_p\} \longrightarrow \mathrm{Vect}/\mathbb{Q}_\ell$

$$p \longmapsto V_\ell(p)$$

and we can have a functor $\{\text{fake motives} / \bar{\mathbb{F}}_p\} \longrightarrow \{\text{free modules} / \mathbb{A}_f^p\}$

$$p \longmapsto V_f^p(p)$$

s.t. $V_\ell(p) = V_f^p(p) \otimes_{\mathbb{A}_f^p} \mathbb{Q}_\ell$ for $\ell \neq p, \infty$.

Isocrystal of a fake motive.

Choose a prime w_p of $\bar{\mathbb{Q}}$ dividing p, take $\mathbb{Q}_p^{un}$, $\bar{\mathbb{Q}}_p$ inside $\bar{\mathbb{Q}}_{w_p}$. Then $\Gamma_p = \mathrm{Gal}(\bar{\mathbb{Q}}_p / \mathbb{Q}_p)$ is a closed subgrp of Γ and $\Gamma_p^{un} = \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ is a quotient of Γ_p.

Prop.

(a) The affine extension $\mathfrak{P}(p)$ arises by pullback and pushout from a $\mathbb{Q}_p^{un}/\mathbb{Q}_p$-affine extension $\mathfrak{P}(p)^{un}$.

(b) There is a homomorphism of $\mathbb{Q}_p^{un}/\mathbb{Q}_p$-extensions $D \to \mathfrak{P}(p)^{un}$ whose restriction to the kernels $\mathbb{G} \to P_{\mathbb{Q}_p}$ corresponds to the map on characters
$$W \longrightarrow \mathbb{Q}, \quad \pi \longmapsto s_\pi(w_p).$$

$$\begin{array}{ccccccccc}
1 \to & \mathbb{G}(\mathbb{Q}_p^{un}) & \to & D & \to & \Gamma_p^{un} & \to 1 \\
& \downarrow & & \downarrow & & \| & \\
1 \to & P(\mathbb{Q}_p^{un}) & \to & \mathfrak{P}(p)^{un} & \to & \Gamma_p^{un} & \to 1 \\
& \downarrow & & \vdots & & \uparrow & \\
1 \to & P(\bar{\mathbb{Q}}_p) & \to & \mathfrak{P}(p) & \to & \Gamma_p & \to 1
\end{array}$$

$\rho: \mathfrak{P} \to E_V \Rightarrow$ rep. of $\mathfrak{P}(p)$

$\rightsquigarrow$ rep. of $\mathfrak{P}(p)^{un}$

$\Rightarrow$ rep. of D

$\Rightarrow$ isocrystal $D(p)$.

CM AV/$\bar{\mathbb{Q}}$ gives AV/$\bar{\mathbb{F}}_p$ and fake AV.

Prop. Let $T/\mathbb{Q}$ be a torus split by a CM field, let μ be a cocharacter of T s.t. $\mu + \iota\mu$ is defined over $\mathbb{Q}$ (ι complex conjugation). Then there is a homomorphism well defined up to isom. $\phi_\mu : \mathfrak{P} \to E_T$.

Let A be AV of CM type (E, Φ) over $\bar{\mathbb{Q}}$, $T = (\mathbb{G}_m)_{E/\mathbb{Q}}$.

Φ defines a cocharacter μ_Φ of T. (T, μ_Φ) satisfies the condition above.

Then $\exists\ \phi : \mathfrak{P} \to E_T$. Let $V = H_1(A, \mathbb{Q})$.

Rep. of T on $V \Rightarrow$ rep. ρ of E_T on $V \Rightarrow \rho\circ\phi$ is a fake AV and $V_\ell(\rho\circ\phi) = H_1(A, \mathbb{Q}_\ell)$, $D(\rho)$ is isom. to the Dieudonne module of the reduction of A by Shimura-Taniyama formula.

$k = \bar{\mathbb{F}}_p$, $Mot(k)$ $\mathbb{Q}$-semisimple category of numerical motives.

Tate's conjecture + Grothendieck's standard conjecture $\Rightarrow \Sigma(Mot(k)) = \Gamma | W$ and endomorphism algebras have the same description as in $Rep(\mathfrak{P})$.

$Mot(k)$ is Tannakian, a choice of a fibre functor over $\bar{\mathbb{Q}}$ identifies $Mot(k)$ with the category of rep. of an affine groupoid scheme.

Tannakian theory describes the kernel P, fibre functors at all ℓ (and a polarization) determine the class of the groupoid scheme in $H^2(\mathbb{Q}_\ell, P)$.

Langlands & Rapoport showed the family of local cohomology classes arises from a global class in $H^2(\mathbb{Q}, P)$, giving the existence of such groupoid scheme.

Under the equivalence of affine groupoid schemes and affine extensions, it is $\mathfrak{P}$.