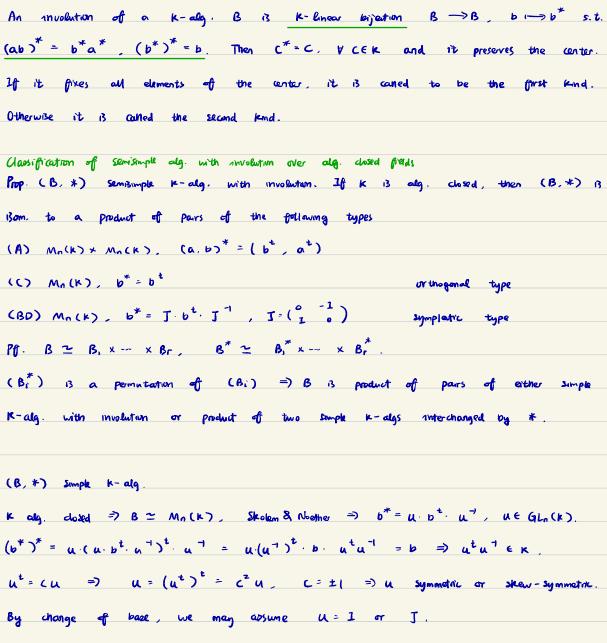
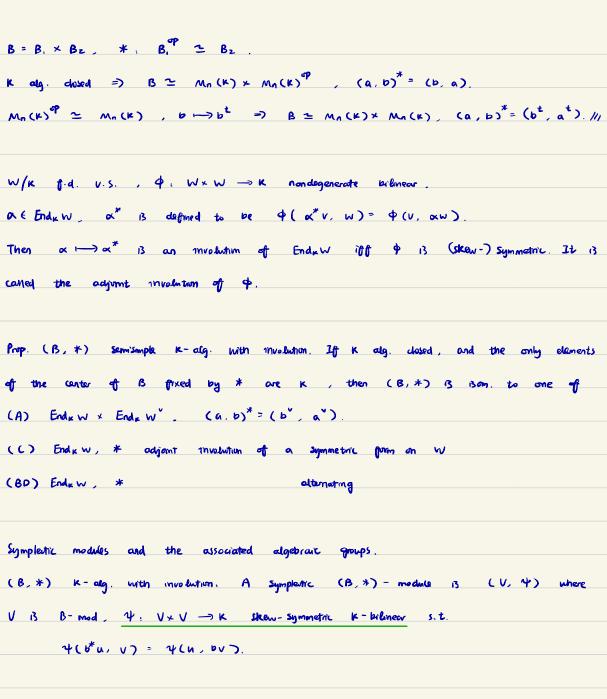
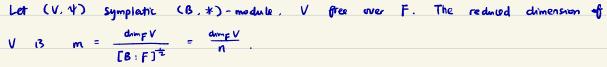
PEL Shimura varieties chark = 0 Algebras with molution. K-alg: mg containing K in center, finite dimensional as K-v.s. Simple k - alg: o, A only two - sided ideals Wedderburn Thm : all simple k-algs are off the form Mn(D), D division K-ala K alg. dosed, D=K, Emple K-algs are Mn(K). A simple k-alg. B has only one simple module M up to isom, and every B-module is isom to direct sum of M. Dⁿ is the only simple Mn(D) - module. Semisimple K-alg B: every B-mod 13 Semisimple. B has only finitely many minimal two-sided ideals B1, ..., Br, each is a simple k = alg, and $B \cong B$, X = X Br. The only simple B-mods are simple B_i -mods M_i via $B \longrightarrow B_i$. The trace map of a B-mod M is $b \longrightarrow tr_{k}(b|_{m})$. Prop. B semisimple k-alg. The B-modes are ison. iffer they have the same tr. Mr. B = Bix + × Br, M = O dimi. tre (ei) (di Mi) = di dume Mi. 111





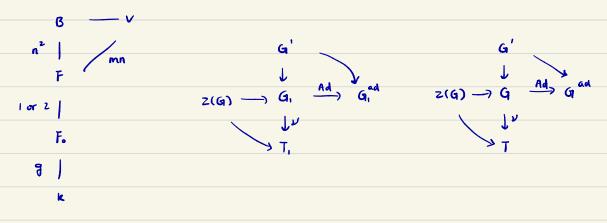
B semisimple k-alg., center of B is $F = TTF_i$, F_i field, $B = TTB_i$ with
$Bi = B \otimes_F Fi$ central simple alg. over Fi
Assume B free F-mod, [Bi: Fi]= n ² . For any B-mod V, there is a reduced
det map \det , $\operatorname{End}_{\mathbf{s}}(\vee) \longrightarrow F$
$If = u \cdot M_n(F) \xrightarrow{\sim} B then det(g) = det(g u(E_{U})V).$
e.g. $B = F$, $V = F^m$, dut : End_F(F ^m) $\rightarrow F$ is the usual dut
$B = M_m(F), V = F^m, \text{ old} : End_{M_m(F)}(F^m) = F \longrightarrow F \text{is identity}$
In general, suppose F field, E/F finite Galoris s.t. $U: M_{h}(E) \xrightarrow{\sim} B^{\oplus}FE$, hence
reduced det _E : End _{elee} (Ve _F E) \rightarrow E. If g ∈ End _e (V) ⊂ End _{Bee} (Ve),
$det(g)$ is fixed by $Gal(E/F)$ so lies in F . Thus get det : $End_B \lor \longrightarrow F$,
it is independent of choice of E/F.

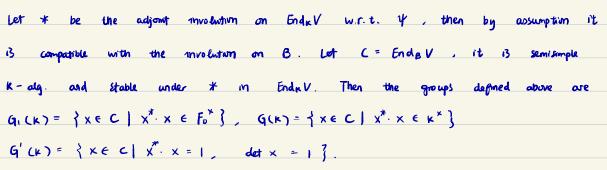
Let
$$(B, *)$$
 semisimple K-alg. with involution, F center of B, Fo subadg.
of * invariants in F. We say $(B, *)$ is of type (A) , (C) , (BD) iff
 $(B^{O}_{F_{n}}, p^{-}_{K}, *)$ is of that type for all K-homomorphisms $p: f_{0} \rightarrow \overline{K}$.
Assume B free F-mod, then F free of rank 2 over Fo in case (A)
and equals Fo in case (C) , (BD) . Let $(B:F] = n^{2}$ and $(F_{0}:K] = g$.
 $(A, *)$ simple k-alg., $Z(A) = F$, A -mod V, $\phi: V \times V \rightarrow A$ (Shaw-) Hermitics iff
 $nondag$.
 $\phi(au, pv) = a \phi(u, v)b^{*}$, $\phi(v, u) = (-) \phi(u, v)^{*} \Rightarrow *\phi$ on $End_{A}V$.
* first kind. $\{\phi \text{ on } V\} \in {}^{(1)}_{C} \{*\phi \text{ on } B\} (F^{*})$



Let
$$G_1$$
, G_2 be algo subgrps of GL_BV s.t.
 $G_1(k) = \{g \in GL_B(V) \mid \Psi(gx, gy) = \Psi(\mu(g)x, y), \mu(g) \in F_0^{\times} \}$
 $G(k) = \{g \in GL_B(V) \mid \Psi(gx, gy) = \mu(g), \Psi(x, y), \mu(g) \in k^{\times} \}$
There is a homomorphism $\mu_1(G_1 \longrightarrow (G_m)F_0/k$ and G_1 is the inverse image of

 $G_m \subset (G_m)_{F_0/K}$. Set $G_1' = ker \mu n ker det , <math>T_1 = G_1/G'$, $T = G_1/G'$. Then





Classify sympletic modules => classify these alg. grps Ex. F/k stale of dag 2 => F = K × k or F/k dag 2 extension.
There is a unique nontrivial muselution * of F fixing K.
Assume $B \simeq M_n(F)$ and $(B, *)$ of type (A) , then $B \simeq End_p W$ for
W simple B-mod, $*$ is the adjunct involution of a Hermitian form $\phi: w \times w \rightarrow F$.
Let Vo free F-mod of finite rank, Yo: Vo × Vo → F skew-Hermitian firm.
Then B auts on $V = W \otimes_F V_0$ and $\Psi : V \times V \longrightarrow K$ defined by
$\Psi(w \otimes v, w' \otimes u') = tr_{F/K} \left(\Phi(w, w') \Psi_{\theta}(v, v') \right) (v, \psi) \longleftrightarrow (v_{\theta}, \psi_{\theta})$
=> (V, Y) is a symplectic (B, *) - module.
Conversely let (V, 4) be a symplectic (B, *)-module. As B-mod, V is
durent sum off copies off W, so V = W@F Vo for some Vo free F-mod
of privite rank. Choose $f \in F^{\times}$ s.t. $f^{*} = -f$. Then there exists a unique
Hermitian form 里:V×V →F s.t. Y(u,v)= trF/x (f星(u,v)) and
$\Psi(b^*u, v) = \Psi(u, bv)$. The adjoint involution of Ψ preserves $End_e v \cong End_p V_o$.
Choose skew-Hennitian form 40 : Vo × Vo → F whose adjoint shrobution is the
restriction of *z. The form f E(-,-) is skew-Hermitian and after
possibly scaling ψ_0 , $f \overline{\Psi} = \phi \otimes \psi_0$.

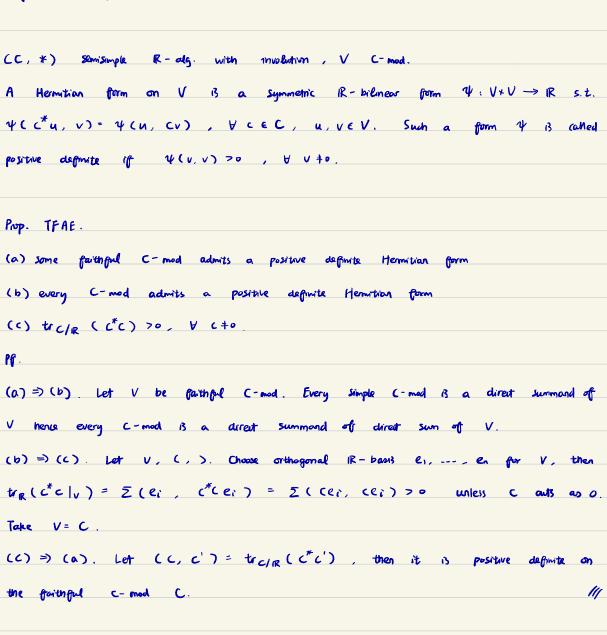
Ex. Assume $B \cong M_n(k)$ and (B, *) is off type CC). Then $B \cong End_k W$, W is Himple B-mod and * is adjoint involution of a symmetric bilinear form $\varphi: W \times W \rightarrow K$. Let V_0 be k - v.s., $\psi_0: V_0 \times V_0 \rightarrow K$ alternating. Let $\psi = \phi \otimes \psi_0$ on $V = V_0 \otimes W$. Then (V, 4) is a symplectic (B, *)-mod and every symplectic mod arises like this. $C = End_B V \simeq End_R V_{\circ}$ and * and on C as the adjoint involution of 4_{\circ} . Hence $G_1 = G = G(Sp(V_{\circ}, 4_{\circ}))$ and $G' = Sp(V_{\circ}, 4_{\circ})$.

Prop. Let (B, *) Semisimple K-adg. with miseluition, (V, 4) symplectic (B, *)-mod.
Let F be arter of B., Fo subally of F of elements fixed by *.
Assume V. B free over F, [B:F] =
$$n^2$$
, [Fo:K] = 9, dom FV = mn. Then
Case (A): G, G₁ connected, reductive, G' semisimple simply connected
F quadratic extension of Fo
for all g \in G₁(K), dot(g) · dut(g)^{*} = $\mu(g)^m$
iff K adg. doted, G' \simeq 5Lm⁹
iff m even, m = 2l, then $(det^{-1} \circ \mu l, \Lambda)$ gives Borns
T₁ \simeq ker ((Gm)F/k \xrightarrow{Nm}) (Gm)F/k) × (Gm)
iff m add, m = 2l , then $(det^{-1} \circ \mu l, \Lambda)$ × Gim
iff m add, m = 2l , then $K = dat^{-1} \circ \mu^{-1}$ gives Borns
T₁ \simeq (Gm)F/k and $\mu = K \cdot K^*$.
Case (C): G, G₁ are connected, reductive, G' semismple, Simply connected
m B even, m = 2l, dat(g) = $\mu(g)^{l}$, \forall g ∈ G₁(K).
T₁ \xrightarrow{L} (Gm)F/k, T \xrightarrow{L} > Gm
iff K adg. doted, G' \simeq $\frac{pm}{m}$.

$$\begin{split} \rho / k \quad quadramic , V \in P^{m} , End_{p} V = M_{m}(F) , a^{q} = \overline{k} , q_{0} , V + V \rightarrow \rho F \quad skew-kem \\ U_{1,5} = (^{1/r} , _{1,3_{1}}) , < c_{1}, p^{2} = a^{2}U_{1,5} \overline{\beta} , g \in m_{m}(F) : c_{3}n \cdot gp > a^{2} g^{2}U_{1,5} \overline{\beta} , g^{2}U_{1,5} \overline{\beta} , a \in n \\ \mu (g)^{2} = a \\ \overline{F} = K \times K , V : \overline{F}^{m} : K^{m} \times K^{m} , End_{F} V = M_{m}(F) : M_{m}(K) \times M_{m}(K) : du(g)^{2} : duff \\ (a, b)^{d} = (b, a) , q_{0} : V \times V \rightarrow F \quad skew - herm \quad \neg I_{r,5} : (-1_{r,1}) \in M_{m}(K) \\ < (\alpha, \beta) , (\alpha', \beta') > = (\alpha^{2}U_{1,5} g', -\beta^{2}U_{1,5} n') , du(g) - dufg' - ldeg ^{2} + ldeg ^{2} = a^{m} \\ (q, h) , < (q, h)(a, \beta) , (q, h)(\alpha', \beta') > = (\alpha^{2}(q^{2}U_{1,5}h)\beta', -\beta^{2}(h^{2}U_{1,5}g)\alpha') \\ dut (q, h) = (dut q, dut h) , g^{1}U_{r,5}h = a I_{r,5} , h^{1}U_{r,5}B = a I_{r,5} , \mu(g, h) = (a, a) \\ dut (q, h)^{d} = (dut q, dut h) , g^{1}U_{r,5}h = du(q, h)^{d'} : (dut q dut h) = (a^{m}, a^{m}) \\ g^{3}U_{r,5}h = I_{r,5} , h = I_{r,6}(g^{1})^{m}I_{r,5} , G' = SI_{m} g^{3}U_{r,5}B = U_{r,5} , g(U^{r,5}) \\ (g_{m}, \rho)_{r}(k) = k^{m} \times k^{m} - 2 g_{m}(k) = k^{m} \\ dut (g, h)^{-1} (a^{d}, a^{d'}) G' = SU^{r,5} \\ \mu^{4}(q, h) = (a^{d}, a^{d}) G' = SU^{r,5} \\ \mu^{4}(q, h) = (a^{d}, a^{d'}) G' = SU^{r,5} \\ \mu^{4}(q, h) = (a^{d}, a^{d'}) G' = SU^{r,5} \\ \mu^{4}(q, h) = (a^{d}, a^{d'}) G' = SU^{r,5} \\ \mu^{4}(q, h) = (a^{d}, a^{d'}) G' = SU^{r,5} \\ \mu^{4}(q, h) = (a^{d}, a^{d'}) G' = SU^{r,5} \\ \mu^{4}(q, h) = (a^{d}, a^{d'}) G' = SU^{r,5} \\ \mu^{4}(q, h) = (a^{d}, a^{d'}) G' = SU^{r,5} \\ \mu^{4}(q, h) = (a^{d}, a^{d'}) G' = SU^{r,5} \\ \mu^{4}(q, h) = (a^{d}, a^{d'}) G' = SU^{r,5} \\ \mu^{4}(q, h) = (a^{d}, a^{d'}) G' = G^{d}(q^{d}) \\ \mu^{4}(q, h) = (a^{d}, a^{d'}) G' = G^{d}(q^{d}) \\ \mu^{4}(q, h) = (a^{d}, a^{m'}) = (a, a) \\ = \mu^{4}(q)^{r,h} G' \\ \mu^{4}(q, h) = (a^{d}, a^{d'}) G' = G^{d}(q^{d}) \\ \mu^{4}(q, h) = (a^{d}, a^{d'}) G' \\ \mu^{4}(q, h) = (a^{d}$$

k olg. dosed, $G' = (SO_m)^{(f \setminus K)}$.





Pef. An muchdum substituting the above equivalent conditions is called positive.
Prop. Let (8, *) Semisimple R-alg. with positive muchation , (U, 4) symplectic mod.
Assume (8, *) is of type (A) or (C), let C = End_B(V) ⊂ End_R(V). Then
there exists a homomorphism of R-alg. h: C → C s.t.
• h(Z) = h(Z)^{*}
• (u, V) = 4(u, h(i)V) is symmetric, positive differta.
Pf. To give such on h substats to specify J ∈ C s.t.

$$T^2 = -1$$
, $4(Ju, Jv) = 4(u, v)$, $4(u, Ju) > 0$ if $u \neq 0$.
Type (A): (B, *, V, 4) decorposes into a product of (Ala(F), *, Uo, 40)
where $F = R \times R$ or C.
Supple $F = C$, then
• B \cong End_EW
• * adjust muchtum of a positive differe Hermitian form $\Phi : W \times W = C$
• $V \cong W \otimes_C V_0$, V. C-v.s.
• $4 = trc/R$ (40%+) with 4. Show - Herm. form an V.
C $\cong End_C V_0$, be may find a basis (ej) for Uo s.t.
(4.(ej, eu)) = drag(i, ..., i, -i, ..., -i)). Then define $Je_j = \begin{cases} i^{ej}_{j=i}^{j \neq i} \\ -iej_{j} > r \end{cases}$
 $= J^{2}_{j} = -1$, $4_{0}(Ju, Jv) = 4_{0}(u, v)$, $4_{0}(u, Ju) > 0$, $u \neq 0$.
 $4_{0}(e_{j}, Te_{j}) = T_{0}(e_{j}(e_{j}, e_{j}) = T_{0}(e_{j}, e_{j}) =$

Suppose	$F = \mathbb{R} \times \mathbb{R}$, then $B \cong \operatorname{End}_F W \cong M_n(F) \cong M_n(\mathbb{R}) \times M_n(\mathbb{R})$. The
muolution is	$(g, h)^* = (h^{t}, g^{t})$ and $(g, h)^* (g, h) = (h^{t}g, g^{t}h)$. In this
case troin ($(g,h)^*(g,h) = \overline{\Sigma}(h^t g)_{ij} + \overline{\Sigma}(g^t h)_{ij} = \overline{\Sigma}(h^t g)_{ij}$ could be negative.

Type (C): (B, *, V, 4) decomposes into a priduct of (Mn(IR), *, Vo, 40) where B = End_R W, * adjoint molution off a positive definite symmetric bilinear form $\phi: W \times W \longrightarrow \mathbb{R}$ and to alternating on Vo, ψ on V = W $\otimes_{\mathbb{R}}$ Vo is given by $\psi = \phi \otimes \psi_{\circ}$. Under suitable basis, ψ_{\circ} is of the form $\psi_{\circ} = \begin{pmatrix} \circ & -1 \\ 1 & \circ \end{pmatrix}$ and define $J = -4_0 \in End_{\mathbb{R}} V_0 \simeq End_{\mathbb{R}} V = C$. Then $J^2 = -[1, 4_0(J_0, J_0)]^2 + 4_0(u, v)$. and tho (n, Ju) > o for n + o. 111

Let (B, *) semismple R-alq. with positive modulism, (V. 4) symplectic module. Let $h: S \rightarrow G_{IR}$ s.t. (V, h) is off type (-1, 0), (0, -1) and $(u, v) \longrightarrow \psi(u, h(i)v)$ is symmetric and positive definite. Let J = h(i). Then = canomical ison. off C-V.S. $(V, J) \simeq V_{c}/F_{h}^{*}V_{c} \simeq V^{\dagger/\circ}$ compartible with the aution of B. For $b \in B$, define $t(b) = tr_{0}(b|(V, J))$.

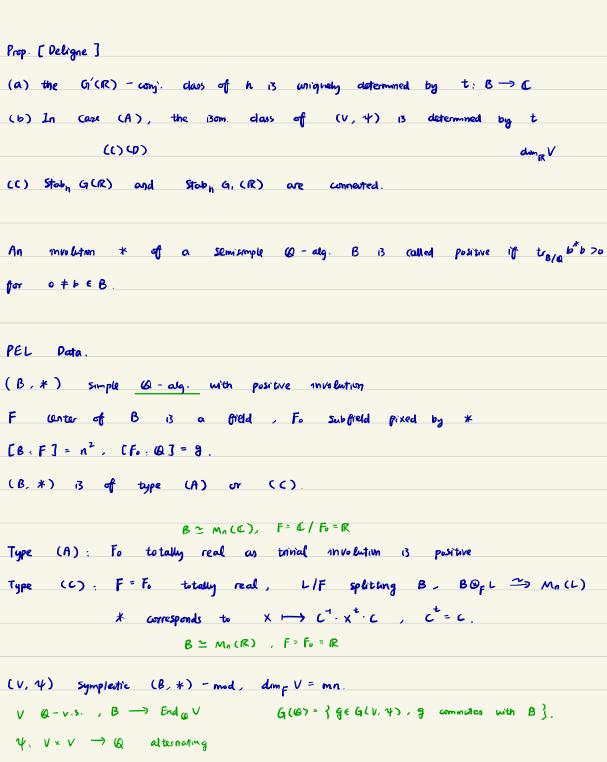
(c)
$$G(R) = GSP_{2n}, G'(R) = SP_{2n}$$

 $B \simeq End_R W, V = W \otimes_R V_0, V_0 \simeq R^{2n}, V_0 = < ie_j, f_j >_R, J | < e_j, f_j >_R = (-1 - 0)$
 $V = W \otimes_R V_0 = \bigoplus_{m} W \otimes_R < e_j, f_j > , Je_j = f_j, Jf_j = -e_j$
 $b \in B, V = w_1 \otimes e_j + w_2 \otimes f_j, b \cdot V = bw_1 \otimes e_j + bw_2 \otimes f_j$
 $(\alpha + \beta J)(w_1 \otimes e_j + w_2 \otimes f_j') = (\alpha w_1 - \beta w_2) \otimes e_j + (\beta w_1 + \alpha w_2) \otimes f_j = b \cdot V =) \alpha = b, \beta = 0$
 $\Rightarrow tr_C (b((U, J)) = m tr_C (b|W)$

(A)

$$g \in G(R), \mu(g) > 0, J' = g_J g^{-1} \rightarrow J'^2 = -1, \psi_0(J'u, J'v) = \psi_0(u, v), \psi_0(u, J'u) > 0 ip u = 0$$

 $e_j' = ge_j = 0, J'e_j' = \begin{cases} ig_j' & j = r \\ -ie_j' & j = r \end{cases}$ $tre(b|(v, J')) = tre(b|(v, J)).$

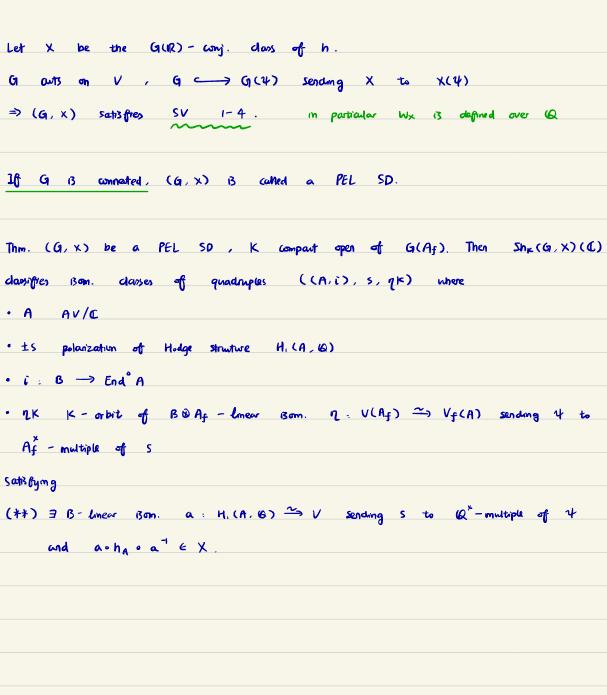


Prop.
$$\exists h: S \rightarrow GR$$
 s.t. (V, h) is of type $(-1, o), (o, -1)$
 $2\pi i \Psi$ is a polarization of (V, h)
Such h is unique up to $G(R)$ - cong.
Pf $VR \cong \bigoplus V \otimes_{\overline{E}, \sigma} R$ or \overline{F}_0 totally real.
 $\sigma \cdot \overline{b} \hookrightarrow R$ $V \otimes_{\overline{E}, \sigma} R$ or \overline{F}_0 totally real.
For each σ , we can find $J: V \otimes_{\sigma} R \rightarrow V \otimes_{\sigma} R$ with $J^2 = -1$, J
commutes with the autim of $B \otimes_{\sigma} R$ and $\Psi_{\sigma} (J u, J u) = \Psi_{\sigma} (u, v)$,
 $\Psi_{\sigma} (u, J u) > o$ for $u \neq 0 = >$ we can find $h\sigma$.
Then $h = \bigoplus h\sigma$.
 $\sigma \cdot \overline{F} \hookrightarrow R$, fix $f_{\sigma}: \overline{F} \hookrightarrow C$ extends σ . $\overline{F} \otimes_{Q} R \cong \prod_{P_{\sigma}} C$
 $T_{\sigma} = \Gamma_{P_{\sigma}} - m \cdot r_{\sigma} = (\overline{F}_{\sigma}, + t(b)) = \sum_{P_{\sigma}} \Gamma_{P} \rho(trb) + (m \cdot r_{\sigma}) \overline{F}_{\sigma}(trb)$
 $Type (A):$ answere for $\sigma \cdot \overline{F} \circ \Longrightarrow R$, Ψ_{σ} , has the form $\begin{pmatrix} \sigma - iI \end{pmatrix}$
then for $b \in B$, $t(b) = \forall_{\sigma} (b) \vee R = \sum_{P \in F \cap \sigma} \Gamma_{P} (trb)$ where trb B the
reduced norm. As h is determined by t , here b_{σ} if $G(R)$.

Type (C) $t(b) = \frac{m}{2} tr_{B/Q} b$

///

$$G \longrightarrow G(\Psi) , X \longrightarrow X(\Psi).$$
Let X be the $G(R)^{-} \operatorname{coy}^{1}$ class of h , then (G, X) subspices
$$SV = 1 + a_{0} = (V, h) + a_{1} = a_{1} + a_{2} + a_$$



For
$$h \in X$$
, the map $b \mapsto tr(b|V^{1,\circ})$, $B \to C$ B independent of h , and
denoted by tr_X .
RM(μ . Consider $(A, i, s, \eta \kappa)$, then $(**)$ implies that
(a) $s(bu, v) = s(u, b^*v)$
(b) $tr(i(u)|T_{u}A) = tr_{X}(b)$, $b \in B \otimes C$
as $T_{u}A \simeq H_{1}(A, C)/F^{o} \xrightarrow{a} V_{c}/V_{h}^{o} B - Booss$.
Prop. For type (A) with reduced drin of V oven , and type (C), $(a)(b)=b(i*)$.
Jemma. Lot $Q/(B$ reductive alg. qp , G^{der} surply connected , $T = Q/Q^{der}$. If
 $H'(u, T) \longrightarrow T H'(ue, T)$ is implayive, then so is $H'(u, Q) \rightarrow T H'(ue, Q)$
 $Pf: G^{der}$ surply connected \Rightarrow $H'(be, G^{der})=o$, $l < \infty$ and $H'(u, Q^{der}) \rightarrow H'(R, Q^{der})$.
Then consider the commutative diagram
 $T(u) \rightarrow H'(u, G^{der}) \rightarrow H'(ue, G) \rightarrow T'H'(ue, T)$
 \downarrow I
 $q(R) \rightarrow T(R) \rightarrow H'(R, Q^{der}) \rightarrow T'H'(ue, G) \rightarrow T'H'(ue, T)$
 $I = Q(R) \rightarrow T(R) = T(u) T(R)^{1}$ and $T(uR)^{+} C$. In $Q(R)$. (11)

Lemma. (G, X) simple PEL SD of type (A) with reduced dow of V even, at
type (C),
$$T = G/G^{dV}$$
. Then $H'(G, T) \rightarrow T$ $H'(Gl, T)$ is myoshive.
Pf.
Type (A) even, $T = ker((Gm)F/G \xrightarrow{Nm} (Gm)F/G) \times Gm$, then $H'(Gl, Gm)=o$
and $F_o^X/NmF^X \rightarrow T$ F_o^X/NmF^X is myoshive as F/F_o cyclic.
 $V \leq \infty$
Type (C), $T = Gm$, $H'(G, T)=o$.
 H/P
Reaf of Pop.
Let $W = H_1(A, Ga)$, q implies $dm_{GL}V = dm_{GL}W$.
Then there exists a $BO_{GL}G = -3m$. $\alpha : V_{GL} \rightarrow W_{GL}$ sending S to
 $G_{X}^X = mHiple of P = 00$ over GL , sympletic $(B, *) = modules off same
dimension are ison.
For $\sigma \in God(G/B)$, $Obsume \sigma = \alpha \circ a_F$ for some $a_{\sigma} \in G(GL)$. Then
 $\sigma \mapsto a_F$ is a 1-cocycle. If it is autivally a 1-cobaindory, i.e.
 $a_F = a^{-1} \cdot \sigma a$, then $\alpha \circ a^{-1}$ is fixed by God(G/B), here defined
over GL . By the Lemmas cobare, suffice to Know it is trivial m
 $H'(Gl, G)$ for each for the cycle of completed by q , and brind m
 $H'(R, G)$, which is mplied by (b) and Rep. [Deligne].
 $a \circ h_B \circ a^{-1} \in X$ B also implied by (b) and Rep. [Deligne].
 $H'/R$$