General Shimura varieties.

generalize : abelian varieties $\rightsquigarrow$ abelian motives

Abelian motives.

$Hod(\mathbb{Q})$ category of polarizable rational Hodge structures.

It is abelian subcat. of the cat. of all rational Hodge structures closed under tensor products and duals.

It is semisimple as the polarization allows to define complement to sub HS.

$V = \coprod V_i$, V_i AV $/\mathbb{C}$, $H^*(V^i, \mathbb{Q}) \simeq \mathrm{Hom}_{\mathbb{Q}}(\wedge H_1(V^i, \mathbb{Q}), \mathbb{Q})$ implies $H^*(V, \mathbb{Q})$ has polarizable HS.

(W, h) rational HS, endomorphism e is called idempotent if $e^2 = e$. Then $(W, h) = \mathrm{Im}\, e \oplus \mathrm{Im}(1-e)$.

An abelian motive over $\mathbb{C}$ is a triple (V, e, m), e idempotent of the rational HS $H^*(V, \mathbb{Q})$, $m \in \mathbb{Z}$.

A AV $/\mathbb{C}$, $H^*(A, \mathbb{Q}) \twoheadrightarrow H^i(A, \mathbb{Q})$ is idempotent e^i, write $h^i(A)$ for $(A, e^i, 0)$.

The morphisms from (V, e, m) to (V', e', m') are maps $H^*(V,\mathbb{Q}) \to H^*(V',\mathbb{Q})$ of the form $e' \circ f \circ e$ where $f : H^*(V,\mathbb{Q}) \to H^*(V',\mathbb{Q})$ of degree $d = m' - m$.

$(V, e, m) \oplus (V', e', m) = (V \amalg V', e \oplus e', m)$

$(V, e, m) \otimes (V', e', m') = (V \times V', e \otimes e', m + m')$

$(V, e, m)^\vee = (V, e^t, d - m)$ if V is purely of dim d

and e^t the transpose of e as correspondence coming from $X \times X \xrightarrow{(p_2, p_1)} X \times X$.

(V, e, m) abelian motive $/\mathbb{C}$, let $H(V, e, m) = e H^*(V, \mathbb{Q})(m)$.

The functor $(V, e, m) \longmapsto H(V, e, m)$ is a functor from the category of abelian motives AM to $\mathrm{Hod}(\mathbb{Q})$ commuting with $\oplus$, $\otimes$, dual.

A rational HS is called abelian if it lies in the essential image of this functor, in particular it is polarizable.

$\mathbb{Q}(1) \simeq \Lambda^2 H_1(A, \mathbb{Q})$, A elliptic curve $\Rightarrow$ $\mathbb{Q}(1)$ abelian.

$\mathrm{Hod}^{ab}(\mathbb{Q})$ full subcat. of abelian HS in $\mathrm{Hod}(\mathbb{Q})$.

Prop. $\mathrm{Hod}^{ab}(\mathbb{Q})$ is the smallest strictly full subcat. of $\mathrm{Hod}(\mathbb{Q})$ containing $H_1(A, \mathbb{Q})$ for each A AV$/\mathbb{C}$ and closed under direct sum, subquotients, duals and tensor products. $H : \mathrm{AM} \longrightarrow \mathrm{Hod}^{ab}(\mathbb{Q})$ is an equivalence of cat.

SV of abelian type.

(V,ψ) symplectic space over $\mathbb{Q}$ $\rightsquigarrow$ conn. SD $(S(\psi), X(\psi)^+)$

Def.

- Conn. SD (H, X^+) is of primitive abelian type if H simple, $\exists (V,\psi)$ symplectic space over $\mathbb{Q}$ s.t. $H \hookrightarrow S(\psi)$ sending X^+ into $X(\psi)$.
- Conn. SD (H, X^+) is of abelian type if $\exists$ conn. SD (H_i, X_i^+) of primitive abelian type and isogeny $\prod H_i \to H$ sending $\prod X_i^+$ into X^+.
- SD (G, X) is of abelian type if (G^{der}, X^+) is.
- $Sh(G,X)$ is of abelian type if (G,X) is.

Prop. (G,X) SD, assume

- SV 4 and SV 6 (w_X defined over $\mathbb{Q}$, Z° splits over a CM field)
- $\exists\ \nu : G \to \mathbb{G}_m$ s.t. $\nu \circ w_X = -2$

If (G,X) is of abelian type, then $(V, \rho\circ h)$ is abelian HS for all representation (V,ρ) of G and $h \in X$. Conversely if there exists a faithful rep. (V,ρ) of G s.t. $(V, \rho\circ h)$ is abelian HS for all $h \in X$ then (G,X) is of abelian type.

For such (G,X) of abelian type, let $\rho : G \to GL_V$ be a faithful rep. of G. Assume $\exists\ \psi : V \times V \to \mathbb{Q}$ s.t.

- $g\psi = \nu(g)^m \psi$ for some fixed m
- ψ is a polarization of $(V, \rho\circ h)$, $\forall\, h \in X$

Then there exists $t_i : V \times \cdots \times V \to \mathbb{Q}(r_i)$ s.t. G is the subgrp of GL_V whose elements satisfy $g\psi = \nu(g)^m \psi$ and fix t_i.

Thm. With the above notation, $Sh_K(G, X)(\mathbb{C})$ classifies the isom. class of triples $(A, (s_i), \eta K)$ where

- A abelian motive over $\mathbb{C}$
- $\pm s_0$ is polarization for the rational HS $H(A)$
- $s_1, \ldots, s_n$ tensors for A
- ηK K-orbit of $\mathbb{A}_f$-linear isom. $V(\mathbb{A}_f) \xrightarrow{\sim} V_f A$ sending ψ to some $\mathbb{A}_f^\times$-multiple of s_0 and t_i to s_i

Satisfying

(**) $\exists$ isom. $a : H(A) \to V$ sending s_0 to $\mathbb{Q}^\times$-multiple of ψ and s_i to t_i, and h to an element in X.

Classification of SV of abelian type.

Deligne : (G, X^+) conn. SD, G simple

G^{ad} is of type A, B, C $\Rightarrow$ (G, X^+) abelian type

G^{ad} is of type E_6, E_7 $\Rightarrow$ (G, X^+) not abelian type as there are no symplectic embeddings

G^{ad} is of type D, (G, X^+) may or may not be.

It is hoped that all Shimura varieties with rational weight classify Isom. classes of motives with additional structure. For a rational rep. $\rho: G \to GL_V$, we have a family of HS $\{\rho_{\mathbb{R}} \circ h\}_{h \in X}$ on V. When the weight of (G, X) is defined over $\mathbb{Q}$, it is hoped that these Hodge structures always occur in the cohomology of algebraic varieties.

Ex. simple SV of type A_1.

(G, X) SD attached to B, quaternion algebra over F totally real.

$G(\mathbb{R}) \simeq \prod_{I_c} \mathbb{H}^\times \times \prod_{I_{nc}} GL_2(\mathbb{R})$

(a) $B \simeq M_2(F)$, (G, X) is of PEL type, $Sh_K(G, X)(\mathbb{C})$ classifies Isom. classes of quadruples $(A, i, t, \eta K)$ where A AV$/\mathbb{C}$ of dim $d = [F : \mathbb{Q}]$ and $i: F \to \mathrm{End}(A) \otimes \mathbb{Q}$ and more.

The boundary of Baily-Borel compactification are indexed by proper parabolic subgroups of G and Sh_K projective / compact $\iff$ no boundary $\iff$ G has no proper parabolic subgroups $\iff$ no embedding $\mathbb{G}_{m,\mathbb{Q}} \to G^{der}$

These SV are called Hilbert varieties and generalize the elliptic modular curves.

(b) B division algebra, $I_c = \phi$, (G, X) is of PEL type, $Sh_K(G, X)(\mathbb{C})$ classifies isom. classes of quadruples $(A, i, t, \eta K)$ where A is $AV/\mathbb{C}$ of dim $2[F:\mathbb{Q}]$ and $i: B \to \operatorname{End} A \otimes \mathbb{Q}$ and more. In this case, the varieties are projective.

(c) B division algebra, $I_c \neq \phi$, (G, X) is of abelian type but the weight is not defined over $\mathbb{Q}$. $Sh_K(G, X)$ classifies certain isom. classes of HS with additional structures, but they are neither rational nor motivic.

(d) $|I_{nc}| = 1$, Shimura curves.

Shimura stack, moduli variety.

$$\begin{tikzcd}[column sep=small]
 & \text{SV} \arrow[dl, dash] \arrow[dr, dash] & \\
\text{SV abelian type} \arrow[dr, dash] & & \text{SV rational weight} \arrow[dl, dash] \\
 & \text{SV abelian type rational weight} & \text{(moduli variety)} \\
\text{SV PEL type} \arrow[ur, dash] \arrow[dr, dash] & & \text{moduli varieties of PEL type} \arrow[dl, dash] \\
 & \text{SV simple PEL type A, C} & \text{(moduli variety)}
\end{tikzcd}$$