Locally Symmetric Varieties.

Study Herm. Sym. Dom. quotient by discrete subgroups.

Slogan: $\Gamma \backslash D$ is alg. variety, $\Gamma \subset \mathrm{Hol}(D)^+$ torsion free arithmetic.

Quotients of Hermitian Symmetric Domains by Discrete Groups.

Prop. Let D be a HSD, $\Gamma \subset \mathrm{Hol}(D)^+$ discrete. If Γ torsion free, then Γ acts freely on D and there is a unique complex structure on $\Gamma\backslash D$ s.t. $\pi: D \to \Gamma \backslash D$ is a local isom. Then any $\varphi: \Gamma\backslash D \to Y$, Y complex manifold, φ holo. $\Leftarrow$ $\varphi \circ \pi$ holo.

Pf. $\mathrm{Hol}(D)^+ / K_p \simeq D$ so

- $\{ g \in \Gamma \mid gp = p \}$ is finite
- $\exists$ nbhd U of p s.t. $\forall g \in \Gamma - \{1\}$, $gU \cap U = \phi$
- $\forall p, q \in D$, $p \notin \Gamma q$, $\exists$ nbhd U of p, V of q s.t. $\forall g \in \Gamma$, $gU \cap V = \phi$.

$\Rightarrow$ $\Gamma \backslash D$ Hausdorff, has a manifold structure.

$U \subset \Gamma\backslash D$ open, $\mathcal{O}(U) = \{ f : U \to \mathbb{C} \mid f \circ \pi$ holo. on $\pi^{-1}U \}$. $\mathcal{O}$ structure sheaf of holo. functions. ///

Γ torsion free, write $D(\Gamma) = \Gamma \backslash D$. D is universal covering of $D(\Gamma)$ and Γ fundamental group: $\forall p \in D$, $\Gamma \simeq \pi_1(D(\Gamma), \pi(p))$.

$$g \longmapsto \pi(p \rightsquigarrow gp)$$

D has Riemannian metric g hence volume form Ω which in local coordinates $\Omega = \sqrt{\det g(x)}\, dx^1 \wedge \cdots \wedge dx^n$ and invariant under Γ $\rightsquigarrow$ $\int_{\Gamma \backslash D} \Omega$.

Ex. $D = \mathcal{H}$, $\Gamma = PSL_2(\mathbb{Z})$, $\int_{\Gamma\backslash D} \Omega = \iint_F \frac{dx\,dy}{y^2} \leq \int_{\sqrt{3}/2}^{\infty} \frac{dy}{y^2} < +\infty$, F fundamental domain $\{ z \in \mathcal{H},\ -\frac{1}{2} < \operatorname{Re} z < \frac{1}{2},\ |z| > 1 \}$.

G real Lie group has a left invariant Haar measure μ, $\Gamma \subset G$ discrete is said to have finite covolume if $\Gamma \backslash G$ has finite volume. For $\Gamma \subset \operatorname{Hol}(D)^+$ torsion free discrete, $|\Gamma \backslash \operatorname{Hol}(D)^+| < +\infty \overset{\text{Fubini}}{\iff} |\Gamma \backslash D| < +\infty$.

Arithmetic subgroups

Two subgroups S_1, S_2 of H are called commensurable if $S_1 \cap S_2$ has finite index in both S_1, S_2. Commensurability is an equi. relation.

$G/\mathbb{Q}$ alg. grp., $\Gamma \subset G(\mathbb{Q})$ is called arithmetic if it is commensurable with $G(\mathbb{Q}) \cap GL_n(\mathbb{Z})$ for some embedding $G \hookrightarrow GL_n$

$\iff$ any embedding $G \hookrightarrow GL_{n'}$.

Prop. Let $p: G \twoheadrightarrow G'$ be surj. map of alg. grps $/\mathbb{Q}$. If $\Gamma \subset G(\mathbb{Q})$ arithmetic so is $p(\Gamma) \subset G'(\mathbb{Q})$.

Thm. $G/\mathbb{Q}$ reductive, $\Gamma \subset G(\mathbb{Q})$ arithmetic.

(a) $|\Gamma \backslash G(\mathbb{R})| < +\infty \iff \mathrm{Hom}(G, \mathbb{G}_m) = 0$. In particular holds for G semisimple.

(b) $\Gamma \backslash G(\mathbb{R})$ compact $\iff \mathrm{Hom}(G, \mathbb{G}_m) = 0$ and $G(\mathbb{Q})$ has no nontrivial unipotent element.

rational unipotent elements correspond to cusps

Ex. B quaternion algebra $/\mathbb{Q}$ s.t. $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R})$ and $G/\mathbb{Q}$ alg. grp s.t. $G(\mathbb{Q}) = \{ b \in B \mid \mathrm{Nm}\, b = 1 \}$. The choice of $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R})$ determines an isom. $G(\mathbb{R}) \xrightarrow{\sim} SL_2(\mathbb{R})$ hence an action of $G(\mathbb{R})$ on $\mathcal{H}$. Let $\Gamma \subset G(\mathbb{Q})$ arithmetic.

$B \simeq M_2(\mathbb{Q})$, $G \simeq SL_2$ semisimple, $\Gamma \backslash SL_2(\mathbb{R})$, $\Gamma \backslash \mathcal{H}$ has finite volume. But $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Q})$ unipotent $\Rightarrow \Gamma \backslash \mathcal{H}$ not compact.

B division algebra, $G(\mathbb{Q})$ contains no nontrivial unipotent element otherwise $B^\times$ would have a nilpotent element. $\Gamma \backslash G(\mathbb{R})$ compact.

Let k be a subfield of $\mathbb{C}$. An automorphism of a k-v.s. V is called neat if its eigenvalues in $\mathbb{C}$ generate a torsion free subgroup of $\mathbb{C}^\times$. For example, no nontrivial automorphism of finite order is neat.

$G/\mathbb{Q}$ alg. group, $g \in G(\mathbb{Q})$ is called neat if $\rho_0(g)$ is neat for some faithful rep. $G \hookrightarrow GL(V)$, in which case $\rho(g)$ neat for every rep. ρ of G over a subfield of $\mathbb{C}$ as all rep. of G can be constructed from ρ_0.

A subgroup of $G(\mathbb{Q})$ is called neat if all elements are.

- neat $\Rightarrow$ torsion free
- $\Gamma \subset G(\mathbb{Q})$ neat $\Rightarrow \forall\, \Gamma' \subset \Gamma$ neat
- $\Gamma \subset G(\mathbb{Q})$ neat, $\forall\, k \subset \mathbb{C}$ subfield, $h: G(k) \to H$, $h(\Gamma)$ torsion free
- $g \in G(\mathbb{Q})$ neat $\Rightarrow \pi(g) \in G^{ad}(\mathbb{Q})$ neat.

Prop. $G/\mathbb{Q}$ alg. group, $\Gamma \subset G(\mathbb{Q})$ arith. Then Γ contains a neat subgrp Γ' of finite index, and Γ' can be defined by congruence condition $\{g \in \Gamma \mid g \equiv 1 \mod N\}$ for some embedding $G \longrightarrow GL_n$ and N.

Let H be a connected real Lie group, $\Gamma \subset H$ is called arithmetic if there exists $G/\mathbb{Q}$ alg. grp, $G(\mathbb{R})^+ \twoheadrightarrow H$ with compact kernel and $\Gamma_0 \subset G(\mathbb{Q})$ arithmetic s.t. $\Gamma_0 \cap G(\mathbb{R})^+ \twoheadrightarrow \Gamma$. Note that if H semisimple, we can take G semisimple.

Prop. Let H be a semisimple real Lie group admitting a faithful finite dimensional rep. Every arithmetic $\Gamma \subset H$ is discrete of finite covolume and it contains a torsion free subgrp of finite index.

Pf. $\alpha : G(\mathbb{R})^+ \twoheadrightarrow H$, $\Gamma_0 \subset G(\mathbb{Q})$.

$\ker\alpha$ compact $\Rightarrow$ α proper $\Rightarrow$ α closed.

$\Gamma_0 \subset G(\mathbb{R})$ discrete $\Rightarrow$ $\exists\, U \subset G(\mathbb{R})^+$ open, $U \cap \Gamma_0 \ker\alpha = \ker\alpha$ $\Rightarrow$ $\alpha(G(\mathbb{R})^+ - U)$ closed in H, whose complement intersects Γ in $\{1\}$ $\Rightarrow$ Γ discrete in H.

$\Gamma_0 \backslash G(\mathbb{R})^+ \twoheadrightarrow \Gamma \backslash H$ $\Rightarrow$ finite volume.

$\Gamma_1 \subset \Gamma_0$ neat of finite index, image of Γ_1 in H has finite index in Γ. ///

$$1 \to \ker\alpha \to G(\mathbb{R})^+ \xrightarrow{\alpha} H \to 1$$

$$\Gamma_1 \cap G(\mathbb{R})^+$$

$$a \longmapsto \bar{a}$$

$\bar{a}^n = 1 \Rightarrow a^n \in \ker\alpha \cap \Gamma_1$ discrete

$\Rightarrow a^{nl} = 1$, $a \in \Gamma_1$ torsion free $\Rightarrow a = 1$.

Algebraic varieties and complex manifolds.

Prop. $\exists !$ functor $(V, \mathcal{O}_V) \longmapsto (V^{an}, \mathcal{O}_{V^{an}})$ from nonsingular varieties $/\mathbb{C}$ to complex manifolds satisfying

(a) $V = V^{an}$ as sets, every Zariski open subset is open for complex topology and every regular function is holo.

(b) $V = \mathbb{A}^n$ then $V^{an} = \mathbb{C}^n$

(c) $V \to W$ etale then $V^{an} \to W^{an}$ local Isom.

Pf. $\mathbb{A}^n$, opens in $\mathbb{A}^n$, any V admits an etale map to some open in $\mathbb{A}^n$. Every nonsingular variety has a Zariski open covering by such V. ///

Rmk. The functor is faithful, but not full or essentially surjective. It can be extended to all alg. varieties, valueing in complex analytic space, i.e. a ringed space locally of the form $(V, \mathcal{O}_V)$ where $V = V(f_1, \dots, f_r)$ is the zero locus of holo. f_i on connected open $U \subset \mathbb{C}^n$.

Thm. (Chow) Every proj. complex analytic space has a unique structure of a proj. alg. variety and every holo. map of proj. complex analytic spaces is regular for these structures. Complex manifolds give rise to nonsingular alg. varieties.

Lie group

Thm. (Baily & Borel) Let $D(\Gamma) = \Gamma \backslash D$ be the quotient of a Herm. Sym. Dom. D by a torsion free arith. $\Gamma \subset \mathrm{Hol}(D)^+$. Then $D(\Gamma)$ has a canonical realization as Zariski open of a proj. alg. variety $D(\Gamma)^*$ hence has a canonical struc. of an alg. variety. auto. form of high enough weight gives closed immersion into proj. space

smooth quasi-proj.

RMK. In case Γ has torsion, $\Gamma \backslash D$ is a normal complex analytic space and has the structure of a normal alg. variety.

$D(\Gamma)^* = \mathrm{Proj}\left(\bigoplus_{n \geq 0} A_n\right)$ where A_n is the v.s. of auto. forms for n th power of canonical automorphy factor, and if PGL_2 is not a quotient of G, then $D(\Gamma)^* = \mathrm{Proj}\left(\bigoplus_{n \geq 0} H^0(D(\Gamma), \omega^n)\right)$, ω sheaf of differentials of max deg.

An alg. variety $D(\Gamma)$ arises in this way is called a (arithmetic) locally symmetric variety.

Thm. (Borel) Let $D(\Gamma) = \Gamma \backslash D$ be quotient of a Herm. Sym. Dom. D by torsion free arith. $\Gamma \subset \mathrm{Hol}(D)^+$ and V be a nonsingular quasi-proj. variety over $\mathbb{C}$, then every holo. $f : V^{an} \to D(\Gamma)^{an}$ is regular.

The proof is to embed V in a proj. nonsingular variety V^* s.t. $V^* - V$ is divisor with normal crossing. Then extend f to $V^{*,an}$ hence regular by Chow.

Cor. The alg. variety structure on $D(\Gamma)$ is unique.

RMK. Torsion-freeness is necessary. $\Gamma(1)\backslash\mathcal{H} = \mathbb{A}^1$, $\mathbb{C} \xrightarrow{\exp} \mathbb{C}$.

Finiteness of the group $\mathrm{Aut}(D(\Gamma))$.

Def. A semisimple group $G/\mathbb{Q}$ is said to be of compact type if $G(\mathbb{R})$ is compact, of noncompact type if it does not contain a nontrivial normal subgrp of compact type.

$G/\mathbb{Q}$ semisimple, $\exists$ isogeny $G_1 \times \cdots \times G_r \longrightarrow G$, G_i simple. Then G is of compact / noncpt type if each / no $G_i(\mathbb{R})$ is compact. In particular a simply connected or adjoint group is of noncompact type iff no simple factor is of compact type.

Thm. (Borel Density) $G/\mathbb{Q}$ semisimple of noncompact type, then every arithmetic $\Gamma \subset G(\mathbb{Q})$ is Zariski dense in G.

Cor. $G/\mathbb{Q}$ semisimple of noncompact type, $Z = Z(G)$. The centralizer in $G(\mathbb{R})$ of any arith. Γ of $G(\mathbb{Q})$ is $Z(\mathbb{R})$.

Thm. Let $D(\Gamma) = \Gamma \backslash D$, D HSD, Γ torsion free arith. $\subset \mathcal{H}ol(D)^+$. Then $D(\Gamma)$ has only finitely many automorphisms as a complex manifold.

Pf.

Γ torsion free $\Rightarrow$ D universal covering of $\Gamma \backslash D$ and Γ transformation group

$\Rightarrow$ $\alpha : \Gamma \backslash D \to \Gamma \backslash D$ auto. lifts to $\tilde{\alpha} : D \to D$ auto.

$\Rightarrow$ $\tilde{\alpha} \gamma \tilde{\alpha}^{-1} \in \Gamma$, $\forall \gamma \in \Gamma$.

Conversely any auto. of D normalizing Γ gives an auto. of $\Gamma \backslash D$. Hence there is a surj. $\Gamma \backslash N(\Gamma) \twoheadrightarrow \mathrm{Aut}(\Gamma \backslash D)$, $N(\Gamma)$ normalizer in $\mathrm{Aut}(D)$.

$G/\mathbb{Q}$ semisimple alg. grp, $G(\mathbb{R})^+ \twoheadrightarrow \mathcal{H}ol(D)^+$ with compact kernel, $\Gamma_0 \subset G(\mathbb{Q})$ arith. s.t. $\Gamma_0 \cap G(\mathbb{R})^+ \twoheadrightarrow \Gamma$. We may disgard any compact isogeny factors of G and suppose G is of noncompact type. Let N^+ be the identity comp. of $N(\Gamma)$. As Γ discrete, $N(\Gamma)$ acts trivially and N^+ is contained in the image of $Z(\mathbb{R})$ hence finite, $N(\Gamma)$ discrete. $\Gamma \backslash \mathrm{Aut} D$ has finite volume, $\Gamma \backslash N(\Gamma)$ is then finite. ///