Hodge Structures and Their Classifying Spaces

Reductive groups and tensors

G reductive group over k, char $k = 0$.

$\rho : G \longrightarrow GL(V)$ be a representation.

The dual rep. $\rho^\vee : G \longrightarrow GL(V^\vee)$ is $(\rho^\vee(g) f) v = f(\rho(g^{-1}) v)$

ρ is called selfdual if $\rho \simeq \rho^\vee$.

r tensor of V, $t : V^{\otimes r} \longrightarrow k$, $t \in (V^{\otimes r})^\vee$

Let $GL(V)_t = \{ g \in GL(V) \mid t(gv_1, \dots, gv_r) = t(v_1, \dots, v_r) \}$ algebraic subgroup

Prop. $\rho : G \longrightarrow GL(V)$ faithful selfdual representation, $\exists$ T finite set of tensors s.t. $G = \bigcap GL(V)_t$.

Pf. Chevalley's theorem $\Rightarrow$ $\exists$ W rep. of $GL(V)$, $L \subset W$ one-dim subspace s.t. $G = \{ g \in GL(V) \mid gL \subset L \}$.

As rep. of reductive G in char 0, W is semisimple, $W = L \oplus W'$.

Choose $0 \neq t \in L \otimes L^\vee \subset W \otimes W^\vee$, then $G = \mathrm{Stab}_t(GL(V))$. As $W \otimes W^\vee$ is a subrep. of $\sum V^{\otimes r} \otimes (V^\vee)^{\otimes s}$, which is just $\sum (V^{\otimes l})^\vee$, hence $t = \sum t_l$, $t_l \in (V^{\otimes l})^\vee$. ///

Prop. Let $G = \bigcap_T GL(V)_t$. Then $\mathrm{Lie}\, G = \{ g \in \mathrm{End}\, V \mid \sum_j t(v_1, \dots, gv_j, \dots, v_r) = 0, \forall t \}$

Flag Variety.

V n-dim v.s. $/k$.

$G_d(V)$ set of d-dim subspaces of V, $0 < d < n$.

$$G_d(V) \longrightarrow \mathbb{P}^{\binom{n}{d}-1} \quad \text{closed immersion}$$

$$W \longmapsto \wedge^d W$$

$S \subset V$ dim $n-d$ subspace, $G_d(V)_S = \{ W \in G_d(V) \mid W \cap S = \{0\} \}$.

Fix $W_0 \in G_d(V)_S$, $V = W_0 \oplus S$. Consider projection of $G_d(V)_S$ to W_0

$\Rightarrow$ each $W \in G_d(V)_S$ is the graph of $W_0 \to S$

$\Rightarrow$ $G_d(V)_S \simeq \mathrm{Hom}(W_0, S)$

$G_d(V)_S$ open subvariety of $G_d(V)$ is just affine $\mathbb{A}(\mathrm{Hom}(W_0, S))$

$\Rightarrow$ $G_d(V)$ smooth, $T_{W_0}(G_d(V)) \simeq \mathrm{Hom}(W_0, S) \simeq \mathrm{Hom}(W_0, V/W_0)$.

$\mathbf{d} = (d_1, \dots, d_r)$, $n > d_1 > \cdots > d_r > 0$

$G_{\mathbf{d}}(V) = \{ F : V \supset V^1 \supset \cdots \supset V^r \supset 0 \mid \dim V^i = d_i \}$

$$\exists \quad G_{\mathbf{d}}(V) \longrightarrow \prod_i G_{d_i}(V) \subset \prod_i \mathbb{P}(\wedge^{d_i} V) \quad \text{closed immersion}$$

$$F \longmapsto (V^i)$$

$\Rightarrow$ $G_{\mathbf{d}}(V)$ projective variety.

$T_F \, Gr_d(V) = \{ (\varphi^i),\ \varphi^i : V^i \to V/V^i,\ \varphi^i|_{V^{i+1}} \equiv \varphi^{i+1} \bmod V^{i+1} \}$

F flag, $\{e_i\}$ basis of V s.t. $\langle e_1, \ldots, e_{d_i} \rangle = V^i$.

$GL(V)$ acts transitively on basis $\Rightarrow$ $GL(V)$ acts transitively on $Gr_d(V)$

$P(F) = Stab_F(GL(V))$ algebraic subgroup, $GL(V)/P(F) \simeq Gr_d(V)$ projective

$\Rightarrow$ $P(F)$ parabolic.

Hodge Structures

V $\mathbb{R}$-v.s. , complex conjugate on $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ is defined by $\overline{v \otimes z} = v \otimes \bar{z}$.

A Hodge decomposition of V is $V_{\mathbb{C}} = \bigoplus V^{p,q}$ s.t. $\overline{V^{p,q}} = V^{q,p}$.

A Hodge structure is a $\mathbb{R}$-v.s. with a Hodge decomposition.

$\{(p,q) \mid V^{p,q} \neq 0\}$ is called type.

$V_n(\mathbb{C}) = \bigoplus_n V^{p,q}$ is stable under conjugate, hence $V_n(\mathbb{C}) = V_{n,\mathbb{C}}$.

$V = \bigoplus_n V_n$ is called weight decomposition. If $V = V_n$, V is said to have weight n.

A $\mathbb{Z}$ or $\mathbb{Q}$ Hodge structure is a free $\mathbb{Z}$ or $\mathbb{Q}$ module of finite rank with a Hodge decomposition of $V_{\mathbb{R}}$ s.t. the weight decomposition is defined over $\mathbb{Q}$.

Ex. J complex structure on V, $V^{-1,0}$ is i eigenspace of J on $V_{\mathbb{C}}$
$V^{0,-1}$ $-i$

$\Rightarrow V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$. $V^{-1,0} = \{ v\otimes 1 - Jv \otimes i,\ v \in V \}$.

Conversely every real Hodge structure of this type arises from a unique complex structure. If $v\otimes 1 + v' \otimes i \in V^{-1,0}$ then let $Jv = -v'$.

Rational / Integral Hodge structure of type $\{(-1,0),(0,-1)\}$ is the same as a
$\mathbb{Q}$ / $\mathbb{C}$ -v.s. V with complex structure on $V_{\mathbb{R}}$.
with $\Lambda \subset V$ lattice.

Hodge theory in complex geometry

Ex. X nonsingular projective algebraic variety over $\mathbb{C}$. Consider $H^n = H^n(X(\mathbb{C}), \mathbb{C})$.
$H^n = \bigoplus H^{p,q}$ with $H^{p,q} \simeq H^q(X(\mathbb{C}), \Omega^p)$.

Ex. $\mathbb{Q}(m)$ the unique Hodge structure of weight $-2m$, $V = (2\pi i)^m \mathbb{Q}$ and
$\mathbb{Q}(m)_{\mathbb{C}} = \mathbb{Q}(m)^{-m,-m}$. Similar for $\mathbb{R}(m)$, $\mathbb{Z}(m)$.

Hodge filtration associated to a Hodge structure of weight n is
$F^\bullet$: $\cdots \supset F^p \supset F^{p+1} \supset \cdots$, $F^p = \bigoplus_{r \geq p} V^{r,s}$
$F^p \cap \overline{F^q} = V^{p,q}$ if $p+q=n$.

Ex. Hodge structure of type $(-1,0)$, $(0,-1)$

$$F^{-1} \supset F^{0} \supset F^{1} = V_{\mathbb{C}} \supset V^{0,-1} \supset 0$$

the $\mathbb{R}$-linear isom. $V \longrightarrow V_{\mathbb{C}}/F^{0}$ defines the complex structure on V.

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ be the (algebraic) Deligne torus.

$\mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$ and $\mathbb{S}(\mathbb{C}) = \mathbb{C}^{\times} \times \mathbb{C}^{\times}$ with $\overline{(z_1, z_2)} = (\overline{z_2}, \overline{z_1})$.

The weight homomorphism $w: \mathbb{G}_m \longrightarrow \mathbb{S}$ is the map s.t. $w(\mathbb{R}): \mathbb{R}^{\times} \longrightarrow \mathbb{C}^{\times}$
$r \longmapsto r^{-1}$

Next interpretation is important.

$\mathbb{S}_{\mathbb{C}} \simeq \mathbb{G}_m \times \mathbb{G}_m$ and characters of $\mathbb{S}_{\mathbb{C}}$ are homomorphisms $(z_1, z_2) \longmapsto z_1^{r} z_2^{s}$.

Thus if V is a $\mathbb{R}$-rep. of $\mathbb{S}$, $h: \mathbb{S} \longrightarrow GL(V)$, consider

$V^{p,q} = \{ v \in V_{\mathbb{C}} \mid h_{\mathbb{C}}(z_1, z_2) v = z_1^{-p} z_2^{-q} v \}$, $\overline{V^{p,q}} = V^{q,p}$

and $V_n = \{ v \in V \mid w_n(r) v = r^{n} v \}$, $w_n = h \circ w$. $\{V_n\}$ gives the weight decomposition. If $\mu_n: \mathbb{C}^{\times} \longrightarrow GL(V)$, $\mu_n(z) = h_{\mathbb{C}}(z, 1)$ then the Hodge filtration

$F_n^{p} V = \{ v \in V_{\mathbb{C}} \mid \mu_n(z) v = z^{-r} v, \ r \geq p \}$.

Conversely given a Hodge structure V, one can construct the representation $h_V: \mathbb{S} \longrightarrow GL(V)$. Thus $\mathbb{R}$ Hodge structures are the same as $\mathbb{R}$-reps of $\mathbb{S}$. Similarly Hodge structures on $\mathbb{Q}$-v.s. V are the same as homomorphisms $h: \mathbb{S} \longrightarrow GL(V_{\mathbb{R}})$ s.t. w_n is defined over $\mathbb{Q}$.

Ex. A complex structure on V is homomorphism $h: \mathbb{C} \to \mathrm{End}_{\mathbb{R}} V$ of $\mathbb{R}$-algs.

h gives $h: \mathbb{C}^\times \to GL(V)$ is a Hodge structure of type $\{(-1,0),(0,-1)\}$ whose associated complex structure is defined by h.

The Hodge structure $\mathbb{Q}(n)$ corresponds to $h: \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$, $h(z) = (z\bar{z})^n$.

For a Hodge structure $h: \mathbb{S} \to GL(V)$, $C = h(i)$ is called the Weil operator.

If V is of type $(-1,0),(0,-1)$ then C is the complex structure J.

Let V be a Hodge structure of weight 0. Then $V^{0,0}$ invariant under complex conjugate, so $V^{0,0} = V^{00}_{\mathbb{C}}$ where $V^{00} = V^{0,0} \cap V = \ker(V \to V_{\mathbb{C}}/F^0)$.

The tensor product of Hodge structures V and W of weight m, n is a Hodge structure of weight $m+n$: $(V \otimes W)^{p,q} = \bigoplus V^{r,s} \otimes W^{r',s'}$

$h_{V\otimes W} = h_V \otimes h_W$.

A morphism of Hodge structure is a linear map sending $V^{p,q}$ to $W^{p,q}$, i.e. is a morphism of rep. of $\mathbb{S}$.

In general (V,h) Hodge structure, $(V^\vee, h^\vee)$ is dual Hodge structure and $t \in V^{\otimes r} \otimes (V^{\otimes s})^\vee$ s.t. ...

$R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, (V,h) be an R-Hodge structure of weight n. $t \in (V^{\otimes r})^\vee$ is called a Hodge tensor if $V^{\otimes r} \to R(-\frac{nr}{2})$ is a morphism of HS.

$\Leftrightarrow \quad t_{\mathbb{R}}(h(z)v_1, \dots) = (z\bar{z})^{-\frac{nr}{2}} t_{\mathbb{R}}(v_1, \dots), \quad z \in \mathbb{C}, \quad v_i \in V_{\mathbb{R}}$

$\Leftrightarrow$ if $\sum p_i \neq \sum q_i$ then $t_{\mathbb{C}}(v_1^{p_1,q_1}, \dots) = 0$, $v_i^{p_i,q_i} \in V^{p_i,q_i}$.

In particular $t(Cv_1, Cv_2, \dots) = t(v_1, v_2, \dots)$.

Ex. (V,h) Hodge structure of type $(-1,0), (0,-1)$. $t \in (V^{\otimes 2})^\vee$ is Hodge tensor

$\Leftrightarrow t(Ju, Jv) = t(u,v)$.

Let (V,h) be a Hodge structure of weight n. A polarization of (V,h) is a Hodge tensor $\psi : V \times V \longrightarrow \mathbb{R}(-n)$ s.t. $\psi_{\mathbb{C}}(u,v) = (2\pi i)^n \psi(u, Cv)$ is symmetric and positive definite. Then

$$\psi(v,u) = \psi(Cv, Cu) = (2\pi i)^{-n}\psi_c(Cv,u) = (2\pi i)^{-n}\psi_c(u, Cv) = \psi(u, C^2 v)$$
$$= (-1)^n \psi(u,v).$$

A Hodge structure is polarizable if it admits a polarization on each V_n.

Ex. (V,h) Hodge structure of type $(-1,0), (0,-1)$, $J = h(i)$.

$\psi : V \times V \longrightarrow 2\pi i\mathbb{R} = \mathbb{R}(1)$ alternating bilinear form s.t.

$\psi(Ju, Jv) = \psi(u,v)$ and $\frac{1}{2\pi i}\psi(u, Ju) > 0$ if $u \neq 0$.

$X/\mathbb{C}$ nonsingular proj. variety, choose $X \hookrightarrow \mathbb{P}^N$ determines a polarization on the primitive part of $H^n(X(\mathbb{C}), \mathbb{Q})$ for each n. (Hodge-Riemann inequality)

Variation of Hodge structure.

Motivation. $\pi : V \to S$ nonsingular alg. var. $/\mathbb{C}$, V_s nonsingular proj. var.
$H^n(V_s, \mathbb{Q})$ local system of $\mathbb{Q}$-v.s. on $S(\mathbb{C})$, their Hodge decompositions vary continuously and Hodge filtrations vary holomorphically, satisfying Griffiths transversality.

Goal: realize HSD as moduli space for VHS.

S connected cpx mfd, V $\mathbb{R}$-v.s.

For each $s \in S$, we have a Hodge struc. h_s on V of wt n.

$V_s^{p,q} = V_{h_s}^{p,q}$, $F_s^p = F_{h_s}^p V$.

$\{h_s\}_{s\in S}$ is called cont. if $V_s^{p,q}$ varies cont. in s, i.e. $d(p,q)$ is constant and $S \to G_{d(p,q)}(V_{\mathbb{C}})$ cont.

$$s \longmapsto V_s^{p,q}$$

cont. $\{h_s\}_{s\in S}$ is called holo. if $F_s^{\bullet}$ varies holo. in s, i.e. φ holo

$$\varphi : S \longrightarrow G_d(V_{\mathbb{C}}), \quad d = (\cdots d(p) \cdots), \quad d(p) = \sum_{r\geq p} d(r,q)$$
$$s \longmapsto F_s^{\bullet}$$

$\{h_s\}_{s\in S}$ holo. $\Rightarrow$ $d\varphi_s : T_s S \longrightarrow T_{F_s^{\bullet}} G_d(V_{\mathbb{C}}) \subset \bigoplus \mathrm{Hom}(F_s^p, V_{\mathbb{C}}/F_s^p)$

Griffiths transversality: $\mathrm{Im}\, d\varphi_s \subset \bigoplus \mathrm{Hom}(F_s^p, F_s^{p-1}/F_s^p)$, $\forall s$

$\{h_s\}_{s\in S}$ VHS if satisfying Griffiths transversality.

V $\mathbb{R}$-v.s. , T tensors on V including t_0 nondegenerate bilinear.

$d: \mathbb{Z}\times\mathbb{Z} \to \mathbb{N}$ s.t. $\begin{cases} d(p,q)=0 & \text{almost all } (p,q) \\ d(p,q)=d(q,p) \\ d(p,q)=0 & \text{unless } p+q=n \end{cases}$

Define $S(d,T)$ set of all Hodge structures h on V s.t.

- $\dim V_h^{p,q} = d(p,q)$
- $\forall t\in T$, t Hodge tensor for h
- t_0 is polarization for h.

Then $S(d,T)$ acquires topology as a subset of $\prod_{d(p,q)\neq 0} G_{d(p,q)}(V_{\mathbb{C}})$.

Thm. S^+ connected component of $S(d,T)$.

(a) S^+ has a unique complex stru. for which (h_s) is holo.

(b) S^+ is HSD if (h_s) is VHS.

(c) Every irreducible HSD is of the form S^+ for some V, d, T.

Pf.

(a)

Hodge filtration determines Hodge decomposition $\Rightarrow$ $\varphi: S^+ \longrightarrow G_d(V_{\mathbb{C}})$ inj.
$s \longmapsto F_s^{\bullet}$

$G = \cap H$, $H \subset GL(V)$ algebraic subgroup s.t. $h_s(\mathbb{S}) \subset H$, $\forall s\in S^+$.

Choose $h_0 \in S^+$, $\forall g \in G(\mathbb{R})^+$. $\underline{g h_0 g^{-1} \in S^+}$ and $\underline{G(\mathbb{R})^+ \twoheadrightarrow S^+}$ Deligne
$g \longmapsto g h_0 g^{-1}$

Deligne: $G/\mathbb{R}$ alg. grp, X conn. component of $\mathrm{Hom}(\mathbb{S}, G)$. G_1 smallest $\subset G$ s.t. all $h \in X$ factor. Then X is conn. component of $\mathrm{Hom}(\mathbb{S}, G_1)$. $\mathbb{S}$ torus, any two in X are conjugate. X is $G_1(\mathbb{R})^+$-conjugacy class of maps $\mathbb{S}$ to G.

$\Rightarrow S^+ = G(\mathbb{R})^+ \cdot h_0$.

$\mathrm{Stab}_{h_0}(G(\mathbb{R})^+) = K_0$ closed $\Rightarrow S^+ = (G(\mathbb{R})^+/K_0)\cdot h_0 \simeq G(\mathbb{R})^+/K_0$ is smooth mfd.

$\mathfrak{g} = \mathrm{Lie}\, G$, $G \hookrightarrow GL(V) \Rightarrow \mathfrak{g} \hookrightarrow \mathrm{End}(V)$.

h_0 makes the inclusion of Hodge structure of wt 0.

$\Rightarrow \mathrm{Lie}\, K_0 = \mathfrak{g}^{00}$, $T_{h_0} S^+ \simeq \mathfrak{g}/\mathfrak{g}^{00}$.

$$
\begin{array}{ccccc}
T_{h_0} S^+ \simeq \mathfrak{g}/\mathfrak{g}^{00} & \hookrightarrow & \mathrm{End}(V)/\mathrm{End}(V)^{00} & & \\
\wr\wr & & \wr\wr & & \\
\mathfrak{g}_{\mathbb{C}}/F^0 & \hookrightarrow & \mathrm{End}(V)_{\mathbb{C}}/F^0\mathrm{End}(V)_{\mathbb{C}} & \overset{(*)}{\simeq} & T_{h_0} G_d(V_{\mathbb{C}})
\end{array}
$$

the isom. $(*)$ is because $\mathrm{End}\, V_{\mathbb{C}}^{r,s} = \{\varphi \in \mathrm{End}\, V_{\mathbb{C}},\ h(z_1, z_2)\varphi = z_1^{-r} z_2^{-s}\varphi\}$

$= \{\varphi \in \mathrm{End}\, V_{\mathbb{C}},\ \varphi V^{p,q} \subset V^{p+r,\, q+s}\}$

and recall we have $G_d(V_{\mathbb{C}}) \simeq GL(V_{\mathbb{C}})/P(F^\bullet_{h_0})$, $\mathrm{Lie}\, P(F^\bullet_{h_0})_{\mathbb{C}}$ is exactly those in $\mathrm{End}(V)_{\mathbb{C}}$ preserving Hodge filtration, i.e. $F^0\mathrm{End}(V)_{\mathbb{C}}$.

The map from top left to bottom right is $(d\varphi)_{h_0}$, therefore maps $T_{h_0}(S^+)$ as a complex subspace of $T_{h_0}(G_d V_{\mathbb{C}})$, hence φ identifies S^+ with an almost complex submanifold of $G_d V_{\mathbb{C}}$. It is integrable hence gives S^+ the unique complex stru. for φ to be holo.

(b) Let $h, h_0 \in S^+$, $h = g h_0 g^{-1}$, $g \in G(\mathbb{R})^+$.

V_h has weight n, $h(r)$ acts r^{-n} on V, $r \in \mathbb{R}$.

Hence $g h_0(r) g^{-1} = h_0(r)$, $h_0(r) \in Z(G)$.

Let $u_0 : U_1 \longrightarrow G^{ad} = G/Z(G)$ is well-defined.

$z \longmapsto h_0(\sqrt{z})$

Then consider G_0 be the alg. subgrp of G fixing t_0. Clearly u_0 factors through G_0^{ad}, $u_0 : U_1 \to G_0^{ad}$. By Deligne again, $S^+ = G_0^{ad}(\mathbb{R})^+ \cdot u_0$.

The $C = h_0(i) = u_0(-1)$ - polarization t_0 implies that $Ad(u_0(i))$ is a Cartan involution of G_0^{ad}. (In particular G_0^{ad} reductive, center trivial $\Rightarrow$ semisimple).

We have $\mathfrak{g}/\mathfrak{g}^{00} \cong T_{h_0} S^+ \subset T_{h_0} Gd\, V_{\mathbb{C}} \cong End\, V_{\mathbb{C}} / F^0 End\, V_{\mathbb{C}}$ and the Griffiths transversality tells us $\mathfrak{g}/\mathfrak{g}^{00} \subset F^{-1} End\, V_{\mathbb{C}} / F^0 End\, V_{\mathbb{C}}$, hence the action of U_1 on $\mathfrak{g}_{\mathbb{C}}$ only has charaters $1, z, z^{-1}$. Any compact factors of G_0^{ad} can be disgarded. Hence S^+ is just the associated Herm. Sym. Dom. to (G_0^{ad}, u_0).

One can resolve this by allowing Hodge tensors live in $V^{\otimes r} \otimes (V^\vee)^{\otimes s}$.

(C) Let D be an irr. Herm. Sym. Dom., G the connected adjoint group s.t. $G(\mathbb{R})^+ = Hol(D)^+$. Choose a faithful self-dual rep. $G \to GL(V)$.

As V self-dual, $\exists$ nondegenerate bilinear form t_0 on V fixed by G.

We may find a set of tensors T containing t_0 s.t. G is the subgrp of $GL(V)$ fixing $t \in T$. Let $h_0 : \mathbb{S} \longrightarrow U_1 \xrightarrow{u_0} G \longrightarrow GL(V)$

$$z \longmapsto \frac{z}{\bar{z}}$$

Then h_0 defines a Hodge structure on V for which T are Hodge tensors and t_0 is a polarization. Then D is the connected component of $S(d, T)^+$ containing h_0.

///

RMK. $S^+ \longrightarrow G_d V_{\mathbb{C}}$ is an embedding of smooth manifolds, injective smooth maps that are inj. on tangent spaces and maps S^+ homeo. to its image.

Hence if

$$\begin{tikzcd} T \arrow[r, "\beta"] \arrow[dr, hook, "\alpha"'] \arrow[dr, phantom, shift left=3, "\circlearrowright"] & G_d V_{\mathbb{C}} \\ & S^+ \arrow[u] \end{tikzcd}$$

, β smooth $\Rightarrow$ α smooth

β defined by holo. family of Hodge structure on T $\Rightarrow$ α holo.

RMK. Herm. Sym. Dom. can be actually realized as moduli varieties for Hodge structures (in complex manifolds). Also realizing Herm. Sym. Dom. as parameter space for Hodge structures can be done using Tannakian point of view.