The good reduction of Shimura varieties.

Hyperspecial subgroups.

If SV is moduli variety $/\mathbb{C}$, its canonical model $/\bar{\mathbb{Q}}$ should be the moduli variety of the same moduli problem $/\bar{\mathbb{Q}}$.

When SV 5 holds, the canonical model over $E(G,X)$ should be the moduli variety of the same moduli problem over $E(G,X)$ even though $Sh(E)=\phi$.

Def. Let $G/\mathbb{Q}_p$ reductive group, $K\subset G(\mathbb{Q}_p)$ subgroup is called hyperspecial if $\exists$ flat group scheme $\mathcal{G}/\mathbb{Z}_p$ s.t.

1) $\mathcal{G}_{\mathbb{Q}_p}=G$

2) $\mathcal{G}_{\mathbb{F}_p}$ is connected reductive group

3) $\mathcal{G}(\mathbb{Z}_p)=K$

Ex. Let $G=GSp(V,\psi)$, Λ a $\mathbb{Z}_p$-lattice in $V(\mathbb{Q}_p)$, K_p the stabilizer of Λ. If the restriction ψ to $\Lambda\times\Lambda$ takes values in $\mathbb{Z}_p$ and is perfect, then K_p is hyperspecial.

Ex. In the PEL case, to have a hyperspecial group at p, B must be unramified above p, i.e. $B\otimes_{\mathbb{Q}}\mathbb{Q}_p$ is a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$.

Tits 1979: there exists a hyperspecial subgroup in $G(\mathbb{Q}_p)$ $\Longleftrightarrow$ G unramified over $\mathbb{Q}_p$, i.e. quasisplit over $\mathbb{Q}_p$ and split over an unramified ext.

The good reduction of SV.

Fix hyperspecial subgroup $K_p \subset G(\mathbb{Q}_p)$, write $Sh_p(G, X)$ for $\varprojlim_{K^p} Sh_{K^pK_p}$.

Roughly speaking there are two reasons for SV to have bad reduction:

1) G may be ramified at p

2) p may divide the level.

$\Rightarrow$ only consider $Sh_{K^pK_p}$.

Thm. Let (G, X) be of abelian type, $K_p \subset G(\mathbb{Q}_p)$ hyperspecial and $Sh_p(G, X)$ the inverse system $\left(Sh_{K^pK_p}(G, X)\right)$ over $E(G, X)$. Then $Sh_p(G, X)$ has canonical good reduction at every $\mathfrak{p} \mid p$ of $E(G, X)$ unless p lies in a small set of primes depending on (G, X).

RMK. Let $E_\mathfrak{p}$ be the completion, $\mathcal{O}_\mathfrak{p}$ ring of integers, $\kappa(\mathfrak{p})$ residue field. By $Sh_p(G, X)$ having good reduction at $\mathfrak{p}$, we mean the inverse system $\left(Sh_{K^pK_p}\right)$ extends to an inverse system of flat schemes $\mathcal{S}_p = \left(\mathcal{S}_{K^p}\right)$ over $\mathcal{O}_\mathfrak{p}$ together with $G(\mathbb{A}_f^p)$-action such that when modulo $\mathfrak{p}$, the reduction is an inverse system of smooth varieties $\left(\overline{Sh}_{K^pK_p}\right)$ over $\kappa(\mathfrak{p})$ with finite etale transition maps.

By $Sh_p(G, X)$ having canonical good reduction at p we mean for any T over O_p formally smooth, the natural map

$$\mathrm{Hom}_{O_p}(T, \varprojlim_{K^p} S_{K^p}) \longrightarrow \mathrm{Hom}_{E_p}(T_{E_p}, \varprojlim_{K^p} Sh_{K^pK_p})$$

is a bijection. This characterizes the model uniquely.

RMK. In the Siegel case, the theorem was proved by Mumford and S_{K^p}, $\overline{Sh}_{K^pK_p}$ are moduli schemes. For Hodge case, one takes the (normalization of) Zariski closure of $Sh_{K^pK_p} \hookrightarrow Sh_{U^pU_p} \longrightarrow S_{U^p}$ inside the embedding into the Siegel case and the Siegel integral model.

The Langlands - Rapoport Conjecture.

Let (G, X) be SD with SV 4, 5, 6 hold.

For $x \in X$, let $I(x)$ be the subgroup of $G(\mathbb{Q})$ fixing x and let

$S(x) = I(x) \backslash X^p(x) \times X_p(x)$, $X^p(x) = G(\mathbb{A}_f^p)$, $X_p(x) = G(\mathbb{Q}_p)/K_p$.

Then there is a canonical bijection of $G(\mathbb{A}_f^p)$ - sets

$$\coprod_{[x] \in G(\mathbb{Q}) \backslash X} S(x) \longrightarrow Sh_p(\mathbb{C}) = \varprojlim_{K^p} Sh_{K^pK_p}(\mathbb{C})$$

The decomposition has a modular interpretation. In the Hodge type case, $S(x)$ classifies Isom. classes $(A, (s_i), \eta K)$ with $(A, (s_i))$ Isom. to a fixed pair.

$X^p(x)$: prime to p level structure, $X_p(x)$: p level structure

Langlands and Rapoport conjectured a similar description for $\overline{Sh}_p(\overline{\mathbb{F}}_p)$.

Let $\phi : \mathfrak{P} \to \mathcal{E}_G$ be a homomorphism. Such ϕ should be thought of as "pre fake abelian motives with tensors". Also recall ϕ and ϕ' are isom. iff $\exists\ g \in G(\overline{\mathbb{Q}})$ s.t. $\phi' = g \phi g^{-1}$.

$\rho : G \to GL_n$ faithful

Fix $\phi : \mathfrak{P} \to \mathcal{E}_G$. Next we define the set $S(\phi)$.

- $I(\phi)$.

$$I(\phi) = \{ g \in G(\overline{\mathbb{Q}}) \mid Ad(g)\phi = \phi \}$$ $I(\phi) \subset Aut(\rho \circ \phi)$

- $X^p(\phi)$.

Let $\ell \neq p, \infty$ be a prime. Choose a prime w_ℓ of $\overline{\mathbb{Q}}$ lying over ℓ and define $\overline{\mathbb{Q}}_\ell$ and $\Gamma_\ell \subset \Gamma$. Regard Γ_ℓ as an $\overline{\mathbb{Q}}_\ell / \mathbb{Q}_\ell$ - affine extension with trivial kernel and write

$$\xi_\ell : \Gamma_\ell \to \mathcal{E}_G(\ell) = G(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_\ell .$$
$$\sigma \mapsto (1, \sigma)$$

Recall there is a homomorphism $\zeta_\ell : \Gamma_\ell \to \mathfrak{P}(\ell)$. Compose it with $\phi(\ell) : \mathfrak{P}(\ell) \to \mathcal{E}_G(\ell)$ we get another homomorphism $\Gamma_\ell \to \mathcal{E}_G(\ell)$.

Define $X_\ell(\phi) = Isom(\xi_\ell, \phi(\ell) \circ \zeta_\ell)$. $X_\ell(\phi) \subset Isom(V(\mathbb{Q}_\ell), V_\ell(\rho\circ\phi))$

Clearly $Aut(\xi_\ell) = G(\mathbb{Q}_\ell)$ acts on $X_\ell(\phi)$ on the right and $I(\phi)$ acts on the left. If $X_\ell(\phi)$ nonempty, $X_\ell(\phi)$ is a principal homogeneous space for $G(\mathbb{Q}_\ell)$. By choosing ζ_ℓ judiciously we get compact open subspaces of $X_\ell(\phi)$. Define $X^p(\phi)$ to be the restricted product.

• $X_p(\phi)$.

Choose prime w_p of $\bar{\mathbb{Q}}$ lying above p. Let L be the completion of $\mathbb{Q}_p^{un}$ and $\mathcal{O}_L$ the ring of integers. Let σ be the Frobenius automorphism of L acting as $x \longmapsto x^p$ on the residue field.

From ϕ we get $D \longrightarrow \mathfrak{P}(p)^{un} \xrightarrow{\phi(p)^{un}} G(\mathbb{Q}_p^{un}) \rtimes \Gamma_p^{un}$.

For some n, the composite factors through D_n and there is a canonical element in D_n mapping to σ. Let (b, σ) be its image in $G(\mathbb{Q}_p^{un}) \rtimes \Gamma_p^{un}$ and $b(\phi)$ the image of b in $G(L)$. If $b(\phi)'$ also arise in this way then $b(\phi)' = g^{-1} \cdot b(\phi) \cdot \sigma g$. $D(\rho \circ \phi) = (V(L), F: v \longmapsto \rho(b(\phi))\sigma v)$

Recall $c(X)$ the well-defined $G(\bar{\mathbb{Q}})$-conj. class of cocharacters of $G_{\bar{\mathbb{Q}}}$. By the hyperspecial condition, G splits over $\mathbb{Q}_p^{un}$ hence when transfering $c(X)$ to conj. class of cocharacters of $G_{\bar{\mathbb{Q}}_p}$, there is some μ defined over $\mathbb{Q}_p^{un}$. Let $C_p = \mathcal{G}(\mathcal{O}_L) \cdot \mu(p) \cdot \mathcal{G}(\mathcal{O}_L)$, $\mathcal{G}$ as in hyperspecial condition.

Define $X_p(\phi) = \{ g \in G(L)/\mathcal{G}(\mathcal{O}_L) \mid g^{-1} \cdot b(\phi) \cdot g \in C_p \}$. $I(\phi)$ acts naturally on $X_p(\phi)$.

For $g \in X_p(\phi)$, define $\Phi(g) = b(\phi) \cdot \sigma b(\phi) \cdots \sigma^{m-1} b(\phi) \cdot \sigma^m g$, $m = [E_p : \mathbb{Q}_p]$.

• $S(\phi)$.

$$S(\phi) = I(\phi) \backslash X^p(\phi) \times X_p(\phi)$$

$$\begin{array}{cc} \circlearrowleft & \circlearrowleft \\ G(\mathbb{A}_f^p) & \Phi \end{array}$$

We shall only consider the admissible homomorphisms.

- ∞.

Let E_∞ be the extension $1 \to \mathbb{C}^\times \to E_\infty \to \Gamma_\infty \to 1$ associated with the quaternion algebra $\mathbb{H}$, $\Gamma_\infty = Gal(\mathbb{C}/\mathbb{R})$, $E_\infty = \mathbb{C}^\times \amalg \mathbb{C}^\times j$ and $jzj^{-1} = \bar{z}$. It is an affine extension with kernel $\mathbb{G}_m$.

Take $\ell = \infty$ we get local form $\mathfrak{P}(\infty)$ of $\mathfrak{P}$:

$$1 \to P(\mathbb{C}) \to \mathfrak{P}(\infty) \to \Gamma_\infty \to 1.$$

Fact: there is a homomorphism $\zeta_\infty : E_\infty \to \mathfrak{P}(\infty)$ whose restriction to kernels $\mathbb{G}_m \to P_{\mathbb{C}}$ corresponds to the map on characters $\pi \longmapsto wt(\pi)$.

Note for $\forall\, x \in X$, $\xi_x(z) = (w_x(z), 1)$, $\xi_x(j) = (\mu_x(-1)^{-1}, \iota)$ define a homomorphism $E_\infty \to E_G(\infty)$. Replace x with a different point will replace the homomorphism with an isomorphic one. Write ξ_X for the isom. class.

The admissibility condition at ∞ is $\phi(\infty) \circ \zeta_\infty \in \xi_X$.

- $\ell \neq p$.

$X_\ell(\phi)$ nonempty.

• p

$X_p(\phi)$ nonempty.

• Global condition.

Let $\nu : G \to T$ be the quotient by G^{der}.

From X we get a conj. class of cocharacters of $G_{\mathbb{C}}$, hence a well defined cocharacter μ of T. As (G, X) satisfies SV 4, 5, 6, T is split by a CM field and $\mu + \iota\mu$ is defined over $\mathbb{Q}$. Hence there is a homomorphism $\phi_\mu : \mathfrak{P} \to E_T$. We require $\nu \circ \phi \simeq \phi_\mu$.

Now let $LR(G, X) = \coprod S(\phi)$ where the disjoint union is taken over a set of representatives for the isom. classes of admissible homomorphisms $\phi : \mathfrak{P} \to E_G$. There are commuting actions of $G(\mathbb{A}_f^p)$ and Φ.

Conjecture. (Langlands - Rapoport, 1987)

Let (G, X) be SD satisfying SV 4, 5, 6, G^{der} simply connected.

Let $K_p \subset G(\mathbb{Q}_p)$ hyperspecial, $\mathfrak{p} | p$ prime of $E(G, X)$.

Assume Sh_p has good reduction at $\mathfrak{p}$.

Then there is a bijection of sets $LR(G, X) \to \overline{Sh_p}(G, X)(\overline{\mathbb{F}}_p)$ compatible with $G(\mathbb{A}_f^p)$ and Frobenius action.

RMK.

(a) SV5 and SV6 are for simplicity and can be removed.

(b) One can generalize to SD not satisfying SV4 and G^{der} not simply conneted.