DAG.

Def. $\mathcal{G}, \mathcal{G}'$ geom., a functor $f: \mathcal{G} \to \mathcal{G}'$ is called a transformation of geom. if it preserves finite lims, adm. morphisms and adm. coverings.

Def. $\mathcal{G}$ geom., $\mathcal{X}$ ∞-topos. A $\mathcal{G}$-struc. on $\mathcal{X}$ is a left exact functor $\mathcal{O}: \mathcal{G} \to \mathcal{X}$ s.t. for every adm. covering $\{U_\alpha \to U\}$ in $\mathcal{G}$, the induced map $\coprod_\alpha \mathcal{O}(U_\alpha) \to \mathcal{O}(U)$ is an effective epimorphism in $\mathcal{X}$.

(the Cech nerve is a colim diagram)

We call the pair $(\mathcal{X}, \mathcal{O})$ a $\mathcal{G}$-structed topos.

Denote $\mathrm{Str}_{\mathcal{G}}(\mathcal{X}) \subset \mathrm{Fun}(\mathcal{G}, \mathcal{X})$ spanned by $\mathcal{G}$-struc. on $\mathcal{X}$.

Given $\mathcal{O}, \mathcal{O}': \mathcal{G} \to \mathcal{X}$ $\mathcal{G}$-strucs, a natural transformation $\alpha: \mathcal{O} \to \mathcal{O}'$ is called a local transformation of $\mathcal{G}$-strucs if for every adm. $U \to X$ in $\mathcal{G}$, we get

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathcal{O}'(U) \arrow[d] \\ \mathcal{O}(X) \arrow[r] & \mathcal{O}'(X) \end{tikzcd}$$

a pullback in $\mathcal{X}$.

Denote $\mathrm{Str}_{\mathcal{G}}^{loc}(\mathcal{X}) \subset \mathrm{Str}_{\mathcal{G}}(\mathcal{X})$ spanned by local trans.

A morphism $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ of $\mathcal{G}$-struc. topoi consists of a geom. morphism $f^*: \mathcal{Y} \to \mathcal{X}$ of ∞-topoi and a morphism $\alpha: f^*\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ in $\mathrm{Str}_{\mathcal{G}}^{loc}\mathcal{X}$.

We have an ∞-cat. of $\mathcal{G}$-struc. topoi denoted by $\mathcal{L}\mathrm{Top}(\mathcal{G})^{op}$.

A geom. $\mathcal{G}$ is discrete if the adm. morphisms are exactly the equivalences, and the Grothendieck topology on $\mathcal{G}$ is trivial, i.e. a sieve $\mathcal{G}^0_{/X} \subset \mathcal{G}_{/X}$ on $X \in \mathcal{G}$ is a covering sieve if and only if $\mathcal{G}^0_{/X} = \mathcal{G}_{/X}$.

Ex. Let $\mathcal{G}_{zar} = \{$affine schemes of f.t. over $\mathbb{Z}\}$

$$= \{\text{f.g. comm. rings}\}^{op}.$$

We endow it with a struc. of geom. as follows:

(1) $\mathrm{Spec}\,A \to \mathrm{Spec}\,B$ is adm. if it induces an isom. $B_b \simeq A$ for some $b \in B$.

(2) $\{\mathrm{Spec}\,A_{a_i} \to \mathrm{Spec}\,A\}_{i \in I}$ is a covering if $\{a_i\}_I$ generate the unit ideal in A.

For X topological space, $\mathcal{X} = \mathrm{Shv}(X)$ ∞-cat. of sheaves of spaces on X. Then $\mathcal{G}_{zar}$-struc. on $\mathcal{X}$ are exactly sheaves of comm. rings $\mathcal{O}$ on the topological space X which are local.

2.3 Spectrum functor.

Recall: A comm. ring, $\mathrm{Spec}\,A$ is characterized by the universal property: for every locally ringed space $(X, \mathcal{O}_X)$ we have a canonical bijection:

$$\mathrm{Hom}_{\text{locally ringed space}}((X,\mathcal{O}_X), \mathrm{Spec}\,A)$$

$$\simeq \mathrm{Hom}_{\text{comm. ring}}(A, \Gamma(X,\mathcal{O}_X))$$

$$\simeq \mathrm{Hom}_{\text{ringed space}}((X,\mathcal{O}_X), (*, A))$$

Note that locally ringed space $\overset{\text{full}}{\subset} \mathcal{L}\mathrm{Top}(\mathcal{G}_{\mathrm{Zar}})^{op}$

ringed space $\overset{\text{full}}{\subset} \mathcal{L}\mathrm{Top}(\mathcal{G}_{\mathrm{Zar},\,\mathrm{discrete}})^{op}$

Thm. V 2.1.1 Let $f: \mathcal{G} \to \mathcal{G}'$ be a transformation of geom., then the induced functor $\mathcal{L}\mathrm{Top}(\mathcal{G}') \to \mathcal{L}\mathrm{Top}(\mathcal{G})$ admits a left adjoint denoted by $\mathrm{Spec}_{\mathcal{G}}^{\mathcal{G}'}$ called the relative spectrum functor attached to f.

Let $\mathcal{G}$ be a geom. and $\mathcal{G}_0$ the associated discrete geom.

Let $\mathrm{Spec}^{\mathcal{G}}$ denote the composition

$$\mathrm{Ind}(\mathcal{G}^{op}) \simeq \mathrm{Str}_{\mathcal{G}_0}(\mathcal{S}) \to \mathrm{LTop}(\mathcal{G}_0) \xrightarrow{\mathrm{Spec}^{\mathcal{G}}_{\mathcal{G}_0}} \mathrm{LTop}(\mathcal{G})$$

We call $\mathrm{Spec}^{\mathcal{G}}$ the absolute spectrum functor attached to $\mathcal{G}$.

RMK.

1. The analytification functor from algebraic geom. to analytic geom. is an example of relative spectrum functor.

2. Consider the functor $\mathrm{LTop}(\mathcal{G}) \times \mathcal{G} \to \mathcal{S}$

$$((\mathcal{X}, \mathcal{O}), U) \longmapsto \mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, \mathcal{O}(U))$$

it induces a functor $\mathrm{LTop}(\mathcal{G}) \to \mathrm{Fun}(\mathcal{G}, \mathcal{S})$ which factors through $\mathrm{Fun}^{\text{left exact}}(\mathcal{G}, \mathcal{S}) \simeq \mathrm{Ind}(\mathcal{G}^{op})$.

Let $\Gamma_{\mathcal{G}}: \mathrm{LTop}(\mathcal{G}) \to \mathrm{Ind}(\mathcal{G}^{op})$ and call it the $\mathcal{G}$-struc. global section functor. Then $\mathrm{Spec}^{\mathcal{G}}$ is left adjoint to $\Gamma_{\mathcal{G}}$.

3. We also have explicit descriptions for $\mathrm{Spec}^{\mathcal{G}} X$, $X \in \mathrm{Ind}(\mathcal{G}^{op})^{op}$

2.4 Structured schemes.

Def. $\mathcal{G}$ geom., a morphism $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ in $\mathrm{LTop}(\mathcal{G})$ is called etale if

1) the underlying geom. morphism $f^*: \mathcal{X} \to \mathcal{Y}$ of ∞-topoi is etale

2) the induced map $f^* \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{Y}}$ is an equiv. in $\mathrm{Str}_{\mathcal{G}}(\mathcal{Y})$.

Ex. If U is an object of $\mathcal{X}$, let $\mathcal{O}_{\mathcal{X}}|_U$ denote the $\mathcal{G}$-struc. on $\mathcal{X}_{/U}$ by $\mathcal{G} \xrightarrow{\mathcal{O}_{\mathcal{X}}} \mathcal{X} \xrightarrow{\pi^*} \mathcal{X}_{/U}$. Then $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$ is an etale morphism in $\mathrm{LTop}(\mathcal{G})$.

Def. $\mathcal{G}$ geom., a $\mathcal{G}$-struc. topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is an affine $\mathcal{G}$-scheme if it is equiv. to $\mathrm{Spec}^{\mathcal{G}} A$ for some $A \in \mathrm{Pro}(\mathcal{G}) \simeq \mathrm{Ind}(\mathcal{G}^{op})^{op}$.

It is called a $\mathcal{G}$-scheme if there is a collection of objects $\{U_\alpha\}$ of X s.t.

1) $\{U_\alpha\}$ covers X, i.e. $\coprod_\alpha U_\alpha \to 1_X$ is effective epi.

2) $\forall \alpha$, $(X/U_\alpha, \mathcal{O}_X|_{U_\alpha})$ is an affine $\mathcal{G}$-scheme.

Denote $Sch(\mathcal{G}) \subset LTop(\mathcal{G})^{op}$ spanned by $\mathcal{G}$-schemes.

Prop.

1) $Sch(\mathcal{G})$ admits colims along etale morphisms

2) If the Grothendieck topology on $Pro(\mathcal{G})$ is precanonical then $Sch(\mathcal{G})$ admits finite lims.

Ex.

1) $\mathcal{G}_{zar} = \{$ affine schemes of f.t. over $\mathbb{Z}\}$

adm. morphisms are inclusions of principle open subsets

coverings coverings by

Then we have a fully faithful embedding Scheme $\hookrightarrow Sch(\mathcal{G}_{Zar})$

and the essential image consists of $\mathcal{G}_{Zar}$ - schemes with

0-localic underlying ∞ - topos.

2) $\mathcal{G}_{et} = \{$ affine schemes of f.t. over $\mathbb{Z}\}$

adm. morphisms are etale morphisms

coverings coverings

Then we have a fully faithful embedding DM-stacks $\hookrightarrow Sch(\mathcal{G}_{et})$

and the essential image consists of $\mathcal{G}_{et}$ - schemes with

1- localic underlying ∞ - topos.

2.5 Pregeometries.

Idea: generate geom. with simpler data.

Def. A pregeom. is an ∞ - cat. T admitting finite

products, equipped with an admisibility struc.

Def. T pregeom., $\mathcal{X}$ ∞-topos, a T-struc. on $\mathcal{X}$ is a functor $\mathcal{O}: T \to \mathcal{X}$ s.t.

1) $\mathcal{O}$ preserves finite products

2) $\mathcal{O}$ preserves pullbacks along adm. morphisms

3) $\forall$ adm. covering $\{U_\alpha \to X\}$ in T, the induced map $\coprod_\alpha \mathcal{O}(U_\alpha) \to \mathcal{O}(X)$ is an effective epimorphism in $\mathcal{X}$

Denote $\mathrm{Str}_T(\mathcal{X}) \subset \mathrm{Fun}(T, \mathcal{X})$, $\mathrm{Str}_T^{loc}(\mathcal{X}) \subset \mathrm{Str}_T(\mathcal{X})$.

A transformation of pregeom. $f: T \to T'$ is a functor which preserves finite products, adm. morphisms, adm. coverings and pullbacks along adm. morphisms.

A transformation of pregeom. $f: \mathcal{T} \to \mathcal{G}$ exhibits $\mathcal{G}$ as a geom. envelope of $\mathcal{T}$ if

1) $\mathcal{G}$ is a geom. with the coarest struc. s.t. f is a transformation of pregeom.

2) $\forall$ ∞-cat. $\mathcal{C}$ idempotent complete admitting finite lims composition with f induces an equiv.

$$\mathrm{Fun}^{\text{left exact}}(\mathcal{G}, \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}^{\text{ad}}(\mathcal{T}, \mathcal{C})$$

preserve finite products, pullbacks along adm. mor.