A formula for the number of points.

Let (G, X) be SD satisfying SV 4, 5, 6, $K_p \subset G(\mathbb{Q}_p)$ hyperspecial.
Assume G^{der} simply connected, $Sh_p(G, X)$ has canonical good reduction at a prime $\mathfrak{p} | p$ of $E(G, X)$. Write L_n unramified extension of $\mathbb{Q}_p$ of deg n.

Consider triples $(\gamma_0, \gamma, \delta)$ where

- γ_0 semisimple element of $G(\mathbb{Q})$ that is contained in an elliptic torus of $G_{\mathbb{R}}$, i.e. a torus whose image in $G^{ad}_{\mathbb{R}}$ is anisotropic.
- $\gamma = (\gamma(\ell))_{\ell \neq p, \infty} \in G(\mathbb{A}_f^p)$ s.t. for all ℓ, $\gamma(\ell)$ conjugate to γ_0 in $G(\overline{\mathbb{Q}}_\ell)$.
- $\delta \in G(L_n)$ s.t. $N\delta = \delta \cdot \sigma\delta \cdot \cdots \cdot \sigma^{n-1}\delta$ conjugate to γ_0 in $G(\overline{\mathbb{Q}}_p)$.

Two triples $(\gamma_0, \gamma, \delta)$ and $(\gamma_0', \gamma', \delta')$ are equivalent iff γ_0 conj. to γ_0' in $G(\mathbb{Q})$, $\gamma(\ell)$ conj. to $\gamma'(\ell)$ in $G(\mathbb{Q}_\ell)$, δ σ-conj. to δ' in $G(L_n)$.

Given $(\gamma_0, \gamma, \delta)$, let

- $I_0 = G_{\gamma_0}$ centralizer of γ_0 in G
- I_∞ inner form of $I_{0,\mathbb{R}}$ s.t. $I_\infty / Z(G)$ anisotropic
- I_ℓ centralizer of $\gamma(\ell)$ in $G_{\mathbb{Q}_\ell}$
- I_p inner form of $G_{\mathbb{Q}_p}$ s.t. $I_p(\mathbb{Q}_p) = \{ x \in G(L_n) \mid x^{-1} \cdot \delta \cdot \sigma x = \delta \}$.

Assume in addition that

(*) $\exists$ inner form I of I_0 s.t. $I_{\mathbb{Q}_\ell} \simeq I_\ell$ for all ℓ including p, ∞.

Let dx be the Haar measure on $G(\mathbb{A}_f^p)$ s.t. K^p has volume 1. Choose a Haar measure di^p on $I(\mathbb{A}_f^p)$ that gives rational measure to compact open subgroups of $I(\mathbb{A}_f^p)$ and use isoms $I_{\mathbb{Q}_\ell} \simeq I_\ell$ to transport it to a measure on $G(\mathbb{A}_f^p)_\gamma$, the centralizer of γ in $G(\mathbb{A}_f^p)$. Write $d\bar{x}$ for the quotient of dx by di^p. Let $\mathcal{H}_K$ be the Hecke algebra consisting of locally constant K bi-invariant $\mathbb{Q}$-valued functions on $G(\mathbb{A}_f)$. Let $f \in \mathcal{H}_K$ and assume $f = f^p \cdot f_p$, f^p function on $G(\mathbb{A}_f^p)$ and $f_p = \frac{1}{|K_p|} 1_{K_p}$.

Define $$O_\gamma(f^p) = \int_{G(\mathbb{A}_f^p)_\gamma \backslash G(\mathbb{A}_f^p)} f^p(x^{-1}\gamma x)\, d\bar{x}.$$

Let dy be the Haar measure on $G(L_n)$ s.t. $\mathcal{G}(O_{L_n})$ has volume 1. Choose a Haar measure di_p on $I(\mathbb{Q}_p)$ that gives rational measure to compact open subgroups and use isom. $I_{\mathbb{Q}_p} \simeq I_p$ to transport the measure to $I_p(\mathbb{Q}_p)$. Write $d\bar{y}$ for the quotient of dy by di_p. Choose a cocharacter μ in $c(X)$ defined over L_n and let $\varphi = 1_{\mathcal{G}(O_{L_n})\mu(p)\mathcal{G}(O_{L_n})}$.

Define $$TO_\delta(\varphi) = \int_{I(\mathbb{Q}_p)\backslash G(L_n)} \varphi(y^{-1}\delta\,\sigma(y))\, d\bar{y}.$$

Since $I/Z(G)$ anisotropic over $\mathbb{R}$ and we assume SV 5, $I(\mathbb{Q})$ is discrete in $I(\mathbb{A}_f^p)$, we can define the volume of $I(\mathbb{Q})\backslash I(\mathbb{A}_f^p)$. It is a rational number by our assumption.

Define $I(\gamma_0, \gamma, \delta) = \mathrm{vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f^p)) \cdot O_\gamma(I_{K^p}) \cdot TO_\delta(\varphi)$.

Then $(\gamma_0, \gamma, \delta) \sim (\gamma_0', \gamma', \delta') \Rightarrow I(\gamma_0, \gamma, \delta) = I(\gamma_0', \gamma', \delta')$.

An admissible pair (ϕ, γ_0) is an admissible homomorphism $\phi : \mathfrak{P} \to E_G$ and $\gamma_0 \in I(\phi)$ s.t. for some $x \in X_p(\phi)$, $\gamma_0 x = \Phi^r x$, $r = [k(p) : \mathbb{F}_p]$.

An isom. of (ϕ, γ_0) and (ϕ', γ_0') is a $g \in G(\overline{\mathbb{Q}})$ s.t. $\mathrm{Ad}(g)\phi = \phi'$ and $\mathrm{Ad}(g)\gamma_0 = \gamma_0'$.

Let $(T, x) \subset (G, X)$ be a special pair. By our assumptions on (G, X), the cocharacter μ_x of T defines a homomorphism $\phi_x : \mathfrak{P} \to E_T$. Langlands and Rapoport showed that every admissible pair is isom. to a pair (ϕ_x, γ_0') with $\gamma_0' \in T(\mathbb{Q})$. For such a pair, $b(\phi_x)$ is represented by some $\delta \in G(L_n)$. Let γ be the image of γ_0' in $G(\mathbb{A}_f^p)$. Then the triple $(\gamma_0', \gamma, \delta)$ satisfies all the conditions. A triple of this form is called effective.

For any triple $(\gamma_0, \gamma, \delta)$, the kernel of $H^1(\mathbb{Q}, I_0) \to H^1(\mathbb{Q}, G) \oplus \prod_\ell H^1(\mathbb{Q}_\ell, I_0)$ is finite and denote its order by $c(\gamma_0)$.

Conjecture. (Kottwitz, 1990)

$$\# \overline{Sh}_{K^p K_p}(\mathbb{F}_q) = \sum_{\substack{(\gamma_0, \gamma, \delta) \\ \text{effective Born. class}}} c(\gamma_0) \cdot I(\gamma_0, \gamma, \delta), \quad \text{where } q = p^n.$$

Thm. (Milne, 1992)

Suppose (G, X) SD satisfying SV 4, 5, 6, G^{der} simply connected, $K_p \subset G(\mathbb{Q}_p)$ hyperspecial. $\mathfrak{p} | p$ prime of $E(G, X)$, Sh_p has canonical good reduction at $\mathfrak{p}$.

Then LR conj. implies Kottwitz conj.