Shimura Varieties.

Why nonconnected?

① Langlands: reductive groups instead of semisimple groups

② $D(\Gamma)$ have models over number fields, depending on realization of G as derived group of reductive and on Γ, e.g. $\Gamma(N)\backslash\mathcal{H}$ defined over $\mathbb{Q}[\zeta_N]$. How to define them over the same number field?

$Y + i = 0$ in $\mathbb{C}^2$ defined over $\mathbb{Q}[i]$ $\Rightarrow$ $Y^2 + 1 = 0$ defined over $\mathbb{Q}$

Idea: Lower the field of definition by taking other connected components, i.e. disjoint union of its conjugates.

Reductive groups.

$G/\mathbb{Q}$ reductive, $G(\mathbb{R})_+ = \{g \in G(\mathbb{R}) \mid \bar{g} \in G^{ad}(\mathbb{R}) \text{ is in } G^{ad}(\mathbb{R})^+\}$, $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$

e.g. $GL_2(\mathbb{Q})_+ = \{g \in GL_2(\mathbb{Q}) \mid \det g > 0\}$.

In the diagram

$$\begin{tikzcd} & G^{der} \arrow[d] \arrow[dr, "\alpha"] \arrow[r, phantom, "\text{semisimple}"] & \\ Z(G) \arrow[r] \arrow[dr, "\beta"'] & G \arrow[r] \arrow[d, "\nu"] & G^{ad} \\ & T & \end{tikzcd}$$

where T is the largest commutative quotient of G. The vertical and horizontal sequences are short exact, and α, β are isogenies with kernel $Z \cap G^{der}$.

This gives s.e.s.

$$1 \longrightarrow Z \cap G^{der} \longrightarrow Z \times G^{der} \longrightarrow G \longrightarrow 1.$$

G^{ad} acts on G by Adjoint action.

e.g. $G = GL_n$,

$$\begin{tikzcd} & SL_n \arrow[d] \arrow[dr] & \\ \mathbb{G}_m \arrow[r] \arrow[dr, "x \mapsto x^n"'] & GL_n \arrow[r] \arrow[d, "\det"] & PGL_n \\ & \mathbb{G}_m & \end{tikzcd}$$

and $1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \times SL_n \longrightarrow GL_n \longrightarrow 1$.

$H^1(\mathbb{Q}, G) = \{$ set of cont. cross homomorphisms $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow G(\overline{\mathbb{Q}}) \} / \simeq$ where $f \sim f'$ if $\exists\, a \in G(\overline{\mathbb{Q}})$ s.t. $f'(\sigma) = a^{-1} \cdot f(\sigma) \cdot \sigma(a)$. The distinguished element representing principal cross homomorphisms is denoted by e.

Translate action of $G^{ad}(\mathbb{R})^+$ to $G(\mathbb{R})^+$.

Prop. $\varphi : G \longrightarrow H$ surj. morphism of alg. grps $/\mathbb{R}$, then $\varphi(\mathbb{R}) : G(\mathbb{R})^+ \twoheadrightarrow H(\mathbb{R})^+$.

Pf. $\varphi(\mathbb{R})$ can be regarded as a smooth map of smooth manifolds, surj. on tangent space at 1. Hence $\mathrm{Im}\,\varphi(\mathbb{R})$ contains an open nbhd of 1, hence $\mathrm{Im}\,\varphi(\mathbb{R})$ is open subgroup, hence closed and surj. ///

G simply connected alg. grp, $G(\mathbb{C})$ simply connected, but $G(\mathbb{R})$ may not be.

Thm. (Cartan) G semisimple simply connected alg. grp $/\mathbb{R}$, $G(\mathbb{R})$ connected.

Cor. $G/\mathbb{R}$ reductive alg. grp. $G(\mathbb{R})$ has only finitely many connected components.

Thm. (Real approximation) $G/\mathbb{Q}$ connected alg. grp. $G(\mathbb{Q})$ dense in $G(\mathbb{R})$. More generally $G/\mathbb{Q}$ alg. grp, each connected component of G contains a $\mathbb{Q}$-point, then $G(\mathbb{Q})$ dense in $G(\mathbb{R})$.

Shimura Data.

Def. A Shimura datum is a pair (G, X), $G/\mathbb{Q}$ reductive, X $G(\mathbb{R})$-conj. class of $h: \mathbb{S} \to G_{\mathbb{R}}$ satisfying

SV 1: for all $h \in X$, the Hodge structure on Lie $G_{\mathbb{R}}$ by $\mathrm{Ad} \circ h$ is of type $(-1, 1), (0, 0), (1, -1)$ only characters $1, \frac{\bar{z}}{z}, \frac{z}{\bar{z}}$ appear in rep. of $\mathbb{S}$ on Lie $G_{\mathbb{C}}$

SV 2: for all $h \in X$, $\mathrm{Ad}(h(i))$ is Cartan involution of $G^{ad}_{\mathbb{R}}$

SV 3: G^{ad} has no $\mathbb{Q}$-factor on which h is trivial. and $\mathbb{S}$ acts on Lie $Z_{\mathbb{C}}$ trivially.

Ex. $G = GL_2/\mathbb{Q}$, X set of $GL_2(\mathbb{R})$-conj. of $h_0 : \mathbb{S} \to GL_{2,\mathbb{R}}$.

$$a+bi \longmapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

(G, X) is SD. The natural bijection $X \to \mathbb{C} - \mathbb{R}$, h corresponds to

$$g h_0 g^{-1} \longmapsto gi$$

z iff $h(\mathbb{C}^\times) = \mathrm{Stab}_z GL_2(\mathbb{R})$ and $h(z)$ acts on the tangent space at z via $\frac{z}{\bar{z}}$.

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(i+\delta) = \frac{1}{-i-\delta} = \frac{i}{1-i\delta} = i(1+i\delta) = i - \delta$$

action on tangent space $\delta \longmapsto -\delta = \frac{i}{-i}\delta$

Slogan : SD is "finite disjoint union" of conn. SD.

Prop. $G/\mathbb{R}$ reductive, $h: \mathbb{S} \to G$, $\bar{h}: \mathbb{S} \to G \to G^{ad}$. Let X be a $G(\mathbb{R})$-conj. class of $\mathbb{S} \to G$ and $\bar{X}$ be the $G^{ad}(\mathbb{R})$-conj. class of $\mathbb{S} \to G^{ad}$ containing the $\bar{h}$ for $h \in X$. Then

(a) $h \longmapsto \bar{h}$, $X \to \bar{X}$ is injective with image union of connected components

(b) Let X^+ be a connected component of X, $\bar{X}^+$ its image in $\bar{X}$. If (G, X) satisfies SV 1-3 then $(G^{der}, \bar{X}^+)$ satisfies SV 1-3 for connected SD. Moreover $\mathrm{Stab}_{X^+}(G(\mathbb{R})) = G(\mathbb{R})_+$, i.e. $gX^+ = X^+ \iff g \in G(\mathbb{R})_+$.

Pf.

(a) $h: \mathbb{S} \to G$ is determined by projection to T and G^{ad}, and all projections to T are the same. Now $G^{ad}(\mathbb{R})^+$ acts transitively on each connected comp. of $\bar{X}$ and $G(\mathbb{R})^+ \to G^{ad}(\mathbb{R})^+$ surj.

(b) Easy to see $(G^{der}, \bar{X}^+)$ satisfies SV 1-3 (for connected SD). Then Stab in $G^{ad}(\mathbb{R})$ of $\bar{X}^+$ is $G^{ad}(\mathbb{R})^+$, hence $\mathrm{Stab}_{X^+}(G(\mathbb{R})) = G(\mathbb{R})_+$. ///

In particular X^+ is HSD, $G(\mathbb{R})^+ \twoheadrightarrow G^{ad}(\mathbb{R})^+ \twoheadrightarrow \mathrm{Hol}(X^+)^+$.

Cor. (G, X) SD, X^+ connected component of X regarded as a $G(\mathbb{R})^+$-conj. class of $\mathbb{S} \to G^{ad}_{\mathbb{R}}$ (i.e. $\bar{X}^+$ $G^{ad}(\mathbb{R})^+$-conj. class of $\mathbb{S} \to G_{\mathbb{R}}$ by the Prop above). Then (G^{der}, X^+) is a conn. SD. In particular X is a finite disjoint union of HSD.

$\pi_0(G(\mathbb{R}))$ finite

RMK. $\pi_0(X) \hookrightarrow \pi_0(\bar{X})$, $G(\mathbb{R})/G(\mathbb{R})_+ \hookrightarrow G^{ad}(\mathbb{R})/G^{ad}(\mathbb{R})^+$

They are surj. if $H^1(\mathbb{R}, Z) = 0$, but not in general.

(G, X) SD, $h: \mathbb{S} \to G_{\mathbb{R}}$ in X, by SV 1, $\mathbb{R}^\times \to \mathbb{C}^\times$, $r \mapsto r^{-1}$ acts trivially on Lie $G_{\mathbb{C}}$ $\Rightarrow$ $h(r) \in Z(G)(\mathbb{R})$ $\Rightarrow$ $h|_{\mathbb{G}_m}$ is independent of h, denoted by w, called the weight homomorphism.

$\rho: G_{\mathbb{R}} \to GL_V$, $\rho \circ w$ gives $V = \bigoplus V_n$ which is the weight decomposition of the Hodge structure $(V, \rho \circ h)$ for any $h \in X$.

Prop. (G, X) SD. Then X has a unique structure of complex manifold s.t. for every $\rho: G_{\mathbb{R}} \to GL_V$, $(V, \rho \circ h)_{h \in X}$ is a holo. family of Hodge structures and each family is a VHS. In particular X is finite disjoint union of HSD.

Pf. Choose $\rho: G_{\mathbb{R}} \to GL_V$ faithful and repeat the proof for VHS. All other rep. of G can be constructed from ρ. SV 1 implies Griffiths transversality. ///

Shimura Variety.

(G, X) SD.

Lemma. For every conn. component X^+ of X, $G(\mathbb{Q})_+ \backslash X^+ \times G(\mathbb{A}_f) \xrightarrow{\sim} G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)$.

Pf. $G(\mathbb{Q})$ dense in $G(\mathbb{R})$, $G(\mathbb{R})$ acts transitively on X, every $x \in X$ is of the form $g x^+$ for $g \in G(\mathbb{Q})$, $x^+ \in X^+$ $\Rightarrow$ surj.

Let $(x, a), (x', a') \in X^+ \times G(\mathbb{A}_f)$ have same image, then

$\exists\, g \in G(\mathbb{Q})$, $gx = x'$, $ga = a'$. Then g stabilizes X^+ hence $g \in G(\mathbb{Q})_+$. ///

Lemma. For every $K \subset G(\mathbb{A}_f)$ open, $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$ is finite.

Pf. $G(\mathbb{Q})_+ \backslash G(\mathbb{Q}) \hookrightarrow G^{ad}(\mathbb{R})^+ \backslash G^{ad}(\mathbb{R})$ and the latter is finite. ETS $G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/K$ finite. (See later) ///

$Sh_K(G,X) = G(\mathbb{Q}) \backslash G(\mathbb{A})/K_\infty \cdot K$, $K_\infty = \mathrm{Stab}_h(G(\mathbb{R}))$, $G(\mathbb{R})/K_\infty = X$

For $K \subset G(\mathbb{A}_f)$ compact open, consider $\underline{Sh_K(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K}$ in which $g \cdot (x, a) \cdot k = (gx, gak)$, $g \in G(\mathbb{Q})$, $x \in X$, $a \in G(\mathbb{A}_f)$, $k \in K$.

$gx = g \cdot x \cdot g^{-1}$ left-action

$|\pi_0(Sh_K(\mathbb{C}))| = |G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K|$.

Lemma. Let $\mathcal{C}$ be a set of representatives of $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$ and X^+ some connected component of X. Then $\underline{G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K \simeq \coprod_{g \in \mathcal{C}} \Gamma_g \backslash X^+}$ where $\Gamma_g = gKg^{-1} \cap G(\mathbb{Q})_+$. This is homeo. if $G(\mathbb{A}_f)$ given adelic / discrete topology.

Pf. For $g \in \mathcal{C}$, the map

$$\begin{aligned} \Gamma_g \backslash X^+ &\longrightarrow G(\mathbb{Q})_+ \backslash X^+ \times G(\mathbb{A}_f)/K \\ x &\longmapsto (x, g) \end{aligned}$$

is well-defined.

Claim: it is injective and $G(\mathbb{Q})_+ \backslash X^+ \times G(\mathbb{A}_f)/K$ is the disjoint union of their images for different g. The claim implies the lemma.

If $(x, g) = (x', g)$, then $\exists\, q \in G(\mathbb{Q})_+$, $k \in K$, $x' = qx$, $g = qgk$. Then $q \in \Gamma_g$ and $[x] = [x']$.

$\alpha: G^{ad}(\mathbb{R})^+ \twoheadrightarrow Aut(X^+)^+$, $\ker\alpha$ compact. $\Gamma_g = gKg^{-1} \cap G(\mathbb{Q})_+$ neat $\Rightarrow \pi(\Gamma_g) \subset G^{ad}(\mathbb{Q})^+$ neat

$\pi: G(\mathbb{R})_+ \twoheadrightarrow G^{ad}(\mathbb{R})^+$ $\Rightarrow \alpha\circ\pi(\Gamma_g)$ torsion free

Let $(x, a) \in X^+ \times G(A_f)$, $a = qgk$ for some $q \in G(\mathbb{Q})_+$, $g \in \mathcal{C}$, $k \in K$ hence $(x, a) = (q^{-1}x, g)$ lies in the image of $\Gamma_g \backslash X^+$.

Suppose $(x, g) = (x, g')$, $g, g' \in \mathcal{C}$. Then $x' = qx$, $g' = qgk \Rightarrow g = g'$. ///

Γ_g is arithmetic subgrp of $G(\mathbb{Q})$, hence its image in $G^{ad}(\mathbb{Q})$ is arithmetic and by definition its image in $Aut(X^+)^+$ is arithmetic. When K small enough, Γ_g will be neat for all $g \in \mathcal{C}$ and its image in $Aut(X^+)^+$ will be torsion free. Then $\Gamma_g \backslash X^+$ is an arithmetic locally symmetric variety, and $Sh_K(G, X)$ is finite disjoint union of such varieties.

For $K' \subset K$ small open compact subgrps, the natural map $Sh_{K'}(G, X) \to Sh_K(G, X)$ is regular. Thus we get an inverse system of alg. varieties $(Sh_K(G, X))_K$.

$G(A_f)$ acts on the inverse system naturally, $g \in G(A_f)$, $K \to g^{-1}Kg$ and $T(g): Sh_K(G, X) \longrightarrow Sh_{g^{-1}Kg}(G, X)$, $(x, a) \longmapsto (x, ag)$. Clearly this is a right action, $T(gh) = T(h) \circ T(g)$.

Def. (G, X) SD. A Shimura variety relative to (G, X) is a variety of the form $Sh_K(G, X)$ for some small compact open K. The Shimura variety attached to (G, X) is $Sh(G, X)$, the inverse system $(Sh_K(G, X))_K$ endowed with the action of $G(A_f)$.

Def. Let (G, X), (G', X') be SD.

(a) A morphism of SD $(G, X) \to (G', X')$ is a homomorphism $G \to G'$ sending X to X'.

(b) A morphism of SV $Sh(G, X) \to Sh(G', X')$ is an inverse system of regular maps of alg. varieties compatible with the action of $G(\mathbb{A}_f)$.

Thm. $(G, X) \to (G', X')$ induces $Sh(G, X) \to Sh(G', X')$, which is closed immersion if $G \to G'$ injective.

The structure of a Shimura variety. What about G^{der} not simply connected?

Slogan: (G^{der} simply connected) The set of connected components is a "0-dim SV" and each connected component is a connected SV.

(G, X) SD, $Z = Z(G)$, T largest commutative quotient of G, $Z \hookrightarrow G \overset{\nu}{\twoheadrightarrow} T$.

Define $T(\mathbb{R})^{\dagger} = \mathrm{Im}(Z(\mathbb{R}) \to T(\mathbb{R}))$, $T(\mathbb{Q})^{\dagger} = T(\mathbb{Q}) \cap T(\mathbb{R})^{\dagger}$. Z surj. to T hence $T(\mathbb{R})^{+} \subset T(\mathbb{R})^{\dagger}$ and $T(\mathbb{R})^{\dagger}$, $T(\mathbb{Q})^{\dagger}$ are of finite index in $T(\mathbb{R})$, $T(\mathbb{Q})$. For example, $G = GL_2$, $T(\mathbb{Q})^{\dagger} = T(\mathbb{Q})^{+} = \mathbb{Q}_{>0}$.

Thm. Assume G^{der} simply connected, for K small, the natural map

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \longrightarrow T(\mathbb{Q})^{\dagger}\backslash T(\mathbb{A}_f)/\nu(K) \quad \text{induces}$$

$$\pi_0(Sh_K(G,X)) \xrightarrow{\sim} T(\mathbb{Q})^{\dagger}\backslash T(\mathbb{A}_f)/\nu(K) \quad \text{which is finite, and}$$

the connected component over 1 is canonically isom. to $\Gamma\backslash X^{+}$ for some Γ congruence subgrp of $G^{der}(\mathbb{Q})$ containing $K \cap G^{der}(\mathbb{Q})$.

Next we prove the theorem.

Lemma. G^{der} simply connected, then $G(\mathbb{R})_+ = G^{der}(\mathbb{R}) \cdot Z(\mathbb{R})$.

Pf. G^{der} simply connected $\Rightarrow$ $G^{der}(\mathbb{R})$ connected $\Rightarrow$ $G^{der}(\mathbb{R}) \subset G(\mathbb{R})_+$.

Consider exact diagram, $Z' = Z \cap G^{der}$

$$\begin{tikzcd}
1 \arrow[r] & Z'(\mathbb{R}) \arrow[r] \arrow[d, equal] & Z(\mathbb{R}) \times G^{der}(\mathbb{R}) \arrow[r] \arrow[d, "p_2"] & G(\mathbb{R}) \arrow[r] \arrow[d] & H^1(\mathbb{R}, Z') \arrow[d, equal] \\
1 \arrow[r] & Z'(\mathbb{R}) \arrow[r] & G^{der}(\mathbb{R}) \arrow[r] & G^{ad}(\mathbb{R}) \arrow[r] & H^1(\mathbb{R}, Z')
\end{tikzcd}$$

$G^{der} \twoheadrightarrow G^{ad} \Rightarrow G^{der}(\mathbb{R}) \twoheadrightarrow G^{ad}(\mathbb{R})^{+}$.

$g \in G(\mathbb{R})_+ \iff g$ maps to 0 in $H^1(\mathbb{R}, Z')$

$\iff g$ lifts to $Z(\mathbb{R}) \times G^{der}(\mathbb{R})$

$\iff g \in Z(\mathbb{R}) \cdot G^{der}(\mathbb{R})$. ///

Lemma. Let H be a simply connected semisimple alg. grp $/\mathbb{Q}$

(a) For every finite prime ℓ, $H^1(\mathbb{Q}_\ell, H)$ is trivial.

(b) $H^1(\mathbb{Q}, H) \hookrightarrow \prod_{\ell \leq \infty} H^1(\mathbb{Q}_\ell, H)$. (Hasse principle)

Lemma. Assume G^{der} simply connected, $t \in T(\mathbb{Q})$. Then $t \in T(\mathbb{Q})^\dagger \Leftrightarrow t$ lifts to $G(\mathbb{Q})_+$.

Pf. Consider exact diagram

$$\begin{array}{ccccccccc} 1 & \to & G^{der}(\mathbb{Q}) & \to & G(\mathbb{Q}) & \xrightarrow{\nu} & T(\mathbb{Q}) & \to & H^1(\mathbb{Q}, G^{der}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \text{ by above lemma} \\ 1 & \to & G^{der}(\mathbb{R}) & \to & G(\mathbb{R}) & \xrightarrow{\nu} & T(\mathbb{R}) & \to & H^1(\mathbb{R}, G^{der}) \end{array}$$

$t \in T(\mathbb{Q})^\dagger \Rightarrow t_{\mathbb{R}}$ in $T(\mathbb{R})$ lifts to $z \in Z(\mathbb{R}) \subset G(\mathbb{R}) \Rightarrow t$ maps to trivial element in $H^1(\mathbb{Q}, G^{der}) \Rightarrow t$ lifts to $g \in G(\mathbb{Q}) \Rightarrow g_{\mathbb{R}} \cdot z^{-1}$ maps to $1 \in T(\mathbb{R}) \Rightarrow$ $g_{\mathbb{R}} \in G^{der}(\mathbb{R}) \cdot z \subset G(\mathbb{R})_+ \Rightarrow g \in G(\mathbb{Q})_+$.

$t \in T(\mathbb{Q})$ lifts to $a \in G(\mathbb{Q})_+ \Rightarrow a_{\mathbb{R}} = gz$, $g \in G^{der}(\mathbb{R})$, $z \in Z(\mathbb{R}) \Rightarrow a_{\mathbb{R}}$ and z maps to $t_{\mathbb{R}} \Rightarrow t \in T(\mathbb{Q})^\dagger$. ///

Cor. G^{der} simply connected, $T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(K) = \nu(G(\mathbb{Q})_+)\backslash T(\mathbb{A}_f)/\nu(K)$.

$\simeq G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$

Define the natural map

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \simeq G(\mathbb{Q})_+ \backslash X^+ \times G(\mathbb{A}_f)/K \longrightarrow \nu(G(\mathbb{Q})_+)\backslash T(\mathbb{A}_f)/\nu(K)$$

$$(x, g) \longmapsto \nu(g)$$

We now study the fibre over 1.

$g \in G(\mathbb{A}_f)$, $[\nu(g)] = [1]$, $\nu(g) = \nu(\gamma)\nu(k)$, $\gamma \in G(\mathbb{Q})_+$, $k \in K$.

$\nu(\gamma^{-1} g k^{-1}) = 1$, $\gamma^{-1} g k^{-1} \in G^{der}(\mathbb{A}_f)$, $g \in G(\mathbb{Q})_+ \cdot G^{der}(\mathbb{A}_f) \cdot K$.

Hence every element of the fibre is of the form (x, a), $a \in G^{der}(\mathbb{A}_f)$.

Strong approximation $\Rightarrow$ $G^{der}(\mathbb{A}_f) = G^{der}(\mathbb{Q}) \cdot (K \cap G^{der}(\mathbb{A}_f))$. Thus every element in the fibre is represented by $(x, 1)$. The fibre is $\Gamma \backslash X^+$, Γ is the image of $K \cap G(\mathbb{Q})_+$ inside $G^{ad}(\mathbb{Q})^+$. Γ is an arithmetic subgrp of $G^{ad}(\mathbb{Q})^+$ containing the image of the congruence subgrp $K \cap G^{der}(\mathbb{Q})$ of $G^{der}(\mathbb{Q})$.

$G^{der}(\mathbb{R}) \subset G(\mathbb{R})$,

$$\begin{array}{ccccc}
K \cap G^{der}(\mathbb{Q}) & \overset{\text{congruence}}{\subset} & G^{der}(\mathbb{Q}) & \subset & G^{der}(\mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow \\
K \cap G(\mathbb{Q})_+ & \overset{\text{arithmetic}}{\subset} & G(\mathbb{Q}) & \subset & G(\mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow \\
G^{ad}(\mathbb{Q})^+ & \subset & G^{ad}(\mathbb{Q}) & \subset & G^{ad}(\mathbb{R})
\end{array}$$

All such small Γ arise in this way. Hence the inverse system of fibres over 1 is equivalent to the inverse system $Sh^{\circ}(G^{der}, X^+) = (\Gamma \backslash X^+)_\Gamma$.

The fibre over $[t]$ has similar structure once we showed it is nonempty.

Lemma. G^{der} simply connected, $\nu : G(\mathbb{A}_f) \to T(\mathbb{A}_f)$ is surj. and preserves compact open subgrps.

Pf. $\nu : G(\mathbb{Q}_\ell) \longrightarrow T(\mathbb{Q}_\ell)$ surj. for all finite ℓ. ///

$\nu : G(\mathbb{Z}_\ell) \longrightarrow T(\mathbb{Z}_\ell)$ surj. for almost all finite ℓ.

$0 \to \mathcal{G}'_\ell \to \mathcal{G}_\ell \to \mathcal{T}_\ell \to 0$, $H^1(\mathbb{F}_\ell, (\mathcal{G}'_\ell)_{\mathbb{F}_\ell}) = 0 \Rightarrow H^1(\mathbb{Z}_\ell, \mathcal{G}'_\ell) = 0$.

$T(\mathbb{Q})^{\dagger} \backslash T(\mathbb{Q})$ finite as $0 \to \overset{\text{finite}}{Z'(\mathbb{R})} \to Z(\mathbb{R}) \to T(\mathbb{R}) \to \overset{\text{finite}}{H^1(\mathbb{R}, Z')}$

Remains to show $T(\mathbb{Q})^{\dagger} \backslash T(\mathbb{A}_f)/\nu(K)$ finite $\Leftarrow$ $T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/\nu(K)$ finite.

Def. Let T be a torus over $\mathbb{Q}$, set $T(\mathbb{Z}_\ell) = \{ a \in T(\mathbb{Q}_\ell) \mid \chi(a)$ integral, $\forall \chi$ character of $T \}$ and $T(\hat{\mathbb{Z}}) = \prod_\ell T(\mathbb{Z}_\ell)$. The class group of T is defined to be

$$H(T) = T(\mathbb{Q}) \backslash T(\mathbb{R}) \times T(\mathbb{A}_f) / T(\mathbb{R}) \times T(\hat{\mathbb{Z}})$$

Prop. $H(T)$ is finite. Essentially from finiteness of class group of number fields

Assume K small enough s.t. $\nu(K) \subset T(\hat{\mathbb{Z}})$. As $T(\hat{\mathbb{Z}})$ compact, $\nu(K)$ open, $\nu(K)$ has finite index in $T(\hat{\mathbb{Z}})$ $\Rightarrow$ $T(\mathbb{Q}) \backslash T(\mathbb{R}) \times T(\mathbb{A}_f) / T(\mathbb{R}) \times \nu(K)$ is finite, so is its quotient $T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/\nu(K)$. ///

RMK. If Z' satisfies the Hasse principle, then every element in $G(\mathbb{Q})_+ \cap K$ with K small enough lies in $G^{der}(\mathbb{Q}) \cdot Z(\mathbb{Q})$, the image of $K \cap G(\mathbb{Q})_+$ in $G^{ad}(\mathbb{Q})^+$ is the same as the image of $K \cap G^{der}(\mathbb{Q})$, hence $Sh_K(G, X)^\circ = Sh^\circ_{K \cap G^{der}(\mathbb{A}_f)}(G^{der}, X^+)$.

Zero-dimensional Shimura varieties.

Let $T/\mathbb{Q}$ torus, any $h: \mathbb{C}^\times \to T(\mathbb{R})$ defines a SV $Sh(T, \{h\})$. For compact open $K \subset T(\mathbb{A}_f)$, $Sh_K(T, \{h\}) \simeq T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K$ finite discrete.

Let Y be a finite set on which $T(\mathbb{R})/T(\mathbb{R})^+$ acts transitively. Define $Sh(T, Y)$ to be the inverse system of finite sets $Sh_K(T, Y) = T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_f)/K$. Such a system is called a zero-dim SV.

(G, X) SD, G^{der} simply connected, $T = G/G^{der}$, $Y = T(\mathbb{R})/T(\mathbb{R})^+$. As $T(\mathbb{Q})$ dense in $T(\mathbb{R})$, $Y \simeq T(\mathbb{Q})/T(\mathbb{Q})^+$ and $T(\mathbb{Q})^+\backslash T(\mathbb{A}_f)/K \simeq T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_f)/K$

hence $\pi_0(Sh_K(G, X)) \simeq Sh_{\nu(K)}(T, Y)$. In particular the set of connected components of the SV is a zero-dim SV.

Ex. $(G, X) = (GL_2, \mathcal{H}^{\pm})$, $K = K(N)$, $T = \mathbb{G}_m$, $Y = \mathbb{R}/\mathbb{R}^+ \simeq \{\pm 1\}$, thus

$$\pi_0(Sh_K(G, X)) = \mathbb{Q}^\times \backslash \{\pm 1\} \times \mathbb{A}_f^\times / K(N) \simeq (\mathbb{Z}/N\mathbb{Z})^\times \simeq Gal(\mathbb{Q}(\zeta_N)/\mathbb{Q}).$$

Additional Axioms.

(G, X) SD, $w_X : \mathbb{G}_{m,\mathbb{R}} \longrightarrow Z(G)^\circ_{\mathbb{R}} \subset G_{\mathbb{R}}$ is a homomorphism of tori defined over $\mathbb{Q}$.

SV 2^*: for all $h \in X$, $Ad(h(i))$ is a Cartan involution on $G_{\mathbb{R}}/w_X(\mathbb{G}_m)$

SV 4: w_X is defined over $\mathbb{Q}$

SV 5: $Z(\mathbb{Q})$ discrete in $Z(\mathbb{A}_f)$

SV 6: Z° splits over a CM field

Let $G \to GL_V$, then each $h \in X$ gives a Hodge struture on $V(\mathbb{R})$. SV 4 will imply that they are all rational Hodge strutures. It is hoped that these Hodge strutures will all occur in the cohomology of algebraic varieties, and that the Shimura variety will be a moduli variety for motives when SV 4 holds and a fine moduli variety if in addition SV 5 holds.

SV 6 will imply that w is defined over a totally real field, and the field of definition of SV is either totally real or CM.

Ex. B quaternion alg. $/F$ totally real, $G/\mathbb{Q}$ alg. with $G(\mathbb{Q}) = B^\times$.

$B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_v B \otimes_{F,v} \mathbb{R}$, $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{H} \times \cdots \times \mathbb{H} \times M_2(\mathbb{R}) \times \cdots \times M_2(\mathbb{R})$

$$G(\mathbb{R}) \simeq \mathbb{H}^\times \times \cdots \times \mathbb{H}^\times \times GL_2(\mathbb{R}) \times \cdots \times GL_2(\mathbb{R})$$

$$h(a+bi) = 1, \cdots, 1, \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \cdots, \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$w(r) = 1, \cdots, 1, r^{-1} I, \cdots, r^{-1} I$$

Let X be $G(\mathbb{R})$-conj. class of h, then (G, X) satisfies SV 1, 2, and is SD if B splits for at least one real place of F, i.e. at least one $M_2(\mathbb{R})$ appears in $B \otimes_{\mathbb{Q}} \mathbb{R}$ because G^{ad} is simple as an alg. grp $/\mathbb{Q}$.

Let $I = \mathrm{Hom}(F, \overline{\mathbb{Q}}) = \mathrm{Hom}(F, \mathbb{R})$ and let I_{nc} be the set of v where $B \otimes_{F,v} \mathbb{R}$ splits. Then w is defined over the subfield of $\overline{\mathbb{Q}}$ fixed by the auto. of $\overline{\mathbb{Q}}$ stabilizing I_{nc}, the field is always totally real, and equals $\mathbb{Q}$ iff $I = I_{nc}$. In the case there is only one split real place of F, it gives rise to Shimura curves.

Arithmetic subgroup of a tori. (Try to understand SV 5).

Let $T/\mathbb{Q}$ be a torus, $T(\mathbb{Z})$ be an arithmetic subgrp of $T(\mathbb{Q})$, e.g. $T(\mathbb{Z}) = \mathrm{Hom}(X^*(T), O_L^\times)^{\mathrm{Gal}(L/\mathbb{Q})}$, $X^*(T)$ group of charaters of T, L some Galois splitting field of T. Serre 1964 every subgroup of $T(\mathbb{Z})$ of finite index contains a congruence subgrp. Thus the topology induced on $T(\mathbb{Q})$ by $T(\mathbb{A}_f)$ shows that $T(\mathbb{Z})$ is open and the induced topology on $T(\mathbb{Z})$ is the profinite topology. In particular $T(\mathbb{Q})$ discrete $\Leftrightarrow$ $T(\mathbb{Z})$ discrete $\Leftrightarrow$ $T(\mathbb{Z})$ finite.

Ex.

(a) $T = \mathbb{G}_m$, $T(\mathbb{Z}) = \{\pm 1\}$ $\Rightarrow$ $T(\mathbb{Q})$ discrete in $T(\mathbb{A}_f)$.

(b) $T(\mathbb{Q}) = \{a \in \mathbb{Q}(i)^\times \mid \mathrm{Nm}\, a = 1\}$, $T(\mathbb{Z}) = \{\pm 1, \pm i\}$ $\Rightarrow$ $T(\mathbb{Q})$ discrete.

(c) $T(\mathbb{Q}) = \{a \in \mathbb{Q}(\sqrt{2})^\times \mid \mathrm{Nm}\, a = 1\}$, $T(\mathbb{Z}) = \{\pm(1+\sqrt{2})^{\mathbb{Z}}\}$ $\Rightarrow$ $T(\mathbb{Q})$ not discrete.

Thm. $T/\mathbb{Q}$ torus, $T^a = \bigcap \ker(\chi : T \to \mathbb{G}_m)$. Then $T(\mathbb{Q})$ discrete $\Leftrightarrow$ $T^a(\mathbb{R})$ compact.

Pf. Ono: $T(\mathbb{Z}) \cap T^a(\mathbb{Q})$ finite index in $T(\mathbb{Z})$, $T^a(\mathbb{R}) / T(\mathbb{Z}) \cap T^a(\mathbb{Q})$ compact ///

RMK. T/k torus is called anisotropic if there are no charaters $\chi : T \to \mathbb{G}_m$ defined over k. $T/\mathbb{R}$ is anisotropic $\Leftrightarrow$ $T(\mathbb{R})$ compact. T^a is the largest anisotropic subtorus of T, and the theorem says $T(\mathbb{Q})$ discrete iff T^a remains anisotropic over $\mathbb{R}$. SD (G, X) satisfies SV 5 iff Z^a remains anisotropic $/\mathbb{R}$.

A CM field L admits a nontrivial involution ι that becomes complex conjugate after embedding into $\mathbb{C}$. Let $T/\mathbb{Q}$ torus, split over L, then $T_L^+ = \bigcap_{\iota\chi = -\chi} \ker(\chi : T_L \to \mathbb{G}_m)$ is a subtorus of T defined over $\mathbb{Q}$, and is the largest subtorus splitting over $\mathbb{R}$. Then T^+ splits over the maximal totally real subfield of L and $T(\mathbb{Q})$ discrete $\Leftrightarrow$ T^+ splits over $\mathbb{Q}$.

Passage to the limit.

$K \subset G(\mathbb{A}_f)$ compact open, $\overline{Z(\mathbb{Q})}$ the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}_f)$. Then $Z(\mathbb{Q}) \cdot K = \overline{Z(\mathbb{Q})} \cdot K$ in $G(\mathbb{A}_f)$ and

$$\begin{aligned} Sh_K(G, X) &= G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K \\ &= \left(G(\mathbb{Q})/Z(\mathbb{Q}) \right) \backslash X \times \left(G(\mathbb{A}_f)/Z(\mathbb{Q}) \cdot K \right) \\ &= \left(G(\mathbb{Q})/Z(\mathbb{Q}) \right) \backslash X \times \left(G(\mathbb{A}_f)/\overline{Z(\mathbb{Q})} \cdot K \right) \end{aligned}$$

Thm. (G, X) SD, then

$$\varprojlim_K Sh_K(G, X) = \left(G(\mathbb{Q})/Z(\mathbb{Q}) \right) \backslash X \times \left(G(\mathbb{A}_f)/\overline{Z(\mathbb{Q})} \right) \overset{SV5}{=} G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) \quad \text{as topological space}$$

Same argument as before.

RMK. Put $S_K = Sh_K(G, X)$. When varying small K, $(S_K)_K$ is an inverse system of varieties with a right action of $G(\mathbb{A}_f)$ s.t. for $g \in K$, the action $\rho(g)$ on S_K is identity. Therefore if $K' \triangleleft K$, the finite group K/K' acts on $S_{K'}$ and S_K is the quotient of $S_{K'}$ by this action.

$S = \varprojlim S_K$ exists as a scheme locally Noetherian, regular $/\mathbb{C}$, $G(\mathbb{A}_f)$ acts on S on the right s.t. $K \subset G(\mathbb{A}_f)$ small compact open, $S_K = S/K$. Thus the inverse system together with its action by $G(\mathbb{A}_f)$ can be recovered by S with the action of $G(\mathbb{A}_f)$. On $\mathbb{C}$-points, $S(\mathbb{C}) = \varprojlim G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$.

RMK. Every arithmetic locally symmetric alg. variety arises as a connected component of a SV. Actually for every $H/\mathbb{Q}$ semisimple alg. grp, $\bar{h}: \mathbb{S}/\mathbb{G}_m \to H_{\mathbb{R}}^{ad}$ satisfying SU 1-3, there exists $G/\mathbb{Q}$ reductive, $G^{der} = H$, $h: \mathbb{S} \to G_{\mathbb{R}}$ lifting $\bar{h}$ satisfying SV 1-6, Z^*.