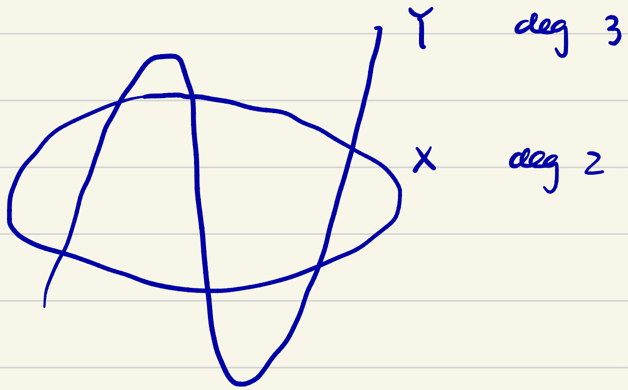
Intro. to Derived Geometry.

I. Motivations

1. Bezout theorem

$X, Y \subset \mathbb{CP}^2$ smooth alg. curves of deg a, b



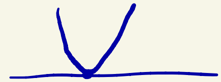
then if X and Y meet transversely

then $X \cap Y$ has $a \cdot b$ points

$$\# X \cap Y = \deg X \cdot \deg Y$$

Q: Non-transverse intersection?

Two cases: proper case



$\dim X \cap Y = 0$

non-proper case — self-intersection

$\dim X \cap Y > 0$

If we want Bezout still holds, we need to reinterpret $X \cap Y$ i.e. equip $X \cap Y$ with more strutures.

<u>Proper case</u>: equip each intersection point with mult.

$$\text{mult}(p) = \dim \mathcal{O}_{X,p} \otimes_{\mathcal{O}_{\mathbb{CP}^2,p}} \mathcal{O}_{Y,p}$$

then $\displaystyle\sum_{p \in X \cap Y} \text{mult}(p) = \deg X \cdot \deg Y$

The above formula is wrong in higher dim for singular var. and should be corrected by Tor.

Assume $X, Y \subset \mathbb{CP}^n$ subvar., $\dim X + \dim Y = n$

$\dim X \cap Y = 0$

For every $p \in X \cap Y$, define

$$\text{mult}(p) = \sum (-1)^i \dim \text{Tor}_i^{\mathcal{O}_p} (\mathcal{O}_{X,p}, \mathcal{O}_{Y,p})$$

$\Rightarrow$ Serre's inter. formula

$i > 0$, Tor_i: correction term

$$\sum_{p \in X \cap Y} \text{mult}(p) = \deg X \cdot \deg Y$$

Non-proper case.

Recall Tor: choose proj. resol. of A

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$

$$\Rightarrow \quad A \overset{\mathbb{L}}{\underset{R}{\otimes}} B = P_\cdot \underset{R}{\otimes} B$$

Rewrite $\quad \operatorname{mult}(p) = \chi (\mathcal{O}_{X,p} \otimes^{\mathbb{L}} \mathcal{O}_{Y,p})$

$$\chi (\mathcal{O}_X \otimes^{\mathbb{L}} \mathcal{O}_Y) = \deg X \cdot \deg Y$$

This reformulation generalizes to arbitrary cases.

Ex. (self-intersection).

$C \subset \mathbb{CP}^2$ of deg d, $C \cap C$.

Compute $\chi (\mathcal{O}_C \otimes^{\mathbb{L}} \mathcal{O}_C)$.

Resolve $\mathcal{O}_C$:

$$0 \to \mathcal{O}(-C) \to \mathcal{O} \to \mathcal{O}_C \to 0$$

$\otimes \mathcal{O}_C$: $0 \to \mathcal{O}(-C) \otimes \mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O}_C \to 0$

$$\chi (\mathcal{O}_C \otimes^{\mathbb{L}} \mathcal{O}_C) = \chi (\mathcal{O}_C) - \chi (\mathcal{O}(-C) \otimes \mathcal{O}_C)$$

Use Riemann-Roch

$$= (1-g) - (-d^2 - g + 1) = d^2$$

Moral: for full generality of Bezout theorem, equip set-theoretic inter. $X \cap Y$ with $\mathcal{O}_X \otimes^{\mathbb{L}} \mathcal{O}_Y$.

RMK. When A, B are algs. / R, we can enhance $P_\bullet$ to get a commutative differential graded alg. structure (cdga):

(i) multiplications $P_m \otimes P_n \to P_{m+n} \rightsquigarrow \bigoplus P_n$ comm. graded

$$xy = (-1)^{|x||y|} yx$$

(ii) differential $d: P_n \to P_{n-1}$

$$d(xy) = (dx)y + (-1)^{|x|} x(dy)$$

$\Rightarrow P_\bullet \otimes B = A \otimes^{\mathbb{L}} B$ inherits cdga struc. (well-def. up to htpy)

Def. A dg scheme / $\mathbb{C}$ is $(X, \mathcal{O}_X)$ consists of a topological space X and a sheaf of cdga / $\mathbb{C}$ $\mathcal{O}_X$ on X s.t.

(a) $(X, H_0 \mathcal{O}_X)$ is a scheme (truncation)

(b) $H_n \mathcal{O}_X$ is QCoh sheaf on the scheme (a)

(c) $H_n \mathcal{O}_X$ vanish for $n < 0$.

RMK.

1) Need ∞-cat. as objs. up to quasi-isom.

2) cdga are not well-behaved over char p

$\rightsquigarrow$ use simplicial rings $\rightsquigarrow$ derived scheme

3) multiplications on cdga and simplicial rings are <u>strictly comm. and associative</u>

Not natural or convenient $\rightsquigarrow$ relax by requiring everything up to quasi-isom. $\Rightarrow$ E_∞-rings

spectral scheme

Over char 0, cdga, simplicial rings and E_∞-rings are equivalent.

2. Enumerative geometry.

Problem: Given $X / \mathbb{C}$ sm. proj. var., $\beta \in H_2(X, \mathbb{Z})$

want to count alg. curves in X with class β satisfying some constrains

Ex. • rational curves in $\mathbb{P}^2$ of deg d passing through $3d-1$ general points

d	1	2	3	4	5	
#	1	1	12	620	87304	...

- deg d rational curves on a general quintic $X \subset \mathbb{CP}^4$

d	1	2	...
#	2875	609250	...

Idea: $\mathcal{M}$ moduli space of such objects

want $\#\mathcal{M}$.

Similar to Bezout: points in $\mathcal{M}$ may have multiplicities, $\dim \mathcal{M} > 0$

$\Rightarrow$ should endow $\mathcal{M}$ with extra structure: derived struc.

History: perfect obstruction theory (before derived geom.)

3. Homotopy theory.

cdga and E_∞-rings occur naturally.

de Rham: M sm. mfd.

$H^*(M, \mathbb{R})$ can be computed by de Rham complex $\Omega^\bullet_M$ (cdga)

More generally $\forall$ X topological space

Sullivan: poly. de Rham cpx $C^*_{dR}(X; \mathbb{Q})$

cdga $\sim$ singular cpx

Thm. (Sullivan) X simply conn., $\dim H^n(X, \mathbb{Q}) < \infty$

then rational htpy type of X can be recovered from $C^*_{dR}(X; \mathbb{Q})$.

More precisely $X \longrightarrow X_{\mathbb{Q}} = \mathrm{Map}_{cdga}(C^*_{dR}(X;\mathbb{Q}), \mathbb{Q})$

is a rational htpy equivalence.

Reformulate : $\hat{X} = \mathrm{Spec}\, C^*_{dR}(X;\mathbb{Q})$ dg scheme

" schematization of X "

then $X_{\mathbb{Q}} = \hat{X}(\mathbb{Q})$ rational htpy equiv. to X.

More generally $\forall X$, field k, the singular chain cpx $C^*(X;k)$ has the struc. of an $\mathbb{E}_\infty$-alg. $/k$

Thm. (Mandell) X simply conn. $\dim H^*(X, \mathbb{F}_p) < \infty$

then the canonical map $X \to X_p^\wedge = \mathrm{Map}_{\mathbb{E}_\infty}(C^*(X;\mathbb{F}_p), \bar{\mathbb{F}}_p)$

is an isom. on $\mathbb{F}_p$-coh. and $\pi_n X_p^\wedge \simeq$ p-adic

completion of $\pi_n X$.

4. Derived Cat.

Fourier - Mukai transform

$E/\mathbb{C}$ elliptic curve

$$\begin{tikzcd} & E \times E \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\ E & & E \end{tikzcd}$$

P : line bundle on $E \times E$ corresponding to $\Delta - \{e\} \times E - E \times \{e\}$

Consider $\mathrm{QCoh}(E) \longrightarrow \mathrm{QCoh}(E)$

$$\mathcal{F} \longmapsto \pi_{1*}(P \otimes \pi_0^* \mathcal{F})$$

not exact and faithful

$\rightsquigarrow$ get improvement passing to derived cat.

FM transform : $D\mathrm{QCoh}(E) \longrightarrow D\mathrm{QCoh}(E)$

$$\mathcal{F} \longmapsto R\pi_{1*}(P \otimes \pi_0^* \mathcal{F})$$

gives an equivalence (also for AV).

Thm. (Bondal - Orlov) X/k sm. proj. var. Assume K_X ample or antiample, then X is determined by $D^b coh(X) \subset DQcoh(X)$.

Base change theorem.