Connected Shimura Varieties.

Congruence subgroups.

$G/\mathbb{Q}$ reductive, choose $G \hookrightarrow GL_n$ and let

$$\Gamma(N) = G(\mathbb{Q}) \cap \{ g \in GL_n(\mathbb{Z}) \mid g \equiv I_n \mod N \}$$

A congruence subgroup of $G(\mathbb{Q})$ is one containing $\Gamma(N)$ as a subgrp of finite index. In particular congruence subgroups are arithmetic.

RMK. Arithmetic groups may not be congruence subgroups. The image of congruence subgroup under an isogeny may not be congruence subgroup.

Given $V/\mathbb{Q}$ variety, V has flat models over $\operatorname{Spec} \mathbb{Z}$ and any two of which are isom. over a nonempty open subset of $\operatorname{Spec} \mathbb{Z}$, hence we can define $V(\mathbb{A}_f) = \prod_\ell (V(\mathbb{Q}_\ell), \tilde{V}(\mathbb{Z}_\ell))$. This is well-defined.

$V/\mathbb{Q}$ affine, $V \xhookrightarrow{\alpha} \mathbb{A}^n_{\mathbb{Q}}$, $V_\alpha = \bar{V}$ in $\mathbb{A}^n_{\mathbb{Z}}$. For $V \xhookrightarrow{\beta} \mathbb{A}^m_{\mathbb{Q}}$

$V_{\alpha,\mathbb{Q}} \simeq V \simeq V_{\beta,\mathbb{Q}}$ extends to $V_\alpha \simeq V_\beta$ on $\operatorname{Spec} \mathbb{Z}[\frac{1}{d}]$, d being s.t. coefficients of all polynomials defining V are in $\mathbb{Z}[\frac{1}{d}]$.

congruence subgroups $\overset{G(\mathbb{Q})}{\subset}$ come from congruence condition on compact open subgroups $\overset{G(\mathbb{A}_f)}{\subset}$

Prop. K compact open $\subset G(\mathbb{A}_f)$, $K \cap G(\mathbb{Q})$ congruence subgroup. All congruence subgroup is of this form.

Pf. Choose embedding $G \hookrightarrow GL_n$. We get surj. $\mathbb{Q}[GL_n] \to \mathbb{Q}[G]$ and $\mathbb{Q}[G] = \mathbb{Q}[x_{11}, \ldots, x_{nn}, t]$. Hence $G(\mathbb{Z}_\ell) = G(\mathbb{Q}_\ell) \cap GL_n(\mathbb{Z}_\ell)$.

For $N > 0$, let $K(N) = \prod_\ell K_\ell$, $K_\ell = \begin{cases} G(\mathbb{Z}_\ell) & \ell \nmid N \\ \{g \in G(\mathbb{Z}_\ell) \mid g \equiv I_n \bmod \ell^{r_\ell}\} & r_\ell = \mathrm{ord}_\ell N \end{cases}$

Then $K(N)$ compact open in $G(\mathbb{A}_f)$, $K(N) \cap G(\mathbb{Q}) = \Gamma(N)$.

Then

$\{$compact open subgrps of $G(\mathbb{A}_f)$ containing $K(N)\} \cap G(\mathbb{Q})$

$= \{$congruence subgroups of $G(\mathbb{Q})$ containing $\Gamma(N)\}$

and every compact open subgrp of $G(\mathbb{A}_f)$ contains some $K(N)$. ///

RMK. The topology on $G(\mathbb{Q})$ defined by letting congruence subgrps as a fundamental system of nbhds of 1 coincides with the topology defined by the diagonal embedding $G(\mathbb{Q}) \to G(\mathbb{A}_f)$.

Connected Shimura Data.

D is a connected component.

Def. A conn. SD is (G, D), G semisimple alg. grp / $\mathbb{Q}$, D is a $G^{ad}(\mathbb{R})^+$ - conjugacy class of homomorphisms $u: U_1 \longrightarrow G^{ad}_{\mathbb{R}}$ s.t.

SU 1: for all $u \in D$, only the characters $1, z, z^{-1}$ appear in the rep. of U_1 on Lie $G^{ad}_{\mathbb{C}}$ via $Ad \circ u$.

SU 2: for all $u \in D$, $Ad(u(-1))$ is a Cartan involution on $G^{ad}_{\mathbb{R}}$

SU 3: G^{ad} has no $\mathbb{Q}$-factor H s.t. $H(\mathbb{R})$ compact.

SU 3: Borel density, strong approximation

Ex. $u: U_1 \longrightarrow PGL_2(\mathbb{R})$, D set of conjugates of u by $SL_2(\mathbb{R})$

$$z = (a+bi)^2 \longmapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \bmod \pm I$$

Then (SL_2, D) is a conn. SD.

RMK. If $u: U_1 \to G^{ad}(\mathbb{R})$ satisfies SU 1, 2 then so does any conjugate of it by $G^{ad}(\mathbb{R})^+$. Thus a pair (G, u) satisfying SU 1, 2, 3 gives a conn. SD.

Lemma. $H/\mathbb{R}$ adjoint Lie group, $u: U_1 \to H$ satisfying SU 1, 2. TFAE.

(a) $u(-1) = 1$

(b) u trivial

(c) H compact.

Pf.

$u(-1)=1 \Rightarrow u$ factors through $U_1 \xrightarrow{(\ )^2} U_1$ so $z^{\pm 1}$ cannot occur in the rep. of U_1 on Lie $H_{\mathbb{C}}$ $\Rightarrow U_1$ acts trivially on Lie $H_{\mathbb{C}}$ $\Rightarrow u$ trivial.

H compact $\Leftrightarrow Ad(u(-1)) = id_H \Leftrightarrow u(-1)=1$. ///

Relation with HSD.

Prop. To give a conn. SD is the same as to give (G, D) s.t.

- $G/\mathbb{Q}$ semisimple alg. grp. of noncompact type $\quad G^{ad}$ has no $\mathbb{Q}$-factor H s.t. $H(\mathbb{R})$ cpt.
- D Herm. Sym. Dom.
- $G(\mathbb{R})^+$ acts on D by a surj. homomorphism $G^{ad}(\mathbb{R})^+ \longrightarrow Hol(D)^+$ with compact kernel.

Pf. Let (G, D) be a conn. SD, $u \in D$.

$G^{ad}(\mathbb{R})^+ = H_1(\mathbb{R})^+ \times \cdots \times H_s(\mathbb{R})^+$

$G^{ad}_{\mathbb{R}} = H_1 \times \cdots \times H_s$ product of simple factors, $u = (u_1, \dots, u_s)$.

G^{ad} has no $\mathbb{Q}$-factor on which u trivial

$u_i = 1 \Leftrightarrow H_i$ compact. but possibly has $\mathbb{R}$-factor on which u trivial.

u_i nontrivial, $\exists$ irr. Herm. Sym. Dom. D_i' s.t. $H_i(\mathbb{R})^+ = Hol(D_i')^+$ and D_i' is naturally bijective to the set D_i of $H_i(\mathbb{R})^+$ conjugates of u_i.

$D' = \prod D_i'$ is a Herm. Sym. Dom. on which $G(\mathbb{R})^+$ acts by a surj. $G^{ad}(\mathbb{R})^+ \longrightarrow Hol(D)^+$ with compact kernel. $D' = D = \prod D_i$.

Conversely given $(G, D, G(\mathbb{R})^+ \xrightarrow{\rho} Hol(D)^+)$, decompose $G^{ad}_{\mathbb{R}}$.

Let H_c product of compact factors.

H_{nc} noncompact

$G^{ad}(\mathbb{R})^+ \twoheadrightarrow Hol(D)^+ \Rightarrow H_{nc}(\mathbb{R})^+ \twoheadrightarrow Hol(D)^+$, kernel is compact normal, hence trivial.

$G(\mathbb{R})^+$ acts on D gives $H_{nc}(\mathbb{R})^+ \simeq Hol(D)^+$ and $\{u_p, p \in D\}$ is an $H_{nc}(\mathbb{R})^+$ - conjugacy class of $U_1 \to H_{nc}(\mathbb{R})^+$ satisfying SU 1, 2. Then

$\{(1, u_p) : U_1 \to H_c(\mathbb{R}) \times H_{nc}(\mathbb{R}) \mid p \in D\}$ is a $G^{ad}(\mathbb{R})^+$ - conjugacy class of homomorphisms $U_1 \to G^{ad}(\mathbb{R})$ satisfying SU 1, 2. ///

Prop. Let (G, D) be a conn. SD, X the $G^{ad}(\mathbb{R})$ - conj. class of maps $\mathbb{S} \to G_{\mathbb{R}}$ containing D. Then D is a conn. component of X and stab. of D in $G^{ad}(\mathbb{R})$ is $G^{ad}(\mathbb{R})^+$.

Next we build varieties.

Definition of a connected Shimura variety.

Let (G, D) be a conn. SD, D Herm. Sym. Dom. with $G(\mathbb{R})^+$ action.

As $G^{ad}(\mathbb{R})^+ \twoheadrightarrow Hol(D)^+$ has compact kernel, $\forall\ \Gamma \subset G^{ad}(\mathbb{Q})^+$ arithmetic, $\bar{\Gamma} = Im(\Gamma)$ arithmetic and $\Gamma \to \bar{\Gamma}$ has finite kernel. If $\bar{\Gamma}$ torsion free, by Baily & Borel, Borel, $D(\Gamma) = \Gamma \backslash D = \bar{\Gamma} \backslash D$ has a unique structure of an alg. variety and for any $\Gamma' \subset \Gamma$, $D(\Gamma') \to D(\Gamma)$ is regular.

Def. Let (G, D) be a conn. SD, a connected Shimura variety relative to (G, D) is an alg. variety of the form $D(\Gamma)$, $\Gamma \subset G^{ad}(\mathbb{Q})^+$ arithmetic containing the image of a congruence subgrp of $G(\mathbb{Q})^+$ s.t. $\bar{\Gamma}$ torsion free.

The inverse system of such alg. varieties, $Sh^{\circ}(G, D)$ is called the attached connected Shimura variety.

$G(\mathbb{R})^+ \twoheadrightarrow Hol(D)^+$ has compact kernel

$\Rightarrow$ preserves discrete torsion free subgroups

Fact: $N \geq 2$, $n \geq 3$, $\Gamma(n) \subset GL_N(\mathbb{Z})$ is torsion free.

$\Rightarrow$ $\Gamma \backslash D$, Γ principal congruence subgrp of $G(\mathbb{Q})^+$, $n \geq 3$ are cofinal in $Sh^\circ(G, D)$.

RMK. (G, D) conn. SD, τ topology on $G^{ad}(\mathbb{Q})$ for which the images of congruence subgrps of $G(\mathbb{Q})$ form a fundamental system of nbhds of 1. The conn. SV relative to (G, D) are $\Gamma \backslash D$, Γ open arith. $\subset G^{ad}(\mathbb{Q})^+$ s.t. $\bar{\Gamma}$ torsion free.

$g \in G^{ad}(\mathbb{Q})^+$ defines a holo. $D \longrightarrow D$ hence $\Gamma \backslash D \longrightarrow g\Gamma g^{-1} \backslash D$. This is again holo. hence regular. Conjugation by g on $G^{ad}(\mathbb{Q})^+$ is homeo. for the τ topology, so $G^{ad}(\mathbb{Q})^+$ acts on $Sh^\circ(G, D)$, which extends to an action of the completion $\widehat{G^{ad}(\mathbb{Q})^+}$ of $G^{ad}(\mathbb{Q})^+$ for the τ topology.

The varieties $\Gamma \backslash D$, Γ congruence subgrp of $G(\mathbb{Q})^+$ are cofinal in $Sh^\circ(G, D)$.

$\Gamma \subset G^{ad}(\mathbb{Q})^+$ arithmetic containing $\pi(\Gamma')$, $\Gamma' \subset G(\mathbb{Q})^+$ congruence, $\bar{\Gamma}$ torsion free

$\Rightarrow$ $\pi(\Gamma') \subset \Gamma$, $\overline{\pi(\Gamma')} \subset \bar{\Gamma}$ torsion free.

Prop. $\pi: G(\mathbb{Q})^+ \longrightarrow G^{ad}(\mathbb{Q})^+$, $\Gamma \subset G^{ad}(\mathbb{Q})^+$ arithmetic. TFAE.

(a) $\pi^{-1}\Gamma$ congruence subgrp of $G(\mathbb{Q})^+$

(b) $\pi^{-1}\Gamma$ contains a congruence subgrp of $G(\mathbb{Q})^+$

(c) Γ contains the image of a congruence subgrp of $G(\mathbb{Q})^+$

(c) $\Rightarrow$ (a): $\pi(\Gamma')$ finite index in Γ, $Z(G)$ finite

RMK. $\pi: G(\mathbb{Q})^+ \longrightarrow G^{ad}(\mathbb{Q})^+$ is usually not surj. The image of $SL_2(\mathbb{Q}) = SL_2(\mathbb{Q})^+ \longrightarrow PGL_2(\mathbb{Q})^+$ consists of elements rep. by matrix with det in $(\mathbb{Q}^\times)^2$. Thus $\pi(\pi^{-1}\Gamma)$ is usually not Γ.

Ex.

(a) $G = SL_2$, $D = \mathcal{H}$. $Sh^\circ(G, D)$ is the family of elliptic modular curves $\Gamma\backslash\mathcal{H}$ with Γ torsion free arith. $\subset PGL_2(\mathbb{Q})^+$ containing the image of some $\Gamma(N)$.

(b) $G = PGL_2$, $D = \mathcal{H}$. $Sh^\circ(G, D)$ is the (smaller) family of $\Gamma\backslash\mathcal{H}$ with Γ torsion free congruence $\subset PGL_2(\mathbb{Q})^+$.

(c) Let B be a quaternion alg. over totally real F, then

$$B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{v: F \hookrightarrow \mathbb{R}} B \otimes_{F,v} \mathbb{R}$$

and each $B \otimes_{F,v} \mathbb{R}$ is isom. to $\mathbb{H}$ or $M_2(\mathbb{R})$. Let G be the semisimple alg. group $/\mathbb{Q}$ s.t. $G(\mathbb{Q}) = \ker(Nm: B^\times \to F^\times)$. Then $G(\mathbb{R}) = \prod \mathbb{H}^{1,\times} \times \prod SL_2(\mathbb{R})$ where $\mathbb{H}^{1,\times} = \ker(Nm: \mathbb{H}^\times \to \mathbb{R}^\times)$. Assume at least one $SL_2(\mathbb{R})$ appears in $G(\mathbb{R})$ and D corresponding products of $\mathcal{H}$. Then $G(\mathbb{R})$ acts on D hence (G, D) is a conn. SD. If $B \simeq M_2(F)$, $G(\mathbb{Q})$ has unipotent elements $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ so $D(\Gamma)$ is not compact and $D(\Gamma)$ is called Hilbert modular variety.

Def. A semisimple group G is simply connected if every isogeny $G' \to G$ with G' connected is an isom.

SL_2 simply connected, PGL_2 not.

Thm. $G/\mathbb{Q}$ semisimple, simply connected, noncpt type alg. grp, $G(\mathbb{Q})$ dense in $G(\mathbb{A}_f)$.

Adelic Description of $D(\Gamma)$.

$G/\mathbb{Q}$ simply connected semisimple, $G(\mathbb{R})$ connected, $G(\mathbb{Q}) \subset G(\mathbb{R})^+ = G(\mathbb{R})$.

Prop. (G, D) conn. SD, K cpt open $\subset G(\mathbb{A}_f)$, $\Gamma = K \cap G(\mathbb{Q})$. Then

$$\begin{aligned} \Gamma\backslash D &\xrightarrow{\sim} G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K \\ x &\longmapsto [x, 1] \end{aligned}$$

Double coset spaces are natural in Langlands program

$G(\mathbb{Q})$ acts on D, $G(\mathbb{A}_f)$ from left, K acts on the right:

$g \in G(\mathbb{Q})$, $x \in D$, $a \in G(\mathbb{A}_f)$, $k \in K$, $g(x, a)k = (gx, gak)$.

It is homeo. if $G(\mathbb{A}_f)$ given the adelic topology or discrete topology.

Pf. K open, $G(\mathbb{A}_f) = G(\mathbb{Q}) \cdot K$ by strong approximation.

$\Rightarrow$ elements in $G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K$ are represented by $[x:1]$ and if

$[x:1] = [x':1]$, $\exists\, g \in G(\mathbb{Q})$, $k \in K$, $x' = gx$, $1 = gk$ $\Rightarrow g \in \Gamma \Rightarrow x = x' \in \Gamma\backslash D$.

$$\begin{tikzcd} D \arrow[r, "x \longmapsto (x,1)"] \arrow[d] & D \times G(\mathbb{A}_f)/K \arrow[d] \\ \Gamma\backslash D \arrow[r] & G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K \end{tikzcd}$$

K open, $G(\mathbb{A}_f)/K$ discrete, upper map is homeo. onto image and open. ///

As each Γ acts freely on D and $\bigcap \Gamma = \{1\}$, $D \hookrightarrow \varprojlim \Gamma\backslash D$.

Prop. $\varprojlim_K G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)$.

$G(\mathbb{A}_f)$ acts on inverse system $(\Gamma\backslash D)_\Gamma$
hence on $\varinjlim H^i(\Gamma\backslash D, \mathbb{Q})$

Lemma. G topological group acting continuously on topological space X, (G_i) a directed family of subgroups.

(a) $h: X/\cap G_i \longrightarrow \varprojlim X/G_i$ is cont.

(b) h inj. if $\forall i$, $x \in X$, $\mathrm{Stab}_x(G_i)$ compact

(c) h surj. if $\forall i$, $x \in X$, xG_i compact.

Pf. of Prop. Let $(x, a) \in D \times G(\mathbb{A}_f)$, $K \subset G(\mathbb{A}_f)$ compact open. We have to show $(x, a)K$ is Hausdorff in $G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)$ for K sufficiently small.

Let $\Gamma = G(\mathbb{Q}) \cap aKa^{-1}$. After replacing Γ by its neat subgrp of finite index, we may assume Γ torsion free. Then there exists $x \in V$ open s.t. $gV \cap V = \phi$ for all $g \in \Gamma - \{1\}$. For any $(x, b) \in (x, a)K$, $g(V \times aK) \cap (V \times bK) = \phi$ for all $g \in G(\mathbb{Q}) - \{1\}$, so the images of $V \times aK$ and $V \times bK$ in $G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)$ separate (x, a) and (x, b). ///

connected as inverse lim of connected Noeth. schemes

RMK. The inverse lim of $Sh^\circ(G, D)$ exists as a scheme locally Noeth. and regular over $\mathbb{C}$, and it is possible to recover the inverse system from the limit scheme. The limit scheme behaves like simply connected universal covering of $\Gamma\backslash D$.

$$R \subset \tilde{R} \subset R^{sh}$$

$$\tilde{R} = \mathcal{O}_{\varprojlim D(\Gamma), (s_i)}, \quad R = \mathcal{O}_{D(\Gamma_0), s_0}$$

Alternative definition of connected Shimura data.

$$0 \to \mathbb{G}_m \xrightarrow{w} \mathbb{S} \longrightarrow U_1 \longrightarrow 0$$

$$0 \to \mathbb{R}^\times \longrightarrow \mathbb{C}^\times \longrightarrow U_1 \longrightarrow 0$$

$$r \longmapsto r^{-1}$$

$$z \longmapsto \frac{z}{\bar{z}}$$

$H/\mathbb{R}$ adjoint alg grp. $u : U_1 \to H$ gives $h : \mathbb{S} \to H$. U_1 acts on $\mathrm{Lie}\, H_{\mathbb{C}}$ through char $1, z, z^{-1}$ $\Leftrightarrow$ $\mathbb{S}$ acts on $\mathrm{Lie}\, H_{\mathbb{C}}$ through $1, \frac{z}{\bar{z}}, \frac{\bar{z}}{z}$. Given such $h : \mathbb{S} \to H$, $w(\mathbb{G}_m)$ acts trivially on $\mathrm{Lie}\, H_{\mathbb{C}}$. h comes from some u.

$G/\mathbb{Q}$ semisimple alg. grp, to give a $G^{ad}(\mathbb{R})^+$-conj. class D of $u : U_1 \to G_{\mathbb{R}}^{ad}$ satisfying SU 1, 2 is the same as to give a $G^{ad}(\mathbb{R})^+$-conj. class X^+ of $h : \mathbb{S} \to G_{\mathbb{R}}^{ad}$ s.t.

SV 1: for all $h \in X^+$, only the characters $1, \frac{z}{\bar{z}}, \frac{\bar{z}}{z}$ appear in rep. of $\mathbb{S}$ on $\mathrm{Lie}\, G^{ad}_{\mathbb{C}}$ via $\mathrm{Ad} \circ h$.

SV 2: for all $h \in X^+$, $\mathrm{Ad}(h(i))$ is a Cartan involution of $G_{\mathbb{R}}^{ad}$.

Then a conn. SD is the same (G, X^+) and

SV 3: G^{ad} has no $\mathbb{Q}$-factor on which h trivial.