DAG.

V. Structured spaces.

Goal: generalize the notion of locally ringed spaces to the derived setting.

1. ∞-Topoi.

Idea: generalization of topological spaces

1.1 Giraud's axioms.

Def. An ∞-cat. $\mathcal{X}$ is called an ∞-topos if there exists a small ∞-cat. $\mathcal{C}$ and an accessible left exact localization functor $P(\mathcal{C}) \to \mathcal{X}$.

More intrinsic: HTT 6.1.0.6

∞-cat. $\mathcal{X}$ is ∞-topos $\Leftrightarrow$ $\mathcal{X}$ satisfies the following analogue of Giraud's axioms:

(i) $\mathcal{X}$ presentable

(ii) colims in $\mathcal{X}$ are universal:

for any morphism $f: T \to S$ in $\mathcal{X}$, the associated pullback functor $f^*: \mathcal{X}_{/S} \to \mathcal{X}_{/T}$ preserves colims

($\Leftrightarrow$ $\exists$ a right adjoint f_* by adjoint functor thm)

(iii) coproducts in $\mathcal{X}$ are disjoint:

every coCart. diagram
$$\begin{tikzcd} \phi \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X \amalg Y \end{tikzcd}$$
is also Cart.

(iv) every groupoid object of $\mathcal{X}$ is effective:

Rough idea: equivalence relations $\xleftrightarrow{1:1}$ quotients

Precisely:

Def. $\Delta = \{[n], n \geq 0\}$ cat. of combinatorial simplices

$\Delta_+ = \Delta \cup \{[-1] = \phi\}$, $\Delta_+^{\leq n} \subset \Delta_+$ spanned by $\{[k]\}_{k \leq n}$

$\mathcal{C}$ ∞-cat. A simplicial obj. of $\mathcal{C}$ is a functor $U: \Delta^{op} \to \mathcal{C}$, let $U_n = U([n])$. An augmented simplicial obj. of $\mathcal{C}$ is a functor $U^+: \Delta_+^{op} \to \mathcal{C}$.

A simplicial obj. of $\mathcal{C}$ is a groupoid obj. if for every $n \geq 0$ and every partition $[n] = S \cup S'$ s.t. $S \cap S' = \{s\}$ singleton, then the diagram

$$\begin{array}{ccc} U([n]) & \longrightarrow & U(S) \\ \downarrow & & \downarrow \\ U(S') & \longrightarrow & U(\{s\}) \end{array}$$

is a pullback in $\mathcal{C}$.

HTT 6.1.2.11. $\mathcal{C}$ ∞-cat., $U^+: \Delta_+^{op} \to \mathcal{C}$ augmented simplicial obj. of $\mathcal{C}$. TFAE.

(1) U^+ is a right Kan extension of $U^+|_{(\Delta_+^{\leq 0})^{op}}$

(2) the underlying simplicial obj. U is a groupoid obj. of $\mathcal{C}$ and the diagram $U^+|_{(\Delta_+^{\leq 1})^{op}}$ is a pullback

$$\begin{array}{ccc} U_1 & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ U_0 & \longrightarrow & U_{-1} \end{array} \quad \text{in } \mathcal{C}.$$

In this case, U^+ is called the Cech nerve of $U_0 \xrightarrow{u} U_{-1}$.

Def. A simplicial obj. u of ∞-cat. $\mathcal{C}$ is called an effective groupoid if it can be extended to a colim diagram $U^+ : \Delta_+^{op} \to \mathcal{C}$ s.t. U^+ is a Cech nerve.

1.2. Grothendieck topologies and sheaves.

Def. $\mathcal{C}$ ∞-cat., a sieve on $\mathcal{C}$ is a full subcat. $\mathcal{C}^{(0)} \subset \mathcal{C}$ s.t. for any $D \in \mathcal{C}^{(0)}$, $f : C \to D$ in $\mathcal{C}$, we have $C \in \mathcal{C}^{(0)}$.

For any $C \in \mathcal{C}$, a sieve on C is a sieve on $\mathcal{C}_{/C}$.

A Grothendieck topology on $\mathcal{C}$ consists of, for every $C \in \mathcal{C}$, a collection of sieves on C called covering sieves satisfying:

(1) $\forall\ C \in \mathcal{C}$, the sieve $\mathcal{C}_{/C} \subset \mathcal{C}_{/C}$ on C is a covering sieve

(2) $\forall\ f: C \to D$ in $\mathcal{C}$, f^* preserves covering sieves

(3) $\forall\ C \in \mathcal{C}$, covering sieve $\mathcal{C}^{(0)}_{/C}$ on C, $\mathcal{C}^{(1)}_{/C}$ an arbitrary sieve on C. If for each $f: D \to C$ belonging to $\mathcal{C}^{(0)}_{/C}$, the pullback $f^*\mathcal{C}^{(1)}_{/C}$ is a covering sieve on D then $\mathcal{C}^{(1)}_{/C}$ is a covering sieve on C.

RMK. Grothendieck topology on $\mathcal{C}$ is essentially the same as Grothendieck topology on $h\mathcal{C}$.

Prop. $\mathcal{C}$ ∞-cat., $C \in \mathcal{C}$, $j: \mathcal{C} \to P(\mathcal{C})$ Yoneda embedding

we have a bijection $\{\text{subobjs of } j(C)\} \xrightarrow{\sim} \{\text{sieves on } C\}$

$(\text{monomorphism } i: U \to j(C)) \mapsto \mathcal{C}_{/C}(U)$

$\mathcal{C}_{/C}(U) \subset \mathcal{C}_{/C}$ spanned by $f: D \to C$ s.t. $\exists$

$$\begin{tikzcd} j(D) \arrow[r, "j(f)"] \arrow[d] & j(C) \\ U \arrow[ur, "i"'] & \end{tikzcd}$$

Def. $\mathcal{C}$ ∞-cat., S a collection of morphisms in $\mathcal{C}$. An obj. Z of $\mathcal{C}$ is called S-local if every $s: X \to Y$ belonging to S, composition with s induces a htpy equiv

$$\mathrm{Map}_{\mathcal{C}}(Y, Z) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{C}}(X, Z).$$

Def. $\mathcal{C}$ ∞-cat. equipped with a Grothendieck topology, $j: \mathcal{C} \to P(\mathcal{C})$ Yoneda embedding. Consider S the collection of all monomorphisms corresponding to the covering sieves. A presheaf $\mathcal{F} \in P(\mathcal{C})$ is called a sheaf if it is S-local.

Denote $\mathrm{Shv}(\mathcal{C}) \subset P(\mathcal{C})$ full subcat. spanned by sheaves.

HTT 6.2.2.7 : $\mathrm{Shv}(\mathcal{C})$ is an ∞-topos.

1.3. The ∞-cat. of ∞-topoi.

Def. $\mathcal{X}, \mathcal{Y}$ ∞-topoi. A geom. morphism between $\mathcal{X}$ and $\mathcal{Y}$ is a pair of adjoint functors $\mathcal{X} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{Y}$ s.t. f^* is left exact.

Define subcat. $\mathcal{L}Top$, $\mathcal{R}Top \subset Cat_\infty$ as follows:

(1) objs are ∞-topoi

(2) a functor $f^* : \mathcal{X} \to \mathcal{Y}$ ($f_* : \mathcal{X} \to \mathcal{Y}$) of ∞-topoi belong to $\mathcal{L}Top$ ($\mathcal{R}Top$) if it preserves small colims and finite lims

it has a left exact left adjoint.

RMK. $\mathcal{L}Top \simeq \mathcal{R}Top^{op}$.

HTT 6.3.2-4: $\mathcal{L}Top$, $\mathcal{R}Top$ admit lims and colims.

1.4 Etale morphisms of ∞-topoi.

Prop. $\mathcal{X}$ ∞-topos, $U \in \mathcal{X}$, then

(1) $\mathcal{X}_{/U}$ is ∞-topos

(2) $\pi_! : \mathcal{X}_{/U} \to \mathcal{X}$ has a right adjoint commuting with colims

hence π^* has a right adjoint $\pi_*: \mathcal{X}_{/U} \to \mathcal{X}$ and (π^*, π_*) gives a geom. morphism of ∞-topoi.

Def. Such geom. morphisms are called etale morphisms.

HTT 6.3.5.13 : Let $RTop_{et} \subset RTop$ spanned by all ∞-topoi and etale geom. morphisms. Then $RTop_{et}$ admits small colims, i.e. we can glue ∞-topoi along etale open subsets.

2. Structured sheaves.

2.1 Sheaves with values in an ∞-cat.

"ringed space"

Def. $\mathcal{C}$ ∞-cat., $\mathcal{X}$ ∞-topoi, a $\mathcal{C}$-valued sheaf on $\mathcal{X}$ is a functor $\mathcal{X}^{op} \to \mathcal{C}$ which preserves small lims.

Denote $Shv_{\mathcal{C}}(\mathcal{X}) \subset Fun(\mathcal{X}^{op}, \mathcal{C})$ spanned by $\mathcal{C}$-valued sheaves on $\mathcal{X}$.

Def. T ∞-cat. equipped with a Grothendieck topology.

$\mathcal{C}$ ∞-cat., a functor $\mathcal{O}: T^{op} \to \mathcal{C}$ is called a $\mathcal{C}$-valued sheaf on T if for every $X \in T$, covering sieve $T^{\circ}_{/X}$ the composite map $(T^{\circ}_{/X})^{\triangleleft} \subset (T_{/X})^{\triangleleft} \to T \xrightarrow{\mathcal{O}^{op}} \mathcal{C}^{op}$ is a colim diagram in $\mathcal{C}^{op}$.

Denote $\mathrm{Shv}_{\mathcal{C}}(T) \subset \mathrm{Fun}(T^{op}, \mathcal{C})$.

Ex. $\mathrm{Shv}(T) \simeq \mathrm{Shv}_{\mathcal{S}}(T)$.

Prop. For T as above, $j: T \to P(T)$ Yoneda, we have $L: P(T) \to \mathrm{Shv}(T)$ sheafification functor left adjoint to the inclusion. $\mathcal{C}$ ∞-cat. admitting small lims. Then the composition with $L \circ j$ induces equiv. of ∞-cat.

$$\mathrm{Shv}_{\mathcal{C}}(\mathrm{Shv}(T)) \xrightarrow{\sim} \mathrm{Shv}_{\mathcal{C}}(T).$$

2.2. Geometries.

Idea: in order to make sense of "locally ringed" we need an extra struc. on ∞-cat. $\mathcal{C}$.

Def. $\mathcal{G}$ ∞-cat., an admissibility struc. on $\mathcal{G}$ consists of the following data:

(1) a subcat. $\mathcal{G}^{ad} \subset \mathcal{G}$ spanned by all the objs and admissible morphisms satisfying

(i) pullback of admissible morphism always exists and is again admissible

(ii)
$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd}$$
g, h admissible $\Rightarrow f$ admissible

(iii) retract of admissible morphism is admissible

(2) a Grothendieck topology on $\mathcal{G}$, generated by admissible morphisms in the following sense: any covering sieve $\mathcal{G}^{(0)}_{/X}$ contains a covering sieve generated by collection of adm. mor.

Def. A geom. consists of the following data:

(1) an ∞-cat. $\mathcal{G}$ which admits finite lims and is idempotent complete

(2) an adm. struc. on $\mathcal{G}$.