Shimura varieties of Hodge type.

Generalization of Siegel modular varieties : ··· + fixing tensor condition.

What kind of G has symplectic embeddings (in terms of Dynkin diagram)?

Def. SD (G, X) is called of Hodge type if $\exists$ sp. space $(V, \psi)/\mathbb{Q}$ and $\rho : G \longrightarrow G(\psi)$ injective carrying X into $X(\psi)$. $Sh(G, X)$ is said to be of Hodge type if (G, X) is.

Let $\nu : G \xrightarrow{\rho} G(\psi) \xrightarrow{\nu} \mathbb{G}_m$, $\mathbb{Q}(r)$ the v.s. $\mathbb{Q}$ on which G acts by ν^r.

$h \in X$, $(\mathbb{Q}(r), \nu \circ h)$ is rational Hodge structure of type $(-r, -r)$.

$h \in X$, $\rho h \in X(\psi)$, $\nu \rho h(z) = z\bar{z}$.

Lemma. $\exists$ multilinear maps $t_i : V \times \cdots \times V \to \mathbb{Q}(r_i)$ s.t. G is the subgrp of $G(\psi)$ fixing these t_i.

Pf. Chevalley's thm $\Rightarrow$ $\exists$ tensors $t_i \in V^{\otimes r_i} \otimes (V^\vee)^{\otimes s_i}$ s.t. G is the subgrp of $G(\psi)$ fixing these t_i.

$\psi \Rightarrow V \simeq V^\vee \otimes \mathbb{Q}(1)$ G-equivariant. ///

$$v \longmapsto \Phi(v) = \psi(v, -)$$

$$\Phi(gv)(u) = \psi(gv, u) = \nu(g)\psi(v, g^{-1}u) = (g\Phi(v))(u)$$

(G, X) Hodge type, $(G, X) \hookrightarrow (G(\psi), X(\psi))$, G fixing $t_1, \dots, t_n$.

$\mathcal{H}_K$ the set of triples $((W, h), (s_i)_{0 \le i \le n}, \eta K)$ where

- (W, h) rational Hodge structure of type $(-1, 0), (0, -1)$
- s_0 or $-s_0$ polarization for (W, h)
- $s_1, \dots, s_n$ multilinear $W \times \cdots \times W \to \mathbb{Q}(r_i)$
- ηK K-orbit of isom. $V(\mathbb{A}_f) \xrightarrow{\sim} W(\mathbb{A}_f)$ under which ψ corresponds to

$\mathbb{A}_f^\times$-multiple of s_0 and t_i correspond to s_i

satisfying

$(*)$ $\exists$ isom. $a : W \xrightarrow{\sim} V$ under which s_0 corresponds to $\mathbb{Q}^\times$-multiple of ψ and s_i corresponds to t_i, h corresponds to some element in X.

An isom. $((W, h), (s_i), \eta K) \xrightarrow{\sim} ((W', h'), (s_i'), \eta' K)$ is an isom. of rational Hodge structures $(W, h) \to (W', h')$ sending s_0 to $\mathbb{Q}^\times$-multiple of s_0', s_i to s_i' and ηK to $\eta' K$.

Prop. $\exists$ natural bijection $\mathcal{H}_K / \simeq \longrightarrow Sh_K(\mathbb{C})$.

Pf. $((W, h), (s_i), \eta K)$. Choose $a : W \to V$ as in $(*)$, consider $(ah, a\eta)$ where $ah(z) = a \circ h(z) \circ a^{-1}$, $ah : \mathbb{S} \to GL(V)$.

$ah \in X$ and $a\eta$ is a symplectic similitude of $(V(\mathbb{A}_f), \psi)$ fixing t_i, hence $(ah, a\eta) \in X \times G(\mathbb{A}_f)$.

Different a vary by $G(\mathbb{Q})$ and different η vary by K. Hence $((W, h), (s_i), \eta K) \longmapsto [ah, a\eta] \in G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K$ is well-defined.

Surjectivity is clear. ///

Let $t: V^{\otimes m} \to \mathbb{Q}(r)$ fixed by G, $t(gv_1, \dots, gv_m) = \nu(g)^r t(v_1, \dots, v_m)$.

$h \in X$, t defines a morphism of Hodge structures $(V, h)^{\otimes m} \to \mathbb{Q}(r)$.

If $t \neq 0$, $m = 2r$ by comparing the weights.

A $AV/\mathbb{C}$, $W = H_1(A, \mathbb{Q})$. $H^m(A, \mathbb{Q}) \simeq \mathrm{Hom}(\Lambda^m W, \mathbb{Q})$.

$t \in H^{2r}(A, \mathbb{Q})$ is called a Hodge tensor for A if $W^{\otimes 2r} \to \Lambda^{2r} W \to \mathbb{Q}(r)$ is a morphism of Hodge structures.

$(G, X) \hookrightarrow (G(\psi), X(\psi))$, G fixing $t_1, \dots, t_n$.

Let $\mathcal{M}_K$ be the set of triples $(A, (s_i)_{0 \le i \le n}, \eta K)$ where

- $A/\mathbb{C}$ AV
- s_0 or $-s_0$ polarization for the rational Hodge structure $(H_1(A, \mathbb{Q}), \tilde{h})$
- s_i Hodge tensors for A or powers of A
- ηK K-orbit of $\mathbb{A}_f$-linear isom. $V(\mathbb{A}_f) \xrightarrow{\sim} V_f(A)$ sending ψ to $\mathbb{A}_f^\times$-multiple of s_0 and t_i to s_i

satisfying

$(**)$ $\exists$ isom. $a: H_1(A, \mathbb{Q}) \xrightarrow{\sim} V$ sending s_0 to $\mathbb{Q}^\times$-multiple of ψ, s_i to t_i and $\tilde{h}$ to some element in X.

An isom. $(A, (s_i), \eta K) \xrightarrow{\sim} (A', (s_i'), \eta' K)$ is an isom. in AV° $A \xrightarrow{\sim} A'$ sending s_0 to $\mathbb{Q}^\times$-multiple of s_0', s_i to s_i' and ηK to $\eta' K$.

Thm. $\exists$ natural bijection $\mathcal{M}_K/\simeq \longrightarrow Sh_K(\mathbb{C})$.

$(**)$ hard to check.

If Hodge tensors are endomorphisms of AV, simpler trace condition. (PEL)

RMK. $A(\mathbb{C}) = \mathbb{C}^g/\Lambda$, $H^m(A, \mathbb{Q}) \simeq \mathrm{Hom}(\wedge^m \Lambda, \mathbb{Q})$

$\Lambda \otimes \mathbb{C} = T \oplus \bar{T}$, $T = T_0 A$,

$H^m(A, \mathbb{C}) \simeq \mathrm{Hom}(\wedge^m(\Lambda \otimes \mathbb{C}), \mathbb{C}) \simeq \mathrm{Hom}(\bigoplus_{p+q=m} \wedge^p T \otimes \wedge^q \bar{T}, \mathbb{C}) = \bigoplus H^{p,q}$

$H^{p,q} = \mathrm{Hom}(\wedge^p T \otimes \wedge^q \bar{T}, \mathbb{C}) \simeq H^q(A(\mathbb{C}), \Omega^p_{hol})$.

A Hodge tensor on A is an element of $H^{2r}(A, \mathbb{Q}) \cap H^{r,r} \subset H^{2r}(A, \mathbb{C})$.

Hodge conjecture: the space of Hodge tensors is $\mathbb{Q}$-span of cohomology class of algebraic cycles on A.

Lefschetz $(1,1)$ thm: $0 \to \mathbb{Z} \to \mathcal{O}_A \xrightarrow{\exp} \mathcal{O}_A^\times \to 0$

$\Rightarrow \mathrm{Pic}(A) \to H^2(A, \mathbb{Z}) \to H^2(A, \mathcal{O}_A)$

then $\mathrm{Im}\,\mathrm{Pic}(A) = H^2(A, \mathbb{Z}) \cap H^{1,1}$