PEL Shimura varieties.

char $k = 0$.

Algebras with involution.

k-alg : ring containing k in center, finite dimensional as k-v.s.

Simple k-alg : 0, A only two-sided ideals

Wedderburn Thm : all simple k-algs are of the form $M_n(D)$, D division k-alg.

k alg. closed, $D = k$, simple k-algs are $M_n(k)$.

A simple k-alg. B has only one simple module M up to isom. and every B-module is isom. to direct sum of M. D^n is the only simple $M_n(D)$-module.

Semisimple k-alg B : every B-mod is semisimple.

B has only finitely many minimal two-sided ideals $B_1, \ldots, B_r$, each is a simple k-alg. and $B \simeq B_1 \times \cdots \times B_r$.

The only simple B-mods are simple B_i-mods M_i via $B \to B_i$.

The trace map of a B-mod M is $b \longmapsto tr_k(b|_M)$.

Prop. B semisimple k-alg. Two B-mods are isom. iff they have the same tr.

Pf. $B \simeq B_1 \times \cdots \times B_r$, $M \simeq \oplus d_i M_i$.

$tr_k(e_i | \oplus d_i M_i) = d_i \dim_k M_i$. ///

$*: B \xrightarrow{\sim} B^{op}$.

An involution of a k-alg. B is k-linear bijection $B \longrightarrow B$, $b \longmapsto b^*$ s.t. $(ab)^* = b^* a^*$, $(b^*)^* = b$. Then $c^* = c$, $\forall\, c \in k$ and it preserves the center.

If it fixes all elements of the center, it is called to be the first kind.

Otherwise it is called the second kind.

Classification of semisimple alg. with involution over alg. closed fields

Prop. $(B, *)$ semisimple k-alg. with involution. If k is alg. closed, then $(B, *)$ is

Isom. to a product of pairs of the following types

(A) $M_n(k) \times M_n(k)$, $(a, b)^* = (b^t, a^t)$

(C) $M_n(k)$, $b^* = b^t$ orthogonal type

(BD) $M_n(k)$, $b^* = J \cdot b^t \cdot J^{-1}$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ symplectic type

Pf. $B \simeq B_1 \times \cdots \times B_r$, $B^* \simeq B_1^* \times \cdots \times B_r^*$.

(B_i^*) is a permutation of (B_i) $\Rightarrow$ B is product of pairs of either simple k-alg. with involution or product of two simple k-algs interchanged by $*$.

$(B, *)$ simple k-alg.

k alg. closed $\Rightarrow$ $B \simeq M_n(k)$, Skolem & Noether $\Rightarrow$ $b^* = u \cdot b^t \cdot u^{-1}$, $u \in GL_n(k)$.

$(b^*)^* = u \cdot (u \cdot b^t \cdot u^{-1})^t \cdot u^{-1} = u \cdot (u^{-1})^t \cdot b \cdot u^t u^{-1} = b \Rightarrow u^t u^{-1} \in k$.

$u^t = cu \Rightarrow u = (u^t)^t = c^2 u$, $c = \pm 1 \Rightarrow u$ symmetric or skew-symmetric.

By change of base, we may assume $u = 1$ or J.

$B = B_1 \times B_2$, $* : B_1^{op} \simeq B_2$.

K alg. closed $\Rightarrow$ $B \simeq M_n(K) \times M_n(K)^{op}$, $(a, b)^* = (b, a)$.

$M_n(K)^{op} \simeq M_n(K)$, $b \longmapsto b^t$ $\Rightarrow$ $B \simeq M_n(K) \times M_n(K)$, $(a, b)^* = (b^t, a^t)$. ///

W/K f.d. v.s., $\phi : W \times W \longrightarrow K$ nondegenerate bilinear.

$\alpha \in \mathrm{End}_K W$, α^* is defined to be $\phi(\alpha^* v, w) = \phi(v, \alpha w)$.

Then $\alpha \longmapsto \alpha^*$ is an involution of $\mathrm{End}_K W$ iff ϕ is (skew-) symmetric. It is called the adjoint involution of ϕ.

Prop. $(B, *)$ semisimple K-alg. with involution. If K alg. closed, and the only elements of the center of B fixed by $*$ are K, then $(B, *)$ is isom. to one of

(A) $\mathrm{End}_K W \times \mathrm{End}_K W^\vee$. $(a, b)^* = (b^\vee, a^\vee)$.

(C) $\mathrm{End}_K W$, $*$ adjoint involution of a symmetric form on W

(BD) $\mathrm{End}_K W$, $*$ alternating

Symplectic modules and the associated algebraic groups.

$(B, *)$ K-alg. with involution. A symplectic $(B, *)$-module is (V, ψ) where V is B-mod, $\psi : V \times V \longrightarrow K$ skew-symmetric K-bilinear s.t.

$$\psi(b^* u, v) = \psi(u, bv).$$

B semisimple k-alg., center of B is $F = \prod F_i$, F_i field, $B = \prod B_i$ with $B_i = B \otimes_F F_i$ central simple alg. over F_i.

Assume B free F-mod, $[B_i : F_i] = n^2$. For any B-mod V, there is a reduced det map $\det : \mathrm{End}_B(V) \to F$.

If $\exists\, u : M_n(F) \xrightarrow{\sim} B$ then $\det(g) = \det(g|\, u(E_{1,1})V)$.

e.g. $B = F$, $V = F^m$, $\det : \mathrm{End}_F(F^m) \to F$ is the usual det

$B = M_m(F)$, $V = F^m$, $\det : \mathrm{End}_{M_m(F)}(F^m) = F \to F$ is identity

In general, suppose F field, E/F finite Galois s.t. $u : M_n(E) \xrightarrow{\sim} B \otimes_F E$, hence reduced det $\det_E : \mathrm{End}_{B\otimes_F E}(V \otimes_F E) \longrightarrow E$. If $g \in \mathrm{End}_B(V) \subset \mathrm{End}_{B\otimes E}(V \otimes E)$, $\det(g)$ is fixed by $\mathrm{Gal}(E/F)$ so lies in F. Thus get $\det : \mathrm{End}_B V \to F$, it is independent of choice of E/F.

Let $(B, *)$ semisimple k-alg. with involution, F center of B, F_0 subalg. of $*$ invariants in F. We say $(B,*)$ is of type (A), (C), (BD) if $(B \otimes_{F_0, \rho} \bar{k}, *)$ is of that type for all k-homomorphisms $\rho : F_0 \to \bar{k}$.

Assume B free F-mod, then F free of rank 2 over F_0 in case (A) and equals F_0 in case (C), (BD). Let $[B : F] = n^2$ and $[F_0 : k] = g$.

$(A, *)$ simple k-alg., $Z(A) = F$, A-mod V, $\phi : V \times V \to A$ (skew-) Hermitian if $\phi(au, bv) = a\phi(u,v)b^*$, $\phi(v,u) = (-)\phi(u,v)^*$ $\Rightarrow$ $*_\phi$ on $\mathrm{End}_A V$. biadd. nondeg.

- $*$ first kind. $\{\phi \text{ on } V\} \xleftrightarrow{1:1} \{*_\phi \text{ on } B\}/F^\times$
- $*$ second kind. $\{\text{Hermitian } \phi \text{ on } V\} \xleftrightarrow{1:1} \{\text{extensions of } *|_F \text{ to } \mathrm{End}_A V\}/(F^\times)^*$

Let (V, ψ) symplectic $(B, *)$-module, V free over F. The reduced dimension of V is $m = \dfrac{\dim_F V}{[B:F]^{\frac{1}{2}}} = \dfrac{\dim_F V}{n}$.

Let G_1, G be alg. subgrps of $GL_B V$ s.t.

$G_1(k) = \{ g \in GL_B(V) \mid \psi(gx, gy) = \psi(\mu(g)x, y),\ \mu(g) \in F_0^{\times} \}$

$G(k) = \{ g \in GL_B(V) \mid \psi(gx, gy) = \mu(g) \cdot \psi(x, y),\ \mu(g) \in k^{\times} \}$

There is a homomorphism $\mu : G_1 \longrightarrow (\mathbb{G}_m)_{F_0/k}$ and G is the inverse image of $\mathbb{G}_m \subset (\mathbb{G}_m)_{F_0/k}$. Set $G' = \ker \mu \cap \ker \det$, $T_1 = G_1 / G'$, $T = G/G'$. Then

$$\begin{tikzcd}
B \arrow[r, no head] \arrow[d, no head, "n^2"'] & V \arrow[dl, no head, "mn"] \\
F \arrow[d, no head, "1 \text{ or } 2"'] & \\
F_0 \arrow[d, no head, "g"'] & \\
k &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & G' \arrow[d] \arrow[dr, bend left] & \\
Z(G) \arrow[r] \arrow[dr, bend right] & G_1 \arrow[r, "Ad"] \arrow[d, "\nu"] & G_1^{ad} \\
 & T_1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & G' \arrow[d] \arrow[dr] & \\
Z(G) \arrow[r] \arrow[dr, bend right] & G \arrow[r, "Ad"] \arrow[d, "\nu"] & G^{ad} \\
 & T &
\end{tikzcd}$$

Let $*$ be the adjoint involution on $\mathrm{End}_k V$ w.r.t. ψ, then by assumption it is compatible with the involution on B. Let $C = \mathrm{End}_B V$, it is semisimple k-alg. and stable under $*$ in $\mathrm{End}_k V$. Then the groups defined above are

$G_1(k) = \{ x \in C \mid x^* \cdot x \in F_0^{\times} \}$, $G(k) = \{ x \in C \mid x^* \cdot x \in k^{\times} \}$

$G'(k) = \{ x \in C \mid x^* \cdot x = 1,\ \det x = 1 \}$.

Classify symplectic modules $\Rightarrow$ classify these alg. grps

Ex. F/K étale of deg 2 $\Rightarrow$ $F \simeq K \times K$ or F/K deg 2 extension.

There is a unique nontrivial involution $*$ of F fixing K.

Assume $B \simeq M_n(F)$ and $(B, *)$ of type (A), then $B \simeq \mathrm{End}_F W$ for W simple B-mod, $*$ is the adjoint involution of a Hermitian form $\phi: W \times W \to F$.

Let V_0 free F-mod of finite rank, $\psi_0: V_0 \times V_0 \to F$ skew-Hermitian form. Then B acts on $V = W \otimes_F V_0$ and $\psi: V \times V \to K$ defined by

$$\psi(w \otimes v, w' \otimes v') = tr_{F/K}(\phi(w, w') \psi_0(v, v')) \qquad (V, \psi) \longleftrightarrow (V_0, \psi_0)$$

$\Rightarrow (V, \psi)$ is a symplectic $(B, *)$-module.

Conversely let (V, ψ) be a symplectic $(B, *)$-module. As B-mod, V is direct sum of copies of W, so $V \simeq W \otimes_F V_0$ for some V_0 free F-mod of finite rank. Choose $f \in F^\times$ s.t. $f^* = -f$. Then there exists a unique Hermitian form $\Phi: V \times V \to F$ s.t. $\psi(u, v) = tr_{F/K}(f\Phi(u, v))$ and $\Phi(b^* u, v) = \Phi(u, bv)$. The adjoint involution of Φ preserves $\mathrm{End}_B V \simeq \mathrm{End}_F V_0$. Choose skew-Hermitian form $\psi_0: V_0 \times V_0 \to F$ whose adjoint involution is the restriction of $*_\Phi$. The form $f\Phi(-,-)$ is skew-Hermitian and after possibly scaling ψ_0, $f\Phi = \phi \otimes \psi_0$.

Ex. Assume $B \simeq M_n(K)$ and $(B, *)$ is of type (C). Then $B \simeq \mathrm{End}_K W$, W is simple B-mod and $*$ is adjoint involution of a symmetric bilinear form $\phi: W \times W \to K$.

Let V_0 be K-v.s., $\psi_0: V_0 \times V_0 \to K$ alternating. Let $\psi = \phi \otimes \psi_0$ on $V = V_0 \otimes W$.

Then (V, ψ) is a symplectic $(B,*)$-mod and every symplectic mod arises like this.

$C = \mathrm{End}_B V \simeq \mathrm{End}_k V_0$ and $*$ acts on C as the adjoint involution of ψ_0. Hence

$G_1 = G = GSp(V_0, \psi_0)$ and $G' = Sp(V_0, \psi_0)$.

Prop. Let $(B,*)$ semisimple k-alg. with involution, (V,ψ) symplectic $(B,*)$-mod.

Let F be center of B, F_0 subalg. of F of elements fixed by $*$.

Assume V, B free over F, $[B:F] = n^2$, $[F_0 : k] = g$, $\dim_F V = mn$. Then

Case (A): G, G_1 connected, reductive, G' semisimple simply connected

F quadratic extension of F_0

for all $g \in G_1(k)$, $\det(g) \cdot \det(g)^* = \mu(g)^m$

if k alg. closed, $G' \simeq SL_m^g$

if m even, $m = 2\ell$, then $(\det^{-1} \circ \mu^\ell, \mu)$ gives isoms

$$T_1 \simeq \ker\left((\mathbb{G}_m)_{F/k} \xrightarrow{Nm} (\mathbb{G}_m)_{F_0/k} \right) \times (\mathbb{G}_m)_{F_0/k}$$

$$T \simeq \ker\left((\mathbb{G}_m)_{F/k} \xrightarrow{Nm} (\mathbb{G}_m)_{F_0/k} \right) \times \mathbb{G}_m$$

if m odd, $m = 2\ell - 1$, then $\kappa = \det^{-1} \circ \mu^\ell$ gives isom

$$T_1 \simeq (\mathbb{G}_m)_{F/k} \quad \text{and} \quad \mu = \kappa \cdot \kappa^*.$$

Case (C): G, G_1 are connected, reductive, G' semisimple, simply connected

m is even, $m = 2\ell$, $\det(g) = \mu(g)^\ell$, $\forall g \in G_1(k)$.

$T_1 \xrightarrow[\sim]{\mu} (\mathbb{G}_m)_{F/k}$, $T \xrightarrow[\sim]{\mu} \mathbb{G}_m$

if k alg. closed, $G' \simeq Sp_m^{[F:k]}$.

F/k quadratic, $V = F^m$, $\mathrm{End}_F V = M_m(F)$, $a^* = \bar{a}$, $\psi_0 : V \times V \to F$ skew-herm.

$U_{r,s} = \begin{pmatrix} iI_r & \\ & -iI_s \end{pmatrix}$, $\langle \alpha, \beta \rangle = \alpha^t U_{r,s} \bar{\beta}$, $g \in M_m(F)$, $\langle g\alpha, g\beta \rangle = \alpha^t g^t U_{r,s} \bar{g} \bar{\beta} \Rightarrow g^t U_{r,s} \bar{g} = a U_{r,s}$, $a \in k$

$\mu(g) = a$

$F = k \times k$, $V = F^m = k^m \times k^m$, $\mathrm{End}_F V = M_m(F) = M_m(k) \times M_m(k)$.

$\det(g)^* = \overline{\det g}$

$(a, b)^* = (b, a)$, $\psi_0 : V \times V \to F$ skew-herm. $\Rightarrow I_{r,s} = \begin{pmatrix} I_r & \\ & -I_s \end{pmatrix} \in M_m(k)$

$\langle (\alpha, \beta), (\alpha', \beta') \rangle = (\alpha^t I_{r,s} \beta', -\beta^t I_{r,s} \alpha')$. $\det g \cdot \det g^* = |\det g|^2 = a^m$

(g, h), $\langle (g,h)(\alpha,\beta), (g,h)(\alpha', \beta') \rangle = (\alpha^t (g^t I_{r,s} h) \beta', -\beta^t (h^t I_{r,s} g) \alpha')$

$\det(g, h) = (\det g, \det h)$, $g^t I_{r,s} h = a I_{r,s}$, $h^t I_{r,s} g = a I_{r,s}$, $\mu(g,h) = (a, a)$

$\det(g,h)^* = (\det h, \det g)$, $\det(g,h) \cdot \det(g,h)^* = (\det g \det h, \det g \det h) = (a^m, a^m)$

$g^t I_{r,s} h = I_{r,s}$, $h = I_{r,s} (g^t)^{-1} I_{r,s}$, $G' = SL_m$

$g^t U_{r,s} \bar{g} = U_{r,s}$, $g \in U(r,s)$

$\mu^\ell (g,h) = (a^\ell, a^\ell)$ $G' = SU(r,s)$

$\det(g,h) = (\det g, \det h)$ $G'_{\mathbb{C}} = SL_m$

$\mathbb{G}_{m, F/k}(k) = k^\times \times k^\times \longrightarrow \mathbb{G}_m(k) = k^\times$

$T(k) \simeq (a, a^{-1}, b)$, $a, b \in k^\times$

$a^\ell / \det g$, $a^\ell / \det h$

$T \simeq G / G'$

$\det g \cdot \det h = a^m = a^{2\ell}$.

$\kappa = \det^{-1} \cdot \mu^\ell$

$\kappa \cdot \kappa^* (g,h) = \left(\frac{a^\ell}{\det g}, \frac{a^\ell}{\det h} \right) \cdot \left(\frac{a^\ell}{\det h}, \frac{a^\ell}{\det g} \right) = \left(\frac{a^{m+1}}{\det g \det h}, \frac{a^{m+1}}{\det g \det h} \right) = (a, a)$

$= \mu(g, h)$.

$\kappa(g,h) = 1 \Rightarrow \mu(g,h) = 1 \Rightarrow \det(g,h) = 1$.

inj. + surj.

RMK. The case (BD) splits into (B), m odd and case D, m even. In case (B) the alg. grps are not part of a SD. In case (D), G and G_1 have $2^{[F:k]}$ conn. components and identity comp. is reductive. G' semisimple, not simply conn. k alg. closed, $G' = (SO_m)^{[F:k]}$.

Algebras with positive involution

$(C, *)$ semisimple $\mathbb{R}$-alg. with involution, V C-mod.

A Hermitian form on V is a symmetric $\mathbb{R}$-bilinear form $\psi : V \times V \to \mathbb{R}$ s.t. $\psi(c^* u, v) = \psi(u, cv)$, $\forall c \in C$, $u, v \in V$. Such a form ψ is called positive definite if $\psi(v, v) > 0$, $\forall v \neq 0$.

Prop. TFAE.

(a) some faithful C-mod admits a positive definite Hermitian form

(b) every C-mod admits a positive definite Hermitian form

(c) $\mathrm{tr}_{C/\mathbb{R}}(c^* c) > 0$, $\forall c \neq 0$.

Pf.

(a) $\Rightarrow$ (b). Let V be faithful C-mod. Every simple C-mod is a direct summand of V hence every C-mod is a direct summand of direct sum of V.

(b) $\Rightarrow$ (c). Let V, $(\, , \,)$. Choose orthogonal $\mathbb{R}$-basis $e_1, \dots, e_n$ for V, then $\mathrm{tr}_{\mathbb{R}}(c^* c|_V) = \sum (e_i, c^* c e_i) = \sum (ce_i, ce_i) > 0$ unless c acts as 0. Take $V = C$.

(c) $\Rightarrow$ (a). Let $(c, c') = \mathrm{tr}_{C/\mathbb{R}}(c^* c')$, then it is positive definite on the faithful C-mod C. ///

Def. An involution satisfying the above equivalent conditions is called positive.

Prop. Let $(B, *)$ semisimple $\mathbb{R}$-alg. with positive involution, (V, ψ) symplectic mod. Assume $(B, *)$ is of type (A) or (C), let $C = \mathrm{End}_B(V) \subset \mathrm{End}_{\mathbb{R}}(V)$. Then there exists a homomorphism of $\mathbb{R}$-algs. $h : \mathbb{C} \to C$ s.t.

- $h(\bar{z}) = h(z)^*$
- $(u, v) = \psi(u, h(i)v)$ is symmetric, positive definite.

Pf. To give such an h suffices to specify $J \in C$ s.t.

$J^2 = -1$, $\psi(Ju, Jv) = \psi(u, v)$, $\psi(u, Ju) > 0$ if $u \neq 0$.

Type (A): $(B, *, V, \psi)$ decomposes into a product of $(M_n(F), *, V_0, \psi_0)$ where $F = \mathbb{R} \times \mathbb{R}$ or $\mathbb{C}$.

Suppose $F = \mathbb{C}$, then

- $B \simeq \mathrm{End}_{\mathbb{C}} W$
- $*$ adjoint involution of a positive definite Hermitian form $\phi : W \times W \to \mathbb{C}$
- $V \simeq W \otimes_{\mathbb{C}} V_0$, V_0 $\mathbb{C}$-v.s.
- $\psi = \mathrm{tr}_{\mathbb{C}/\mathbb{R}}(\phi \otimes \psi_0)$ with ψ_0 skew-Herm. form on V_0

$C \simeq \mathrm{End}_{\mathbb{C}} V_0$. We may find a basis (e_j) for V_0 s.t.

$(\psi_0(e_j, e_k)) = \mathrm{diag}(\underbrace{i, \ldots, i}_{r}, -i, \ldots, -i)$. Then define $J e_j = \begin{cases} i e_j & j \leq r \\ -i e_j & j > r \end{cases}$

$\Rightarrow J^2 = -1$, $\psi_0(Ju, Jv) = \psi_0(u, v)$, $\psi_0(u, Ju) > 0$, $u \neq 0$.

$$\psi_0(e_j, Je_j) = \begin{cases} \psi_0(e_j, ie_j) = \bar{i}\, \psi_0(e_j, e_j) = \bar{i} \cdot i = 1 \\ \psi_0(e_j, -ie_j) = \overline{-i}\, \psi_0(e_j, e_j) = \bar{i} \cdot i = 1 \end{cases}$$

Suppose $F = \mathbb{R} \times \mathbb{R}$, then $B \simeq \mathrm{End}_F W \simeq M_n(F) \simeq M_n(\mathbb{R}) \times M_n(\mathbb{R})$. The involution is $(g, h)^* = (h^t, g^t)$ and $(g,h)^*(g,h) = (h^t g, g^t h)$. In this case $tr_{B/\mathbb{R}}((g,h)^*(g,h)) = \Sigma(h^t g)_{ij} + \Sigma(g^t h)_{ij} = 2\Sigma(h^t g)_{ij}$ could be negative.

Type (C): $(B, *, V, \psi)$ decomposes into a product of $(M_n(\mathbb{R}), *, V_0, \psi_0)$ where $B \simeq \mathrm{End}_{\mathbb{R}} W$, $*$ adjoint involution of a positive definite symmetric bilinear form $\phi: W \times W \longrightarrow \mathbb{R}$ and ψ_0 alternating on V_0. ψ on $V = W \otimes_{\mathbb{R}} V_0$ is given by $\psi = \phi \otimes \psi_0$. Under suitable basis, ψ_0 is of the form $\psi_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and define $J = -\psi_0 \in \mathrm{End}_{\mathbb{R}} V_0 \simeq \mathrm{End}_B V = C$. Then $J^2 = -1$, $\psi_0(Ju, Jv) = \psi_0(u, v)$ and $\psi_0(u, Ju) > 0$ for $u \neq 0$. ///

Let $(B, *)$ semisimple $\mathbb{R}$-alg. with positive involution, (V, ψ) symplectic module. Let $h: \mathbb{S} \longrightarrow G_{\mathbb{R}}$ s.t. (V, h) is of type $(-1, 0), (0, -1)$ and $(u, v) \longmapsto \psi(u, h(i)v)$ is symmetric and positive definite. Let $J = h(i)$. Then $\exists$ canonical isom. of $\mathbb{C}$-v.s.

$$(V, J) \simeq V_{\mathbb{C}} / F_h^0 V_{\mathbb{C}} \simeq V^{-1,0}$$

compatible with the action of B. For $b \in B$, define $t(b) = tr_{\mathbb{C}}(b \mid (V, J))$.

$G'(\mathbb{R}) = \{ g \in GL_{\mathbb{C}} V_0 \mid \psi_0(gu, gv) = \psi_0(u,v),\ \det g = 1 \}$

$G(\mathbb{R}) = \{ g \in GL_{\mathbb{C}} V_0 \mid \psi_0(gu, gv) = \mu(g)\psi_0(u,v),\ \mu(g) \in \mathbb{R}^\times \}$

(A)

$B \simeq \mathrm{End}_{\mathbb{C}} W$, $V = W \otimes_{\mathbb{C}} V_0$, $\mathrm{End}_B V = \mathrm{End}_{\mathbb{C}} V_0$, $V_0 = \langle e_j \rangle_{\mathbb{C}}$, $\psi_0 = \begin{pmatrix} iI_r & \\ & -iI_s \end{pmatrix}$

Want: $J \in \mathrm{End}_{\mathbb{C}} V_0$, $J^2 = -1$, $\psi_0(Ju, Jv) = \psi_0(u,v)$, $\psi_0(u, Ju) > 0$ if $u \neq 0$

Define $Je_j = \psi_0(e_j, e_j) e_j$, $V = W \otimes_{\mathbb{C}} V_0 = \bigoplus W \otimes e_j$, $J(w \otimes e_j) = \begin{cases} i\, w \otimes e_j & j \le r \\ -i\, w \otimes e_j & j > r \end{cases}$

$b \in B$, $b \cdot (w \otimes e_j) = bw \otimes e_j = b'(w \otimes e_j) = \begin{cases} b' w \otimes e_j & j \le r \\ \overline{b'} w \otimes e_j & j > r \end{cases} \Rightarrow b' = \begin{cases} b & j \le r \\ \overline{b} & j > r \end{cases}$

$\Rightarrow \mathrm{tr}_{\mathbb{C}}(b \mid (V, J)) = r \cdot \mathrm{tr}_{\mathbb{C}}(b \mid W) + (n-r) \cdot \overline{\mathrm{tr}_{\mathbb{C}}(b \mid W)}$

(C) $G(\mathbb{R}) = GSp_{2m}$, $G'(\mathbb{R}) = Sp_{2m}$

$B \simeq \mathrm{End}_{\mathbb{R}} W$, $V = W \otimes_{\mathbb{R}} V_0$, $V_0 \simeq \mathbb{R}^{2m}$, $V_0 = \langle \{e_j, f_j\} \rangle_{\mathbb{R}}$, $J \mid \langle e_j, f_j \rangle_{\mathbb{R}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$V = W \otimes_{\mathbb{R}} V_0 = \bigoplus_m W \otimes_{\mathbb{R}} \langle e_j, f_j \rangle$, $Je_j = f_j$, $Jf_j = -e_j$

$b \in B$, $v = w_1 \otimes e_j + w_2 \otimes f_j$, $b \cdot v = bw_1 \otimes e_j + bw_2 \otimes f_j$

$(\alpha + \beta J)(w_1 \otimes e_j + w_2 \otimes f_j) = (\alpha w_1 - \beta w_2) \otimes e_j + (\beta w_1 + \alpha w_2) \otimes f_j = b \cdot v \Rightarrow \alpha = b,\ \beta = 0$

$\Rightarrow \mathrm{tr}_{\mathbb{C}}(b \mid (V, J)) = m\, \mathrm{tr}_{\mathbb{C}}(b \mid W)$

(A)

$g \in G(\mathbb{R})$, $\mu(g) > 0$, $J' = gJg^{-1} \Rightarrow J'^2 = -1$, $\psi_0(J'u, J'v) = \psi_0(u,v)$, $\psi_0(u, J'u) > 0$ if $u \neq 0$

$e_j' = ge_j \Rightarrow J' e_j' = \begin{cases} i e_j' & j \le r \\ -i e_j' & j > r \end{cases} \Rightarrow \mathrm{tr}_{\mathbb{C}}(b \mid (V, J')) = \mathrm{tr}_{\mathbb{C}}(b \mid (V, J))$.

if J' is of different signature, $\psi_0(u, Ju) > 0$ will fail $\Rightarrow \exists g$, $J' = gJg^{-1}$ and $g \in G(\mathbb{R})$, $\mu(g) > 0$.

Prop. [Deligne]

(a) the $G'(\mathbb{R})$-conj. class of h is uniquely determined by $t: B \to \mathbb{C}$

(b) In case (A), the isom. class of (V, ψ) is determined by t

(C) (D) $\dim_{\mathbb{R}} V$

(c) $\mathrm{Stab}_h\, G(\mathbb{R})$ and $\mathrm{Stab}_h\, G_1(\mathbb{R})$ are connected.

An involution $*$ of a semisimple $\mathbb{Q}$-alg. B is called positive if $\mathrm{tr}_{B/\mathbb{Q}}\, b^* b > 0$ for $0 \neq b \in B$.

PEL Data.

$(B, *)$ simple $\mathbb{Q}$-alg. with positive involution

F center of B is a field, F_0 subfield fixed by $*$

$[B : F] = n^2$, $[F_0 : \mathbb{Q}] = g$.

$(B, *)$ is of type (A) or (C).

$B \simeq M_n(\mathbb{C})$, $F = \mathbb{C} / F_0 = \mathbb{R}$

Type (A): F_0 totally real as trivial involution is positive

Type (C): $F = F_0$ totally real, L/F splitting B, $B \otimes_F L \xrightarrow{\sim} M_n(L)$

$*$ corresponds to $x \longmapsto c^{-1} \cdot x^t \cdot c$, $c^t = c$.

$B \simeq M_n(\mathbb{R})$, $F = F_0 = \mathbb{R}$

(V, ψ) symplectic $(B, *)$-mod, $\dim_F V = mn$.

V $\mathbb{Q}$-v.s., $B \longrightarrow \mathrm{End}_{\mathbb{Q}} V$ $\qquad G(\mathbb{Q}) = \{ g \in G(V, \psi),\ g \text{ commutes with } B \}$.

$\psi: V \times V \longrightarrow \mathbb{Q}$ alternating

Prop. $\exists\, h : \mathbb{S} \to G_{\mathbb{R}}$ s.t. (V, h) is of type $(-1,0), (0,-1)$

$2\pi i \psi$ is a polarization of (V, h)

Such h is unique up to $G(\mathbb{R})$-conj.

Pf. $V_{\mathbb{R}} \simeq \bigoplus_{\sigma: F_0 \hookrightarrow \mathbb{R}} V \otimes_{F_0, \sigma} \mathbb{R}$ as F_0 totally real.

For each σ, we can find $J: V \otimes_\sigma \mathbb{R} \to V \otimes_\sigma \mathbb{R}$ with $J^2 = -1$, J commutes with the action of $B \otimes_\sigma \mathbb{R}$ and $\psi_\sigma(Ju, Jv) = \psi_\sigma(u, v)$, $\psi_\sigma(u, Ju) > 0$ for $u \neq 0$ $\Rightarrow$ we can find h_σ.

Then $h = \oplus h_\sigma$.

$\sigma: F_0 \hookrightarrow \mathbb{R}$, fix $\rho_\sigma : F \hookrightarrow \mathbb{C}$ extends σ. $F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{\rho_\sigma} \mathbb{C}$

$r_\sigma = r_{\rho_\sigma}$, $m - r_\sigma = r_{\bar{\rho}_\sigma}$, $t(b) = \sum_{\rho_\sigma} r_{\rho_\sigma} \rho_\sigma(\mathrm{tr}\, b) + (m - r_\sigma) \bar{\rho}_\sigma(\mathrm{tr}\, b)$

$= \sum_{\rho} r_\rho\, \rho(\mathrm{tr}\, b)$

Type (A): assume for $\sigma: F_0 \hookrightarrow \mathbb{R}$, $\psi_{\sigma,0}$ has the form $\begin{pmatrix} iI_{r_\sigma} & 0 \\ 0 & -iI \end{pmatrix}$

then for $b \in B$, $\underline{t(b) = \mathrm{tr}_{\mathbb{C}}(b \mid V_{\mathbb{R}}) = \sum_{\rho: F \hookrightarrow \mathbb{C}} r_\rho \cdot \rho(\mathrm{tr}\, b)}$ where $\mathrm{tr}\, b$ is the reduced norm. As h is determined by t, hence by $\{r_\sigma\}$, and $\{r_\sigma\}$ is determined by $(V, \psi) \otimes \mathbb{R}$ up to Isom., which is conjugation from $G(\mathbb{R})$.

Type (C): $\underline{t(b) = \frac{m}{2} \mathrm{tr}^{\circ}_{B/\mathbb{Q}}\, b}$ ///

$G \hookrightarrow G(\psi)$, $X \longrightarrow X(\psi)$.

Let X be the $G(\mathbb{R})$-conj. class of h, then (G, X) satisfies

- SV 1 : as (V, h) is of type $(-1,0)$, $(0,-1)$
- SV 2 : as $h(i) = J$, $J^2 = -1$ lies in center and ψ is J-polarization
- SV 3 : as B simple $\Rightarrow$ G^{ad} simple, and $h(i) \neq 1$.
- SV 4 : as G defined over $\mathbb{Q}$, so is the weight homomorphism.

Def. Such SD (G, X) is called simple PEL data of type (A) or (C).

RMK. For $b \in B$, let t_b be the tensor $(x, y) \longmapsto \psi(x, by)$ of V. $g \in G(\psi)$ fixes t_b iff it commutes with b. Let $\{b_1, \dots, b_s\}$ be a set of generators for B as $\mathbb{Q}$-alg. Then (G, X) is SD of Hodge type attached to $(V, \psi, \{t_{b_i}\})$.

PEL SV.

$(B, *)$ semisimple $\mathbb{Q}$-alg. with involution, (V, ψ) faithful $(B, *)$ symplectic module.

G alg. subgrp of $GL_B V$ s.t.

$$G(\mathbb{Q}) = \{ g \in GL_B V \mid \psi(gx, gy) = \mu(g) \cdot \psi(x, y), \ \mu(g) \in \mathbb{Q}^\times \}$$

Then identity component of G is reductive.

Assume $\exists$ $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ s.t.

- (V, h) has type $(-1,0)$, $(0,-1)$
- $\psi(u, h(i)v)$ is symmetric and positive definite

Let X be the $G(\mathbb{R})$ - conj. class of h.

G acts on V, $G \hookrightarrow G(\psi)$ sending X to $X(\psi)$

$\Rightarrow$ (G, X) satisfies SV 1-4. in particular w_X is defined over $\mathbb{Q}$

If G is connected, (G, X) is called a PEL SD.

Thm. (G, X) be a PEL SD, K compact open of $G(\mathbb{A}_f)$. Then $Sh_K(G, X)(\mathbb{C})$ classifies isom. classes of quadruples $((A, i), s, \eta K)$ where

- A AV$/\mathbb{C}$
- $\pm s$ polarization of Hodge structure $H_1(A, \mathbb{Q})$
- $i : B \longrightarrow \mathrm{End}^0 A$
- ηK K-orbit of $B \otimes \mathbb{A}_f$-linear isom. $\eta : V(\mathbb{A}_f) \xrightarrow{\sim} V_f(A)$ sending ψ to $\mathbb{A}_f^{\times}$-multiple of s

satisfying

$(**)$ $\exists$ B-linear isom. $a : H_1(A, \mathbb{Q}) \xrightarrow{\sim} V$ sending s to $\mathbb{Q}^{\times}$-multiple of ψ

and $a \circ h_A \circ a^{-1} \in X$.

For $h \in X$, the map $b \longmapsto tr(b \mid V^{-1,0})$, $B \to \mathbb{C}$ is independent of h, and denoted by tr_X.

RMK. Consider $(A, i, s, \eta K)$, then $(**)$ implies that

(a) $s(bu, v) = s(u, b^* v)$

(b) $tr(i(b) \mid T_0 A) = tr_X(b)$, $b \in B \otimes \mathbb{C}$

as $T_0 A \simeq H_1(A, \mathbb{C}) / F^0 \overset{a}{\xrightarrow{\sim}} V_{\mathbb{C}} / V_h^0$ B-Homs.

Prop. For type (A) with reduced dim of V even, and type (C), (a)(b) $\Rightarrow$ $(**)$.

Lemma. Let $G/\mathbb{Q}$ reductive alg. grp, G^{der} simply connected, $T = G/G^{der}$. If $H^1(\mathbb{Q}, T) \longrightarrow \prod_{l \le \infty} H^1(\mathbb{Q}_l, T)$ is injective, then so is $H^1(\mathbb{Q}, G) \longrightarrow \prod_{l \le \infty} H^1(\mathbb{Q}_l, G)$.

Pf. G^{der} simply connected $\Rightarrow$ $H^1(\mathbb{Q}_l, G^{der}) = 0$, $l < \infty$ and $H^1(\mathbb{Q}, G^{der}) \hookrightarrow H^1(\mathbb{R}, G^{der})$.

Then consider the commutative diagram

$$\begin{tikzcd}
 & T(\mathbb{Q}) \arrow[r] \arrow[d] & H^1(\mathbb{Q}, G^{der}) \arrow[r] \arrow[d] & H^1(\mathbb{Q}, G) \arrow[r] \arrow[d] & H^1(\mathbb{Q}, T) \arrow[d] \\
G(\mathbb{R}) \arrow[r] & T(\mathbb{R}) \arrow[r] & H^1(\mathbb{R}, G^{der}) \arrow[r] & \prod_{l \le \infty} H^1(\mathbb{Q}_l, G) \arrow[r] & \prod_{l \le \infty} H^1(\mathbb{Q}_l, T)
\end{tikzcd}$$

real approximation $\Rightarrow$ $T(\mathbb{R}) = T(\mathbb{Q}) \cdot T(\mathbb{R})^+$ and $T(\mathbb{R})^+ \subset \operatorname{Im} G(\mathbb{R})$. ///

Lemma. (G, X) simple PEL SD of type (A) with reduced dim of V even, or type (C). $T = G/G^{der}$. Then $H^1(\mathbb{Q}, T) \longrightarrow \prod_{\ell \le \infty} H^1(\mathbb{Q}_\ell, T)$ is injective.

Pf.

Type (A) even, $T = \ker((\mathbb{G}_m)_{F/\mathbb{Q}} \xrightarrow{Nm} (\mathbb{G}_m)_{F_0/\mathbb{Q}}) \times \mathbb{G}_m$, then $H^1(\mathbb{Q}, \mathbb{G}_m) = 0$ and $F_0^\times / Nm F^\times \longrightarrow \prod_{v \le \infty} F_{0v}^\times / Nm F_v^\times$ is injective as F/F_0 cyclic.

Type (C), $T = \mathbb{G}_m$, $H^1(\mathbb{Q}, T) = 0$. ///

Proof of Prop.

Let $W = H_1(A, \mathbb{Q})$. η implies $\dim_{\mathbb{Q}} V = \dim_{\mathbb{Q}} W$.

Then there exists a $B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$-isom. $\alpha : V_{\overline{\mathbb{Q}}} \longrightarrow W_{\overline{\mathbb{Q}}}$ sending s to $\overline{\mathbb{Q}}^\times$-multiple of ψ as over $\overline{\mathbb{Q}}$, symplectic $(B, *)$-modules of same dimension are isom.

For $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, assume $\sigma\alpha = \alpha \circ a_\sigma$ for some $a_\sigma \in G(\overline{\mathbb{Q}})$. Then $\sigma \longmapsto a_\sigma$ is a 1-cocycle. If it is actually a 1-coboundary, i.e. $a_\sigma = a^{-1} \cdot \sigma a$, then $\alpha \circ a^{-1}$ is fixed by $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, hence defined over $\mathbb{Q}$. By the Lemmas above, suffice to know it is trivial in $H^1(\mathbb{Q}_\ell, G)$ for all finite ℓ, which is implied by η, and trivial in $H^1(\mathbb{R}, G)$, which is implied by condition (b) and Prop. [Deligne].

$a \circ h_A \circ a^{-1} \in X$ is also implied by (b) and Prop. [Deligne]. ///