Build SD from (V, ψ) symplectic $/\mathbb{Q}$

Siegel SV classifies AV + polarization + level structure

$\Updownarrow$ Riemann

HS of type $(-1,0), (0,-1)$ + PL

The Siegel modular variety.

V $\mathbb{R}$-v.s., ψ nondegenerate bilinear alternating form on V, J complex struc. on V. $\psi(Ju, Jv) = \psi(u,v) \Leftrightarrow \psi(zu, zv) = |z|^2 \psi(u,v)$, $\forall z \in \mathbb{C}$.

$\Leftrightarrow \psi_J(u,v) = \psi(u, Jv)$ is symmetric

and $\psi(Ju, Jv) = \psi(u,v)$, ψ_J positive definite $\Leftrightarrow \psi$ polarization of (V, h_J)

Symplectic Spaces.

char $k \neq 2$, (V, ψ) sp. space of dim $2n$, i.e. V k-v.s. of dim $2n$ and ψ nondegenerate alternating bilinear form.

$W \subset V$ called totally isotropic if $\psi(W, W) = 0$.

Symplectic basis of V, $\{e_{\pm i}\}_{1 \le i \le n}$ s.t. $\psi(e_i, e_{-i}) = 1$, $\psi(e_i, e_j) = 0$, $i \neq \pm j$.

$\Rightarrow \psi = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Lemma. $W \subset V$ totally isotropic. Then every basis of W can be extended to a symplectic basis of V. In particular V has symplectic basis and any two sp. space of same dimension are isomorphic.

Cor. Every maximal totally isotropic subspace of V has dimension n. They are called Lagrangians.

(V,ψ) nonzero symplectic space, $GSp(\psi)$ be the group of symplectic similitudes of (V,ψ), $GSp(\psi)(k) = \{ g \in GL(V) \mid \psi(gu, gv) = \nu(g)\, \psi(u,v),\ \nu(g) \in k^\times \}$

$\nu : GSp(\psi) \to \mathbb{G}_m$ with kernel $Sp(\psi) = GSp(\psi)^{der}$. As before we have

$$\begin{tikzcd} & Sp(\psi) \arrow[d] \arrow[dr] & \\ \mathbb{G}_m \arrow[r] \arrow[dr] & GSp(\psi) \arrow[r, "Ad"] \arrow[d, "\nu"] & GSp(\psi)^{ad} \\ & \mathbb{G}_m & \end{tikzcd}$$

$\mathbb{G}_m \cap Sp(\psi) = \mu_2$.

Ex. $\dim V = 2$, $GSp(\psi) = GL_2$, $Sp(\psi) = SL_2$.

$\psi = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ under symplectic basis, $Sp(\psi)(k) = \{ g \in GL(V) \mid {}^t g \cdot \psi \cdot g = \psi \}$

$Sp(\psi)$ acts simply transitively on symplectic basis.

(V,ψ) symplectic over $\mathbb{Q}$, $G = G(\psi) = GSp(\psi)$, $S = S(\psi) = Sp(\psi) = G^{der}$.

J complex structure on $V_{\mathbb{R}}$ s.t. $\psi(Ju, Jv) = \psi(u,v)$, $J \in S(\mathbb{R})$.

$z \in \mathbb{C}^\times$, $h_J(z) \in G(\mathbb{R})$ and if $z \in U_1$, $h_J(z) \in S(\mathbb{R})$.

J is called positive / negative if ψ_J is positive / negative definite.

X^+ / X^- the set of positive / negative complex structures on $V_{\mathbb{R}}$ s.t.

$\psi(Ju, Jv) = \psi(u, v)$. $X = X^+ \amalg X^-$.

$G(\mathbb{R})$ acts on X by $g \cdot J = gJg^{-1}$, $\mathrm{Stab}_{G(\mathbb{R})}(X^+) = \{g \in G(\mathbb{R}) \mid \nu(g) > 0\} = G(\mathbb{R})^+$

For $(e_{\pm i})$ symplectic basis of V, define $J e_{\pm i} = \pm e_{\mp i}$, then $J^2 = -1$ and

$J \in X^+$, $(e_{\pm i})$ is orthonormal basis for ψ_J. Conversely $\forall J \in X^+$ has this description.

$$\{\text{Symplectic bases}\} \twoheadrightarrow X^+ \quad S(\mathbb{R})\text{-equivariant}$$

$\Rightarrow$ $S(\mathbb{R})$ acts on X^+ transitively

$\Rightarrow$ $G(\mathbb{R})$ acts on X transitively as $g e_{\pm i} = e_{\mp i}$ interchanges X^+ and X^-.

$J \in X$, $h_J : \mathbb{C}^\times \to G(\mathbb{R})$, $h_{gJg^{-1}}(z) = g h_J(z) g^{-1}$.

$J \mapsto h_J$ identifies X with a $G(\mathbb{R})$-conjugacy class of $h : \mathbb{C}^\times \to G(\mathbb{R})$.

We shall write $X(\psi)$ and $X(\psi)^+$.

RMK. $h \in X(\psi)$, $\nu(h(z)) = z\bar{z}$.

$\dim V = 2g$, a choice of symplectic basis for V identifies X^+ with $\mathcal{H}_g$ as $Sp(\psi)$-set.

$(G(\psi), X(\psi))$ satisfies SV 1-6.

SV 1.

$G(\psi) \subset GL_V \times \mathbb{G}_m$, $\mathrm{Lie}\, G_{\mathbb{R}} \subset \{(f, \lambda) \in \mathrm{End}(V_{\mathbb{R}}) \times \mathbb{R} \mid \psi(f(u), v) + \psi(u, f(v)) = \lambda \cdot \psi(u,v)\}$

The action of $G(\mathbb{R})$ on $\mathrm{Lie}\, G_{\mathbb{R}}$ is by $g(f, \lambda) = (gfg^{-1}, \lambda)$.

As $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$, $\mathrm{End}(V_{\mathbb{C}}) = \mathrm{End}(V_{\mathbb{C}})^{0,0} \oplus \mathrm{End}(V_{\mathbb{C}})^{1,-1} \oplus \mathrm{End}(V_{\mathbb{C}})^{-1,1}$ and correspondingly $\mathrm{Lie}\, G_{\mathbb{C}}$ decomposes the same way.

SV 2.

$J^2 = -1 \in Z(S)(\mathbb{R})$. ψ is J-polarization for $S_{\mathbb{R}}$ iff $J \in X^+$. Hence $\mathrm{Ad}\, J$

$-\psi$ $\quad J \in X^-$

is Cartan involution for $S \Rightarrow \mathrm{Ad}\, J$ Cartan involution for $S^{ad} = G^{ad}$.

SV 3.

$S = Sp(\psi)$ is $\mathbb{Q}$-simple and $G^{ad}(\mathbb{R})$ is not compact.

SV 4.

$r \in \mathbb{R}^\times$, $w_h(r)$ acts on $V^{-1,0}$ and $V^{0,-1}$ by r^{-1}. Thus w_X is the map $\mathbb{G}_{m,\mathbb{R}} \longrightarrow GL(V_{\mathbb{R}})$ which is defined over $\mathbb{Q}$.

$r \longmapsto r^{-1}$

SV 5.

$Z(G) = \mathbb{G}_m$, $\mathbb{G}_m(\mathbb{Q}) = \mathbb{Q}^\times$ discrete in $\mathbb{A}_f^\times$.

SV 6.

$Z^\circ(G) = Z(G)$ already splits over $\mathbb{Q}$.

This is called the Siegel SD.

The Siegel modular variety.

$(G, X) = (G(\psi), X(\psi))$ SD defined by sp. space (V, ψ) over $\mathbb{Q}$.

The Siegel modular variety $Sh(G, X)$.

$V(\mathbb{A}_f) = V \otimes_{\mathbb{Q}} \mathbb{A}_f$, $G(\mathbb{A}_f)$ the group of $\mathbb{A}_f$-linear auto. of $V(\mathbb{A}_f)$ preserving ψ up to $\mathbb{A}_f^\times$.

$K \subset G(\mathbb{A}_f)$ compact open, $\mathcal{H}_K$ the set of triples $((W, h), s, \eta K)$ where

- (W, h) rational Hodge structure of type $(-1, 0), (0, -1)$
- s or $-s$ is a polarization for (W, h)
- ηK is a K-orbit of $\mathbb{A}_f$-linear isom. $V(\mathbb{A}_f) \to W(\mathbb{A}_f)$ under which ψ corresponds to an $\mathbb{A}_f^\times$-multiple of s.

An isom. $((W,h), s, \eta K) \xrightarrow{\sim} ((W',h'), s', \eta' K)$ is an isom.

$b: (W,h) \xrightarrow{\sim} (W', h')$ of rational Hodge structures sending s to cs' for some $c \in \mathbb{Q}^\times$ and s.t. $b \circ \eta = \eta'$ mod K.

The triple $((W,h), s, \eta K)$ is the same as a sp. space (W,s) over $\mathbb{Q}$, a complex structure on W that is positive or negative for s, and ηK.

As $\dim V = \dim W$, (W, s) and (V, ψ) are isom. Choose $a: W \xrightarrow{\sim} V$ s.t. ψ corresponds to a $\mathbb{Q}^\times$-multiple of s, then $ah: z \longmapsto a \circ h(z) \circ a^{-1}$ lies in X and $V(\mathbb{A}_f) \xrightarrow{\eta} W(\mathbb{A}_f) \xrightarrow{a} V(\mathbb{A}_f)$ lies in $G(\mathbb{A}_f)$.

Any other such isom. a' differs from a by some $q \in G(\mathbb{Q})$, $a' = q \circ a$. Replacing a with a' replaces $(ah, a\circ\eta)$ by $(qah, qa\circ\eta)$. Thus we have well-defined map

$$\mathcal{H}_K \longrightarrow G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K$$

$$\bullet \longmapsto [ah, a\circ\eta]_K$$

Prop. The map above induces bijection $\mathcal{H}_K/\simeq \;\longrightarrow G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$.

$$M \longmapsto H_1(M(\mathbb{C}), \mathbb{Z})$$

Slogan : $AV/\mathbb{C} \longleftrightarrow$ polarizable integral Hodge structure of type $(-1,0), (0,-1)$

Complex abelian varieties.

$\Lambda \subset \mathbb{C}^n$ lattice, $M = \mathbb{C}^n/\Lambda$.

$\pi_1(M, 0) = H_1(M, \mathbb{Z}) = \Lambda$, $H^1(M, \mathbb{Z}) = \mathrm{Hom}(\Lambda, \mathbb{Z})$.

Prop. $H^n(M, \mathbb{Z}) \simeq \mathrm{Hom}(\wedge^n \Lambda, \mathbb{Z})$.

Pf. $\mathrm{Hom}(\wedge^n \Lambda, \mathbb{Z}) = \wedge^n \mathrm{Hom}(\Lambda, \mathbb{Z}) = \wedge^n H^1(M, \mathbb{Z})$.

If $M = S^1$, $\wedge^n H^1(M, \mathbb{Z}) = H^n(M, \mathbb{Z})$.

$M = (S^1)^{2n}$, true by Kunneth. ///.

Prop. A linear map $\alpha : \mathbb{C}^n \to \mathbb{C}^m$, $\alpha(\Lambda) \subset \Lambda'$ induces a holo. $\mathbb{C}^n/\Lambda \to \mathbb{C}^m/\Lambda'$ sending 0 to 0, and any holo. $\mathbb{C}^n/\Lambda \to \mathbb{C}^m/\Lambda'$ sending 0 to 0 arises in this way for some unique α.

$\mathbb{R} \otimes \Lambda \simeq \mathbb{C}^n$ defines cpx struc. J on $\mathbb{R} \otimes \Lambda$.

A Riemann form for M is $\psi : \Lambda \times \Lambda \to \mathbb{Z}$ alternating s.t.

$\psi_{\mathbb{R}}(Ju, Jv) = \psi_{\mathbb{R}}(u, v)$, $\psi_{\mathbb{R}}(u, Ju) > 0$ if $u \neq 0$.

M is called polarizable if there is a Riemann form ψ.

Thm. M is projective $\Leftrightarrow$ M is polarizable.

M projective $\Rightarrow$ M compact Kahler, $c_1(\mathcal{O}(1)|_X) \in H^2(X, \mathbb{Z})$ Riemann form

M polarizable $\Rightarrow$ ψ is Kahler form in $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$

$\Rightarrow \psi = c_1(L)$ for some L positive $\overset{\text{Kodaira}}{\Longrightarrow}$ L ample.

Chow : polarizable complex torus is a projective algebraic variety and holo. maps between them are regular.

Conversely the complex manifold associated with an abelian variety is a complex torus, $\exp : T_0(A) \to A(\mathbb{C})$ is surj. with kernel lattice Λ. $A(\mathbb{C}) = T_0A/\Lambda$.

$M = \mathbb{C}^n/\Lambda$, $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{C}^n$, $\Lambda \simeq H_1(M, \mathbb{Z})$ has integral Hodge structure of weight -1, a Riemann form for M is a polarization for the Hodge struc.

Thm. (Riemann) The functor $A \longmapsto H_1(A(\mathbb{C}), \mathbb{Z})$ is an equi. of categories.

$$\{AV/\mathbb{C}\} \longrightarrow \{\text{polarizable integral Hodge struc. of type } {(-1,0) \atop (0,-1)}\}$$

Pf.

$$\{AV/\mathbb{C}\} \xrightarrow{an} \{\text{polarizable complex tori}\} \xrightarrow{H_1(-,\mathbb{Z})} \{\text{polarizable integral HS}, {(-1,0) \atop (0,-1)}\}$$

$$A \longmapsto A^{an} \qquad\qquad M \longmapsto H_1(M, \mathbb{Z})$$

CHOW $\qquad \mathbb{C}^n/\Lambda \to \mathbb{C}^m/\Lambda' \Leftarrow \mathbb{C}^n \to \mathbb{C}^m, \Lambda \to \Lambda'$. ///

AV^0 : $Obj = AV/\mathbb{C}$, $Hom_0(A, B) = Hom(A, B) \otimes \mathbb{Q}$

Cor. $AV^0 \xrightarrow{equi.}$ polarizable rational Hodge structure of type $(-1,0), (0,-1)$.

$T_0A = H_1(A, \mathbb{R}) = H_1(A, \mathbb{C}) / F^0$

A modular description of the points of the Siegel variety

A AV/$\mathbb{C}$, $\quad T_f A = H_1(A, \mathbb{Z}) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} = \varprojlim_n H_1(A, \mathbb{Z}) / n H_1(A, \mathbb{Z})$

$\quad V_f A = H_1(A, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}_f = T_f A \otimes_{\mathbb{Z}} \mathbb{Q}$

$T_f A$ free $\hat{\mathbb{Z}}$-mod of rank $2\dim A$

$$A(\mathbb{C})_n = \ker([n] : A(\mathbb{C}) \to A(\mathbb{C}))$$

$A(\mathbb{C}) = \mathbb{C}^g / \Lambda$, $\quad H_1(A, \mathbb{Z}) = \Lambda$. $\quad A(\mathbb{C})_n = \frac{1}{n}\Lambda / \Lambda = \Lambda / n\Lambda$. $\quad T_f A = \varprojlim A(\mathbb{C})_n$

A AV $/k$, char $k = 0$, $\quad T_f A = \varprojlim A(\bar{k})_n$, $\quad V_f A = T_f A \otimes_{\mathbb{Z}} \mathbb{Q}$.

(V, ψ) sp. space $/\mathbb{Q}$, consider $(A, s, \eta K)$

- A AV/$\mathbb{C}$
- s alternating form on $H_1(A, \mathbb{Q})$ s.t. s or $-s$ is polarization on $H_1(A, \mathbb{Q})$
- $\eta : V(\mathbb{A}_f) \xrightarrow{\sim} V_f A$, ψ corresponds to a multiple of s by $\mathbb{A}_f^{\times}$.

and $(A, s, \eta K) \xrightarrow{\sim} (A', s', \eta' K)$ if $A \xrightarrow{\sim} A'$ in AV° sending s to $\mathbb{Q}^{\times}$-multiple of s' and ηK to $\eta' K$.

moduli ?

Thm. $\exists$ natural bijection $\quad \{(A, s, \eta K)\}/\simeq \quad \longrightarrow \quad Sh_K(G(\psi), X(\psi))(\mathbb{C})$.

Let $(G, X) = (GSp(V,\psi), X(\psi))$, $S = G^{der} = Sp(\psi)$.

Suppose $\exists$ $\mathbb{Z}$-lattice $V(\mathbb{Z})$ in V s.t. ψ restricts to a pairing $V(\mathbb{Z}) \times V(\mathbb{Z}) \to \mathbb{Z}$ with discriminant ± 1. For $N \geq 3$, let

$K(N) = \{ g \in G(\mathbb{A}_f) \mid g$ preserves $V(\hat{\mathbb{Z}})$ and acts as 1 on $V(\hat{\mathbb{Z}})/NV(\hat{\mathbb{Z}}) \}$

Let $S(\mathbb{Z})$ be the set of g in $S(\mathbb{Q})$ s.t. $gV(\mathbb{Z}) = V(\mathbb{Z})$. Then

$K(N) \cap S(\mathbb{Z}) = \Gamma(N) = \{ g \in S(\mathbb{Z}) \mid g$ acts as 1 on $V(\mathbb{Z})/NV(\mathbb{Z}) \}$

Write $V(\mathbb{Z}/N\mathbb{Z})$ for $V(\mathbb{Z})/NV(\mathbb{Z}) = V(\hat{\mathbb{Z}})/NV(\hat{\mathbb{Z}})$

It is a free $\mathbb{Z}/N\mathbb{Z}$-module of rank $\dim V$ with perfect pairing ψ_N.

In this case $\pi_0(Sh_{K(N)}(G,X)) = (\mathbb{Z}/N\mathbb{Z})^\times$.

Let (A, λ) be $AV/\mathbb{C}$ of dim $\frac{1}{2}\dim V$ and λ a principal polarization.

From λ we get a perfect alternating pairing

$$e_N^\lambda : A(\mathbb{C})[N] \times A(\mathbb{C})[N] \to \mu_N \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}$$
$$e^{2\pi i/N} \longmapsto [1]$$

A level N structure on A is an isom. $\eta : V(\mathbb{Z}/N\mathbb{Z}) \xrightarrow{\sim} A(\mathbb{C})[N]$ sending ψ_N to some $(\mathbb{Z}/N\mathbb{Z})^\times$-multiple of e_N^λ. Then the set $Sh_{K(N)}(\mathbb{C})$ classifies isom. classes of $((A,\lambda),\eta)$. The connected component of a pair $((A,\lambda),\eta)$ in $(\mathbb{Z}/N\mathbb{Z})^\times$ is just $\eta_* \psi_N / e_N^\lambda$. The fibre over $[1]$ consists of those $((A,\lambda),\eta)$ s.t. under η, ψ_N and e_N^λ correspond exactly.

$AV/\mathbb{C}$ always isogenous to principally polarized AV