Yihang Zhu : Counting points on SV, II

Last time :

Conj. (G, X) SD, $K \subset G(\mathbb{A}_f)$, p hyperspecial prime

i.e. $K = K^p K_p$, $K^p \subset G(\mathbb{A}_f^p)$

$K_p = \mathcal{G}_p(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$

$\mathcal{G}_p$ reductive group scheme $/ \mathbb{Z}_p$

with generic fibre $G_{\mathbb{Q}_p}$

Then $\exists$ "canonical" smooth integral model $\mathcal{S}_K / \operatorname{Spec} \mathcal{O}_{E,v}$

E reflex field, $v | p$ s.t. $(\mathcal{S}_K)_{E_v} \simeq Sh_K \times_E E_v$

Conj. (Kottwitz) (G^{der} simply connected)

$$\# \mathcal{S}_K(\mathbb{F}_{q^n}) = \sum_{(\gamma_0, \gamma, \delta)} c(\gamma_0, \gamma, \delta) O_\gamma(1_{K^p}) TO_\delta(f_{\mu,n})$$

Thm. (Kottwitz, 1992)

RHS can be stabilized, i.e.

$$RHS = \sum_{H} ST_{ell}^H(f^H)$$

endoscopic group of G

RMK. $\# S_K(\mathbb{F}_{q^n}) = \sum_i (-1)^i \operatorname{tr}(\mathrm{Frob}_q^n \mid H^i_{et}(S_{K,\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell))$

More generally we consider

$$\sum_i (-1)^i \operatorname{tr}(\mathrm{Frob}_q^n \times f \mid H^i_{et}(S_{K,\bar{\mathbb{F}}_q}, \mathcal{L}))$$

f: Hecke operator away from p; $\mathcal{L}$: certain local system

and will get similar Kottwitz conj.

§. Integral Models

- $G = GSp(V, \psi)$, $X = \mathcal{H}^{g,\pm}$ Siegel double space

  $Sh_K / \mathbb{Q}$ moduli of AV with PL

  If p is a hyperspecial prime for K

  can define "similar" moduli problem$/\mathbb{Z}_p$ in $-\otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ category

  it is rep'd by a smooth $\mathbb{Z}_p$-scheme S_K (Mumford)

  S_K is canonical integral model.

It is expected that Sh_K proper $\Rightarrow$ $\mathcal{S}_K$ proper (true for Hodge type)

Sh_K not proper $\Rightarrow$ the Baily-Borel compactification extends

Thm. (Kisin 2010, $p > 2$; Vasiu, Kim, Madapusi Pera, $p = 2$)

Suppose (G, X) is of abelian type, p hyperspecial for K

$\Rightarrow$ Sh_K has a canonical smooth integral model $/ \mathcal{O}_{E,v}$, $v | p$.

Def.

1) SD (G, X) is of Hodge type if $\exists$ $(G, X) \hookrightarrow (GSp(V, \psi), \mathcal{H}^{g,\pm})$

2) SD (G_2, X_2) is of abelian type if $\exists$ (G, X) Hodge type

s.t. $(G_2^{ad}, X_2^{ad}) = (G^{ad}, X^{ad})$

and

$$\begin{array}{ccc} G_2^{der} & \overset{\exists \text{ central isogeny}}{\longleftarrow} & G^{der} \\ \downarrow & & \downarrow \\ G_2^{ad} & = & G^{ad} \end{array}$$

Idea:

1) (G, X) Hodge type $\Rightarrow$ for suitable $K \subset G(\mathbb{A}_f)$

$\exists$ $\mathcal{U} \subset GSp(\mathbb{A}_f)$

s.t. $Sh_K(G, X) \longrightarrow Sh_{\mathcal{U}}(GSp) \times_{\mathbb{Q}} E$

is closed embedding.

2) If (G_2, X_2) abelian type, (G, X) Hodge type as in the definition, then $Sh_{K_2}(G_2, X_2)$, every geometric connected component is a finite quotient of a geometric connected component of $Sh_K(G, X)$.

Idea of Kisin's Thm (Only for Hodge type)

(G, X) Hodge type, $(G, X) \hookrightarrow (GSp, \mathcal{H}^{g,\pm})$

$K = K^p K_p \subset G(\mathbb{A}_f)$

- can arrange $\exists\, \mathcal{U} = \mathcal{U}^p \mathcal{U}_p \subset GSp(\mathbb{A}_f)$, $\mathcal{U}_p$ hyperspecial

s.t. $Sh_K(G, X) \hookrightarrow Sh_{\mathcal{U}}(GSp) \times_{\mathbb{Q}} E$

(such arrangement may need to change the symplectic embedding)

Take $\mathcal{S}_K(G, X)$ to be normalization of the Zariski closure of

$Sh_K(G, X)$ inside $\mathcal{S}_{\mathcal{U}} \times_{\mathbb{Z}_p} O_{E,v}$

(Y. Xu showed the normalization is unnecessary)

Key point: Need to prove $\mathcal{S}_K / O_{E,v}$ smooth.

$\forall\, x \in \mathcal{S}_K(k)$, $k = \mathbb{F}_q$ or $\overline{\mathbb{F}_q}$

want to show $\widehat{\mathcal{S}_{K,x}}$ smooth.

Strategy: $\widehat{\mathcal{S}_{K,x}} \simeq$ Faltings deformation ring of a p-divisible group with crystalline tensors

$\mathcal{G}_p$ reductive group scheme $/ \mathbb{Z}_p \hookrightarrow GSp(V_{\mathbb{Z}_p})$

$V_{\mathbb{Z}_p}$ self-dual $\mathbb{Z}_p$-lattice inside $(V_{\mathbb{Q}_p}, \psi)$

can find some $\mathbb{Z}_p$-linear tensors $s_\alpha \in V_{\mathbb{Z}_p}^{\otimes}$, $\alpha = 1, \dots, n$

s.t. $\mathcal{G}_p$ is the stabilizer of $\{s_\alpha\}$ inside $GL(V_{\mathbb{Z}_p})$

$x \in \mathcal{S}_K(k)$, find a char 0 lift $\tilde{x} \in Sh_K(K)$

K finite totally ramified extension of $W(k)[\frac{1}{p}]$

We get $\mathcal{A}_x$ AV$/k$, $\mathcal{A}_{\tilde{x}}$ AV$/K$

Write $\mathcal{V}_p = H^1_{et}(\mathcal{A}_{\tilde{x}, \bar{K}}, \mathbb{Z}_p)$ $\mathbb{Z}_p$-mod + $Gal(\bar{K}/K)$-action

$\mathcal{V}_0 = H^1_{cris}(\mathcal{A}_x / W(k))$ $W(k)$-module + φ

$\mathcal{V}_p$ p-adic Tate module of $\mathcal{A}_{\tilde{x}}$, $\mathcal{V}_0$ Dieudonné's module of $\mathcal{A}_x$

$\mathcal{V}_p[\frac{1}{p}] \simeq V_{\mathbb{Q}_p}$

$\cup \qquad\qquad \cup$

$\mathcal{V}_p \simeq V_{\mathbb{Z}_p}$

$\{s_\alpha\}$ can be viewed as $\mathbb{Z}_p$ - tensors / $\mathcal{V}_p$

p-adic comparison

$$\mathcal{V}_0[\tfrac{1}{p}] = D_{\mathrm{cris}}(\mathcal{V}_p[\tfrac{1}{p}]) = \left(\mathcal{V}_p[\tfrac{1}{p}] \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}\right)^{\mathrm{Gal}_K}$$

Functoriality $\Rightarrow$ $D_{\mathrm{cris}}(s_\alpha)$ is a tensor on $\mathcal{V}_0[\frac{1}{p}]$

(but Faltings needs tensor on $\mathcal{V}_0$)

Integral comparison

$$\mathcal{V}_0 \simeq \mathfrak{M}(\mathcal{V}_p) \otimes_{W(k)[[u]]} W(k) =: M_{\mathrm{cris}}(\mathcal{V}_p)$$

$\mathfrak{M}(\mathcal{V}_p)$: BK - module of $\mathcal{V}_p$, a module / $W(k)[[u]]$

$\Rightarrow$ $M_{\mathrm{cris}}(s_\alpha)$ is integral tensor on $\mathcal{V}_0$

$\mathcal{G}_p'$ $\mathbb{Z}_p$ - group scheme defined by $\mathcal{V}_0$ + $\{M_{\mathrm{cris}}(s_\alpha)\}$

Fact: $\mathcal{G}_p' \simeq \mathcal{G}_p \times_{\mathbb{Z}_p} W(k)$ hence reductive

(non canonical)

$(\mathbb{M}(\mathcal{V}_p), \{\mathbb{M}(s_\alpha)\})$ defines a group scheme $/ W(k)[\![u]\!]$ and the obstruction for this to be isom. to $\mathcal{G}_p$ is a $\mathcal{G}_p$ - torsor $/ W(k)[\![u]\!]$.

General properties of $\mathbb{M}(-)$: this torsor is trivial away from the closed point in Spec $W(k)[\![u]\!]$.

$\mathcal{G}_p$ reductive $/ \mathbb{Z}_p$ $\Rightarrow$ every $\mathcal{G}_p$ - torsor on Spec $W(k)[\![u]\!]$ which is trivial away from the closed point is trivial.

Faltings' deformation theory:

$\mathcal{V}_0$ has some tensors cutting out a reductive group scheme $/ W(k)$

$\Rightarrow$ a smooth deformation ring of $\mathcal{A}_x[p^\infty]$ + those tensors on $\mathcal{V}_0$

can show this deformation ring $\simeq \widehat{\mathcal{S}_{K,x}}$

RMK.

Kisin - Pappas constructed integral models of $Sh_K(G, X)$ for (G, X) abelian type, K_p parahoric (+ technical conditions)

They are no longer smooth with a concrete description of singularities.

Recently Pappas found a way to uniquely characterize these models.

Recall Kottwitz's conj.

$$\# S_K(\mathbb{F}_{q^n}) = \sum_{(\gamma_0, \gamma, \delta)} c(\gamma_0, \gamma, \delta) O_\gamma(1_{K^p}) TO_\delta(f_{\mu,n})$$

(p hyperspecial, G^{der} simply connected)

Kottwitz 1990 proved for PEL SV for G of type A, C

(integral models are PEL moduli spaces)

Thm. (Kisin - Shin - Zhu)

prove for all abelian type SV (dropped G^{der} simply connected)

+ stabilization of RHS after dropping G^{der} simply connected.

Idea : prove a suitable version of Langlands - Rapoport conj.

§. Original Langlands - Rapoport conj. (1987)

Conj. K_p hyperspecial

$$\varprojlim_{K^p} S_{K^pK_p}(\overline{\mathbb{F}}_q) = \coprod_{\varphi} I_\varphi(\mathbb{Q}) \backslash X(\varphi).$$

- φ runs over "admissible morphisms" up to $G(\overline{\mathbb{Q}})$ - conj.

(For simplicity, assume Z_G cuspidal, i.e. maximal $\mathbb{Q}$ - split subtorus = maximal $\mathbb{R}$ - split subtorus)

Admissible morphism φ : $(\nu_p, \varphi') +$ other conditions where

- a cocharater ν_p of $G_{\mathbb{Q}_p}$ defined over $\mathbb{Q}_p$

  s.t. $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ average of ν_p = weight cocharater of (G, X)

- Let M_φ = maximal $\mathbb{Q}$ - subgroup of G that centralizes ν_p

  A map $\varphi' : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow M_\varphi(\overline{\mathbb{Q}})$

  $$p \longmapsto g_p$$

  s.t. $\{Ad(g_p)\}_p$ gives a descent datum for $M_{\varphi, \overline{\mathbb{Q}}}$

  $\Rightarrow$ get a new $\mathbb{Q}$ - group I_φ

$X(\varphi) = X^p(\varphi) \times X_p(\varphi)$

$X^p(\varphi) \simeq G(\mathbb{A}_f^p)$ noncanonically ($G(\mathbb{A}_f^p)$ - torsor depends on φ)

$X_p(\varphi) = X_\mu(b)$ affine Deligne - Lusztig variety

b associated to φ

$I_\varphi(\mathbb{Q}) \curvearrowright X(\varphi)$ and Frobenius $\Phi \curvearrowright X_\mu(b) = X_p(\varphi)$

Langlands - Rapoport conj : in addition, the bijection is Φ- equivariant

$\Rightarrow$ $\# S_K(\mathbb{F}_{q^n})$ = take Φ^n- fixed points on RHS $/ K^p$

naturally see $(\gamma_0, \gamma, \delta)$, O_γ, TO_δ when group theoritically working on RHS.

Idea of Langlands - Rapoport conj. in Hodge type

φ ; classifies isogeny classes of AV + additional structures

$X^p(\varphi)$ prime - to - p isogenies

$X_p(\varphi)$ p - power isogenies

$I_\varphi(\mathbb{Q})$ self - isogenies

Thm. (Kisin 2017)

proved a weaker version of LR conj.

① Hodge type case $\overset{\text{functoriality}}{\Longrightarrow}$ abelian type case

② In Hodge type case

Bog. class means $x \overset{\text{Bog}}{\sim} x'$

$$\mathcal{A}_x \xrightarrow{\text{Bog.}} \mathcal{A}_{x'}$$

$$H^1_{et}(\mathcal{A}_x, \mathbb{Q}_\ell) \qquad H^1_{et}(\mathcal{A}_{x'}, \mathbb{Q}_\ell)$$

$$H^1_{cris}(\mathcal{A}_x / W(k)) \qquad H^1_{cris}(\mathcal{A}_{x'} / W(k))$$

Bog. preserves the incarnations of $\{s_\alpha\}$ in H_{et}^1 and H_{cris}^1

③ weaker version : the natural action of $I_\varphi(\mathbb{Q})$ on $X(\varphi)$

replaced by

$$I_\varphi(\mathbb{Q}) \hookrightarrow I_\varphi(\mathbb{A}_f) \xrightarrow{Ad(\tau_\varphi)} I_\varphi(\mathbb{A}_f) \overset{\text{naturally}}{\curvearrowright} X(\varphi)$$

for some unknown $\tau_\varphi \in I_\varphi^{ad}(\mathbb{A}_f)$

No more LR conj. $\Rightarrow$ Kottwitz conj.

Conj. (Kisin - Shin - Zhu)

LR Conj. should hold with these τ_φ where τ_φ should satisfy

1) τ_φ depends on φ only via the component V_p

2) (tori-rational)

$\forall$ maximal torus $T \subset I_\varphi$ defined over $\mathbb{Q}$

the image of $\tau_\varphi \in I_\varphi^{ad}(\mathbb{A}_f)$ in

$$I_\varphi^{ad}(\mathbb{A}_f) \longrightarrow H^1(\mathbb{A}_f, Z_{I_\varphi}) \longrightarrow H^1(\mathbb{A}_f, T)$$

should come from a class in $H^1(\mathbb{Q}, T)$ with extra conditions.

Thm. (KSZ)

1) KSZ Conj. $\Rightarrow$ Kottwitz's Conj.

2) For all abelian type, KSZ Conj. is true.

Pf of 2) (for Hodge type)

$T \subset I_\varphi$, $\quad I_{\varphi, \overline{\mathbb{Q}}} = M_{\varphi, \overline{\mathbb{Q}}} \subset G_{\overline{\mathbb{Q}}}$

$i: T_{\overline{\mathbb{Q}}} \longrightarrow G_{\overline{\mathbb{Q}}}$

Can arrange i s.t. it is defined $/\mathbb{Q}$

and extends to a morphism of SD

$$i : (T, \{h_T\}) \hookrightarrow (G, X)$$

Thm. (Tate - Kisin)

In the Isog. class corresponding to φ, $\exists$ a point that is the

reduction of a char 0 point in the image of

$$Sh(T, \{h_T\}) \xrightarrow{i'} Sh(G, X) \quad , \quad i' \neq i .$$

Need to play with T, h_T, i, i' to show the image of τ_φ

in $H^1(\mathbb{A}_f, T)$ comes from $H^1(\mathbb{Q}, T)$.