Yihang Zhu: Counting points on SV, I.

§. The notion of SV

Modular curves

$\mathcal{H} = \{ x+iy \in \mathbb{C},\ y>0 \}$

$\circlearrowleft$

$SL_2(\mathbb{R}) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \dfrac{az+b}{cz+d}$

$\cup$

$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$

$\mathcal{H}/\Gamma$ is a Riemann surface

Adelic language

$SL_2 \rightsquigarrow GL_2$

$\mathcal{H} \rightsquigarrow X = \mathbb{C} - \mathbb{R}$

$h_0 : \mathbb{C}^\times \longrightarrow GL_2(\mathbb{R})$

$a+bi \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$X = \mathbb{C} - \mathbb{R} \longleftrightarrow GL_2(\mathbb{R})$ - conj. class of h_0

$i \longmapsto h_0$

K compact open subgroup $\subset GL_2(\mathbb{A}_f)$, $\mathbb{A}_f = \prod'_{p<\infty} \mathbb{Q}_p$ finite adeles

e.g. $K = \{ g \in GL_2(\hat{\mathbb{Z}}) = \prod_{p<\infty} GL_2(\mathbb{Z}_p) \mid g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \}$

adelic version of modular curve:

$$Sh_K = \underset{\substack{\circlearrowleft \\ \text{acts diagonally}}}{GL_2(\mathbb{Q})} \backslash X \times GL_2(\mathbb{A}_f) / \underset{\substack{\circlearrowleft \\ \text{acts only on } GL_2(\mathbb{A}_f)}}{K}$$

Sh_K is a one-dim cpx mfd with finitely many connected components.

Sh_K is a moduli space

$\forall (h, g) \in X \times GL_2(\mathbb{A}_f)$

$h: \mathbb{C}^\times \longrightarrow GL_2(\mathbb{R}) \Rightarrow \mathbb{R}^2$ has a complex structure via h

$\mathbb{R}^2 / \mathbb{Z}^2 = E_h$ is an elliptic curve

Rational Tate module $\hat{V}(E_h) = \left(\varprojlim_N E_h(N) \right) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an $\mathbb{A}_f$-mod

$\Rightarrow \hat{V}(E_h) \simeq \mathbb{A}_f^2$

compose with g, $\quad \Sigma_{h,g} : \hat{V}(E_h) \simeq \mathbb{A}_f^2 \xrightarrow{g} \mathbb{A}_f^2$

Upshot : $(h, g) \rightsquigarrow (E_h, \Sigma_{h,g})$

up to isogeny — taking up to K-action

Sh_K is the moduli space of such pairs over $\mathbb{C}$

Consider same moduli problem $/\mathbb{Q}$, it is rep'd by a quasi-proj. smooth variety $/\mathbb{Q}$ (assume K small enough)

This is "canonical model" of Sh_K over $\mathbb{Q}$.

1960's Shimura found many generalizations to higher dim cases

Moduli spaces of AV + PEL

1971 Deligne. More abstract point of view

- SD (G, X)

$G/\mathbb{Q}$ reductive group

X $\quad G(\mathbb{R})$-conj. class of some $h_0 : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$

(i.e. $h_0 : \mathbb{C}^\times \to G(\mathbb{R})$ $\mathbb{R}$-algebraic)

polynomials in x, y, where $z = x + iy$.

+ axioms

Most importantly

$$\mathbb{S} \xrightarrow{h_0} G_{\mathbb{R}} \xrightarrow{\mathrm{Ad}} GL(\mathrm{Lie}\, G_{\mathbb{R}})$$

the resulting HS on Lie $G_{\mathbb{R}}$ has type $(-1,1), (0,0), (1,-1)$

" h_0 has to be minuscule "

Take $K \subset G(\mathbb{A}_f)$ compact open subgrp (small enough)

$Sh_K = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K$ complex mfd

(complex structure comes from axioms)

Baily - Borel : Sh_K is quasi-proj. var. $/\mathbb{C}$.

Thm. (Shimura, Deligne, Borovoi, Milne)

Sh_K has a canonical model over reflex field $E = E(G, X)$.

e.g. $G = T$ torus , $Sh_K(\mathbb{C})$ is a finite set

$$\mathbb{S} \xrightarrow{h_0} T_{\mathbb{R}} \implies \mathbb{S}_{\mathbb{C}} \longrightarrow T_{\mathbb{C}}$$

$$\mathbb{S}_{\mathbb{C}} = \mathbb{G}_m \times \mathbb{G}_m$$

$$\mu_{h_0} : \mathbb{G}_m \xrightarrow{z \mapsto (z,1)} \mathbb{S}_{\mathbb{C}} \xrightarrow{h_{0,\mathbb{C}}} T_{\mathbb{C}}$$

As $\mathbb{G}_m$, $T/\mathbb{Q}$, μ_{h_0} is defined over some number field E (reflex field)

$$\mathbb{A}_E^\times / E^\times \xrightarrow{\mu_{h_0}} T(\mathbb{A}_E)/T(E) \xrightarrow{Nm_{E/\mathbb{Q}}} T(\mathbb{A})/T(\mathbb{Q})$$

$$\pi_0(\mathbb{A}_E^\times / E^\times) \longrightarrow \pi_0(T(\mathbb{A})/T(\mathbb{Q}))$$

$\wr\wr$ CFT

$\mathrm{Gal}(E^{ab}/E)$

$\circlearrowright$ $Sh_K(\mathbb{C}) = Sh_K(\bar{E})$

$Sh_K(\mathbb{C}) = T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K$

finite, $T(\mathbb{A})/T(\mathbb{Q})$ - action factors through π_0

By descent, we get zero-dim var. $/E$

($T \rightsquigarrow$ CM AV $\rightsquigarrow$ Shimura - Taniyama formula)

In general, the canonical models of SV are characterized by

1) $G = T$ torus, the model is as above

2) Require some functoriality w.r.t. $(T, X_T) \longrightarrow (G, X)$

e.g. Siegel modular variety

(V, ψ) symplectic space $/\mathbb{Q}$ of dim $2g$

$G = GSp = \{ g \in GL(V) \mid g \text{ preserves } \psi \text{ up to a scalar} \}$

$\dim V = 2 \Rightarrow GSp = GL_2$

$$X = \left\{ h : \mathbb{S} \to G_{\mathbb{R}} \;\middle|\; \begin{array}{l} \text{the HS on } V \text{ via } h \text{ has type } (-1,0),(0,-1) \\ V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R},\ \psi(-, h(i)-) \text{ symmetric} \\ \qquad\qquad \text{positive / negative definite} \end{array} \right\}$$

$= \mathcal{H}^{g,\pm}$

$E = E(G, X) = \mathbb{Q}$

Sh_K is the moduli space of g-dim AV with polarization and level structure K

§. Hasse - Weil Zeta Function for SV

X smooth proj. var. $/\mathbb{Q}$

for almost all p, could find $\mathcal{X}_p$ good integral model $/\mathbb{Z}_p$

for $s \in \mathbb{C}$, $\mathrm{Re}\, s \gg 0$, $\underline{\zeta_p(X, s) = \exp\left(\sum_{n=1}^{\infty} \# \mathcal{X}_p(\mathbb{F}_{p^n}) \cdot \frac{p^{-ns}}{n} \right)}$

$$\overset{\text{Lefschetz trace formula}}{\underset{\text{smooth proper BC}}{=\!=\!=}} \underline{\prod_{i=0}^{2\dim X} \det\left(1 - \mathrm{Frob}_p \cdot T \,\middle|\, H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)\right)^{(-1)^{i+1}}_{T = p^{-s}}}$$

$\underline{\zeta(X, s) = \prod_p \zeta_p(X, s)}$ (for almost all p), $\mathrm{Re}\, s \gg 0$

(Frob_p action on inertia invariant part of etale cohomology of generic fibre)

Ultimate Conjecture : $\zeta(X, s)$ has meromorphic continuation to $\mathbb{C}$

Thm. (Eichler - Shimura)

$X = X_0(N)$ modular curve of $\Gamma_0(N)$ level

$$\underbrace{\zeta(X, s)}_{} = \underset{H^0}{\zeta(s)}\ \underset{H^2}{\zeta(s-1)} \cdot \underbrace{\prod_{i=1}^{g=g(X)} L(f_i, s)^{-1}}_{H^1}$$

$\{f_i\}$ eigenbasis of $S_2(\Gamma_0(N))$, $L(f_i, s)$ L - function of f_i

(Hecke : $L(f_i, s)$ has mero. cont. to $\mathbb{C}$)

$\zeta(X, s)$ has mero. cont. to $\mathbb{C}$ (and satisfy a functional equation)

RMK. Replace $H^i_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ by $H^i_{et}(X_{\bar{\mathbb{Q}}}, \mathcal{L})$ for $\mathcal{L}$ suitable local system on X built from representations of $G = GL_2$, we will see higher weight modular forms in analogue of $\zeta(X, s)$.

Expect :

Hasse - Weil Zeta of $Sh_K(G, X)$ $\longleftrightarrow$ L - functions of automorphic forms on G

Langlands' idea :

$\zeta(Sh_K, s)$ - - - - - - - - - - automorphic L-functions

$\#\, Sh_K(\mathbb{F}_{p^m})$ spectral information about $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$

?

trace formula

counting points by orbital integral

orbital integrals on G

RMK. When Sh_K not proper, $\zeta(Sh_K, s)$ should be defined using intersection cohomology of canonical Baily-Borel compactification. Also in this case, $G(\mathbb{Q}) \backslash G(\mathbb{A})$ is noncompact, so for $f \in C_c^\infty(G(\mathbb{A}))$, $tr(f \mid L^2(G(\mathbb{Q}) \backslash G(\mathbb{A})))$ does not make sense without truncation.

RMK. Actually one wants to understand the commuting action of $\mathrm{Gal}(\bar{E}/E) \times \mathcal{H}(K\backslash G(\mathbb{A}_f)/K)$ on $H^i_{et}(Sh_{K,\bar{E}}, \mathbb{Q}_\ell)$.

Fix $f \in \mathcal{H}(K\backslash G(\mathbb{A}_f)/K)$, need $tr(f \times \mathrm{Frob}_p^a \mid H^i_{et})$.

Assume $K = K^p K_p$, $f = f^p f_p$, $f^p \in \mathcal{H}(K^p \backslash G(\mathbb{A}_f^p)/K^p)$ and $f_p = \mathbb{1}_{K_p}$. By linearity, assume $f^p = \mathbb{1}_{K^p g K^p}$. Then

$\sum_i (-1)^i tr(f \times \mathrm{Frob}_p^a \mid H^i_{et}) = \#$ fixed points of the correspondence

$$\begin{tikzcd} S_{(K^p \cap g^{-1} K^p g) K_p} \arrow[r, "\mathrm{Frob}_p^a"] \arrow[d, "g"'] & S_{(K^p \cap g^{-1} K^p g) K_p} \arrow[d, "g"'] \\ S_{K^p K_p} & S_{K^p K_p} \end{tikzcd}$$

Kottwitz's precise conjecture

(G, X) SD, $E = E(G, X)$.

For simplicity, assume $E = \mathbb{Q}$, G^{der} simply connected.

$K \subset G(\mathbb{A}_f)$

fix p prime s.t. $K = K^p K_p$, $K^p \subset G(\mathbb{A}_f^p)$

$K_p \subset G(\mathbb{Q}_p)$ hyperspecial

(i.e. $\exists\, \mathcal{G}_p$ reductive group scheme $/\mathbb{Z}_p$

$\mathcal{G}_{p,\mathbb{Q}_p} = G_{\mathbb{Q}_p}$

$K_p = \mathcal{G}_p(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$)

(for fixed K, all but finitely many K_p are hyperspecial)

Conj. For such p, Sh_K has a smooth "canonical" model $S_K / \mathbb{Z}_p$ and $\{ S_{K^p K_p} \}_{K^p}$ is a finite étale system with an extended $G(\mathbb{A}_f^p)$-action.

Conj. (Kottwitz)

$$\# S_K(\mathbb{F}_{p^n}) = \sum_{(\gamma_0, \gamma, \delta)} \underbrace{c(\gamma_0, \gamma, \delta)}_{\text{volume}} \cdot \underbrace{O_\gamma(1_{K^p}) \cdot TO_\delta(f_{\mu,n})}_{\text{(twisted) orbital integral}}$$

- $(\gamma_0, \gamma, \delta)$ Kottwitz triple

γ_0 : semisimple element in $G(\mathbb{Q})$ up to $G(\overline{\mathbb{Q}})$-conj.

γ_0 is contained in an elliptic maximal torus in $G(\mathbb{R})$

compact after mod center

γ : element of $G(\mathbb{A}_f^p)$ up to $G(\mathbb{A}_f^p)$-conj.

s.t. γ is conj. to γ_0 inside $G(\mathbb{A}_f^p \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$

δ : element of $G(\mathbb{Q}_{p^n})$ up to σ-conj.

σ Frob in $Gal(\mathbb{Q}_{p^n} / \mathbb{Q}_p)$

σ-conj : $g \cdot \delta \cdot \sigma(g)^{-1}$, $g \in G(\mathbb{Q}_{p^n})$

s.t. $\delta \cdot \sigma\delta \cdot \dots \cdot \sigma^{n-1}\delta \in G(\mathbb{Q}_{p^n})$

and it is conj. to γ_0 inside $G(\overline{\mathbb{Q}}_p)$

Additional : a certain cohomological invariant defined by $(\gamma_0, \gamma, \delta)$ should vanish

(collect all local obstructions coming from conj. / $\overline{\mathbb{Q}}$, $\overline{\mathbb{Q}}_p$. should come from some global obstruction)

- $c(\gamma_0, \gamma, \delta)$ is a certain volume term
- $O_\gamma(1_{K^p}) = \int_{C_\gamma(G(\mathbb{A}_f^p))\backslash G(\mathbb{A}_f^p)} 1_{K^p}(g^{-1}\gamma g)\, dg$ orbital integral
- $TO_\delta(f_{\mu,n}) = \int_{\{g\in G(\mathbb{Q}_{p^n}),\, g\delta\sigma(g)^{-1}=\delta\}\backslash G(\mathbb{Q}_{p^n})} f_{\mu,n}(g^{-1}\delta\sigma(g))\, dg$

$f_{\mu,n} : G(\mathbb{Q}_{p^n}) \longrightarrow \{0,1\}$

characteristic function of $\underbrace{\mathcal{G}_p(\mathbb{Z}_{p^n})\cdot\mu(p)\cdot\mathcal{G}_p(\mathbb{Z}_{p^n})}^{\text{does not depend on } \mu} \subset G(\mathbb{Q}_{p^n})$

where μ is a cocharacter $\mathbb{G}_m \longrightarrow \mathcal{G}_p$ defined over $\mathbb{Z}_p$

s.t. over $\overline{\mathbb{Q}}_p$, it is conj. to μ_h for some $h\in X$.

For $G=GL_n$, elements conj. in larger field are already conj. in original field $\Rightarrow$ γ is determined by γ_0, (γ,γ_0) is determined by δ

$\Rightarrow$ sum over certain δ

Kottwitz's conj. is bad for trace formula as

- range is too complex
- twisted and untwisted orbital integral at the same time
- $O_\gamma(1_{K^p})$ depends on $G(\mathbb{A}_f^p)$-conj. class of γ, not $G(\mathbb{A}_f^p\otimes_{\mathbb{Q}}\overline{\mathbb{Q}})$-conj. class

Def. A stable conj. class in $G(\mathbb{A}_f^p)$ / $G(\mathbb{Q}_v)$, $G(\mathbb{A})$, $G(\mathbb{A}_f)$ ...

is the union of actual conj. classes which become conjugate after base change to $\overline{\mathbb{Q}}$

Problem: RHS of Kottwitz is not based on stable orbital integrals

Thm. (Kottwitz, 1992)

$$\sum_{(\gamma_0,\gamma,\delta)} c(\gamma_0,\gamma,\delta)\cdot O_\gamma(1_{K^p})\cdot TO_\delta(f_{\mu,n})$$

$$= \sum_{H} \sum_{\gamma_H \text{ semisimple}} SO_{\gamma_H}(f^H)$$

stable orbital integral

endoscopic group of G

finite sum

$f^H : H(\mathbb{A}) \longrightarrow \mathbb{C}$ depends on (G,X), 1_{K^p}, $f_{\mu,n}$

construction of f^H away from p,∞ depends on

$f^H = f_p^H \cdot f_\infty^H \cdot f^{H,p,\infty}$

$H(\mathbb{Q}_p)$ $H(\mathbb{R})$ $H(\mathbb{A}_f^p)$

Langlands-Shelstad transfer

& ($\Leftarrow$) fundamental lemma

$$\sum_{\gamma_H} SO_{\gamma_H}(f^H) = ST^H_{ell}(f^H)$$

elliptic part of stable trace formula for H

Arthur

In ideal cases

(*) Sh_K is proj.

Conj. $\forall H$, $ST_{ell}(f^H) = ST(f^H)$ for our specific f^H

$ST(f^H) \rightsquigarrow$ automorphic L-function

$\sum_H ST^H(f^H) \longrightarrow$ automorphic L-functions on G

conditional on Arthur's multiplicity conj.

Upshot :

Assume Kottwitz conj. + (*) + Arthur's multiplicity conj.

$\Rightarrow \zeta(Sh_K, s)$ is automorphic.