DAG.

Motivation.

5. Deformation theory.

X sm. alg. var. $/\mathbb{C}$ then

(a) $\forall$ 1st-order deformation X_1 of X

$\{$aut. of X_1, id on $X\} \simeq H^0(X, T_X)$

$T_X = (\Omega_X^1)^\vee$

(b) $\{$Isom. classes of $X_1\} \simeq H^1(X, T_X)$

Question: beyond smooth? L_X occur in DAG

Answer: cotangent complex instead of Kahler differentials

replace $H^0(X, T_X)$ by $\mathrm{Hom}(L_X, \mathcal{O}_X)$

$H^1(X, T_X)$ $\qquad$ $\mathrm{Ext}^1(L_X, \mathcal{O}_X)$

Note that $H^0(L_X) \simeq \Omega_X^1$ and $L_X \simeq \Omega_X^1$ if X sm.

Ex. X proj. var. $/\mathbb{C}$, $\mathcal{F}$ QCoh on X

$\rightsquigarrow$ quot scheme Quot by Grothendieck

classifying quotients of $\mathcal{F}$, i.e. exact

sequences $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$

in QCoh.

Want to compute cotangent complex of Quot at a

point $[\mathcal{F}'']$:

0th-coh. = Zar. cotangent space

$= \mathrm{Hom}(\mathcal{F}', \mathcal{F}'')$

higher coh. difficult to compute

as Quot is usually singular

Moral: The cotangent complex of Quot is not the

right object, instead study the derived

enhancement Quot^+ and its cotangent complex which at a point $[\mathcal{J}'']$ is simply

$$\mathrm{RHom}(\mathcal{J}', \mathcal{J}'').$$

Ex. Obstructions.

X sm. alg. var. / $\mathbb{C}$

canonical obstruction class map

$$\rho : \{1\text{st-order def. of } X\} \longrightarrow H^2(X, T_X)$$

and a 1st-order def. can be extended to 2nd-order def. $\iff$ vanish under ρ

Questions: H^0, H^1 has geom. meanings. what about H^2 as well as higher H^n ?

Prop. Let R be the square-zero extension $\mathbb{C} \oplus \mathbb{C}[n]$

(a) $\forall$ def. X' of X over R

$\{$aut. of X', id on $X\}$ / htpy $\simeq H^0(X, T_X)$

(b) $\{$def. of X over $R\}$ / htpy $\simeq H^{n+1}(X, T_X)$

Relation with obs. class map.

We have pullback of cdga:

$$\begin{tikzcd} \mathbb{C}[\varepsilon]/\varepsilon^3 \arrow[r] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C}[\varepsilon]/\varepsilon^2 \arrow[r] & \mathbb{C} \oplus \mathbb{C}[1] \end{tikzcd}$$

$\rightsquigarrow$ pushout of dg schemes

$$\begin{tikzcd} \operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^3 & \operatorname{Spec} \mathbb{C} \arrow[l] \\ \operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2 \arrow[u] & \operatorname{Spec} \mathbb{C} \oplus \mathbb{C}[1] \arrow[l] \arrow[u] \end{tikzcd}$$

$\Rightarrow$ 1st - order def. X_1 extends to a 2nd - order def. $\Leftrightarrow$ pullback of X_1 to $\mathrm{Spec}\ \mathbb{C} \oplus \mathbb{C}[1]$ is a trivial def. of X

This pullback is equiv. to the obs. class map.

II. Infinity cat.

In derived geom. always need to consider things up to htpy equiv. In order to keep track of all htpy involved, use language of infinity cat.

We will use sSet as models for htpy types.

1. sSet.

Def. $[n] = \{0 < 1 < \cdots < n\}$ linearly ordered set.

Cat. of combinatorial simplices Δ :

obj : $[n]$, $n \geq 0$

Mor : $[n] \to [m]$ non-decreasing

A sSet is a functor $S_\cdot : \Delta^{op} \to$ sets

$S_\cdot([n]) = S_n$ set of n-simplices

S_0 set of vertices

S_1 set of edges

$S_\cdot([m] \to [n]) : S_n \to S_m$ describe glueing

Set_Δ has all lims and colims

$\mathrm{Set}_\Delta = \mathrm{Fun}(\Delta^{op}, \mathrm{set})$ cat. of sSet.

Ex. $\Delta^n = \mathrm{Hom}_\Delta(-, [n]) \in \mathrm{Set}_\Delta$ standard n-simplex

$\forall S_\cdot$, Yoneda $\Rightarrow S_n = \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, S_\cdot)$

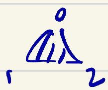
$0 < i < n$

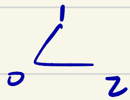
ith horn Λ^n_i

boundary $\partial \Delta^n$ the simplicial subset of Δ^n whose

m-simplices are non-decreasing non-surj. $[m] \to [n]$.

s.t. $\mathrm{Im}([m]) \cup \{i\} \neq [n]$

Δ^2 : 

Λ^2_0 : 

Def. A simplicial set $S_\bullet$ is a Kan cpx if $\forall\ 0 \le i \le n$, $\forall$ map $\sigma_0 : \Lambda^n_i \to S_\bullet$ can be extended to an n-simplex $\sigma : \Delta^n \to S_\bullet$.

Kan : cat. of Kan cpxes.

$S_\bullet, T_\bullet \in \mathrm{Set}_\Delta$, $f, g : S_\bullet \to T_\bullet$, a simplicial htpy from f to g is a map of simplicial sets $h : S_\bullet \times \Delta^1 \to T_\bullet$ s.t. $h|_{S_\bullet \times \{0\}} = f$ and $h|_{S_\bullet \times \{1\}} = g$. f and g are called simplicially htpy.

If T. Kan cpx, this is an equiv. relation.

Htpy cat. of Kan cpxes hKan: Kan cpxes with Mor being simplicial htpy classes of maps.

2. ∞-cat.

Question: How to incorporate htpy data into an ordinary cat.?

Recall: $\mathcal{C}$ cat. $\rightsquigarrow$ simplicial set $N(\mathcal{C})$ nerve

$$N(\mathcal{C})_n = \mathrm{Fun}(\underbrace{\cdot \to \cdot \to \cdots \to \cdot}_{n+1}, \mathcal{C})$$

$$= \{x_0 \to \cdots \to x_n,\ x_i \in \mathcal{C}\}$$

Face maps are compositions

Deg maps are insertions of id.

Prop. $\mathcal{C} \longmapsto N(\mathcal{C})$ gives a fully faithful embedding

Cat. $\longrightarrow$ Set_Δ. The essential image

cat. of small cat.

consists of S. satisfying 0031

$\forall\, 0 < i < n$, $\forall$

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & S. \\ \Delta^n \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

Def. An ∞-cat. $\mathcal{C}$ (modeled as a weak Kan cpx)

is a sSet s.t. $\forall\, 0 < i < n$, $\forall$

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

uniqueness dropped

as we want to

consider ntpy.

Ex. $i=1,\ n=2$

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^2 \arrow[ur, dotted] & \end{tikzcd}$$

existence of composition but non-unique

Basic notions.

$$\begin{tikzcd} & D \arrow[d, "\mathrm{id}_D"] \\ C \arrow[ur, "f"] \arrow[r, "g"'] & D \end{tikzcd}$$

Def. $\mathcal{C}$ ∞-cat. $C, D \in \mathcal{C}$

$f, g: C \to D$, a htpy from f to g is a 2-simplex σ of $\mathcal{C}$ satisfying

$d_0 \sigma = \mathrm{id}_D$

$d_1 \sigma = g$

$d_2 \sigma = f$

$\mathcal{C}$ ∞-cat.

0-simplices are called objs

1- mors

01J5

$\forall X, Y \in \mathcal{C}$, mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is the Kan cpx whose n-simplices are maps $\Delta^n \times \Delta^1$ to $\mathcal{C}$ sending $\Delta^n \times \{0\}$ to vertex X

$\times \{1\}$ to Y.

relation between htpy of morphisms of sSet: $S, T \in$ sSet. view $S \to T$ as vertices in $\mathrm{Fun}(S, T)$ then htpy notions agree.

$\mathcal{C} \rightsquigarrow h\mathcal{C}$ htpy cat.

$\mathcal{C}$ ∞-cat, $\mathcal{D}$ ord. cat.

$h\mathcal{C}$ obj same as $\mathcal{C}$

$\mathrm{Hom}_{\mathrm{Cat}}(h\mathcal{C}, \mathcal{D}) \simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathcal{C}, N\mathcal{D})$

Mor: $\underbrace{\pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y)}_{\text{set of htpy classes of morphisms } X \to Y \text{ in } \mathcal{C}}$

$\mathcal{H}$-enriched htpy cat. obj same as $\mathcal{C}$

Mor: $[\mathrm{Map}_{\mathcal{C}}(X, Y)] \in \mathcal{H}$

$\uparrow$

htpy cat. of spaces

Def. A morphism in $\mathcal{C}$ is called an equiv. if its image in $h\mathcal{C}$ is an isom. Two objs are equiv. if there is an equiv. morphism.

$\mathcal{C}, \mathcal{D}$ ∞-cat., a functor $\mathcal{C} \to \mathcal{D}$ is a map of simplicial sets. The ∞-cat. of functors

$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) = \mathrm{Map}_{\mathrm{Set}_\Delta}(\mathcal{C}, \mathcal{D})$, its set of n-simplices are $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathcal{C} \times \Delta^n, \mathcal{D})$.

$F : \mathcal{C} \to \mathcal{D}$ is an equiv. of cat. iff $\exists$

$G : \mathcal{D} \to \mathcal{C}$ s.t. $F \circ G$, $G \circ F$ is equiv. to id

in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$, $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$.

fully faithful

$F : \mathcal{C} \to \mathcal{D}$ is essentially surj. iff $hF : h\mathcal{C} \to h\mathcal{D}$

is ess. surj. 01JG

fully faithful on $\mathcal{H}$-enriched htpy cat.

i.e. $\forall X, Y \in \mathcal{C}$

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))$$

is htpy equiv.

Equiv. $\Leftrightarrow$ fully faithful + ess. surj.

$\mathcal{C}$ ∞-cat. $(h\mathcal{C})' \subset h\mathcal{C}$ subcat.

$$\Rightarrow \begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ N((h\mathcal{C})') & \longrightarrow & N(h\mathcal{C}) \end{array}$$ pullback of sset

$\mathcal{C}'$ is called subcat. of $\mathcal{C}$ spanned by $(h\mathcal{C})'$

$\mathcal{C}' \subset \mathcal{C}$ is called full subcat. if $(h\mathcal{C})' \subset h\mathcal{C}$ is.

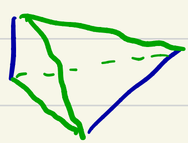
$\mathcal{C}$ ∞-cat.

$X \in \mathcal{C}$ is called $\begin{array}{c}\text{final}\\ \text{initial}\end{array}$ if $\forall\, Y \in \mathcal{C}$, $\begin{array}{c}\mathrm{Map}_{\mathcal{C}}(Y, X)\\ \mathrm{Map}_{\mathcal{C}}(X, Y)\end{array}$ contractible

3. Limits and colims.

Def. S, S' sSet $\Rightarrow$ $S \star S'$: $(S \star S')_n = S_n \cup S'_n \cup \bigcup_{i+j=n-1} S_i \times S'_j$

Ex. $\Delta^i \star \Delta^j \simeq \Delta^{i+j+1}$

$| \star / \simeq$ 

Prop. (Joyal) The join of 2 weak Kan is weak Kan.

Def. K sset, left cone $K^{\triangleleft} = \Delta^0 \star K$

right cone $K^{\triangleright} = K \star \Delta^0$

Prop. (Joyal) $p: K \to S$ map of sset, then $\exists$ sset $S_{/p}$ with universal property:

$$\mathrm{Hom}_{\mathrm{set}_\Delta}(Y, S_{/p}) = \mathrm{Hom}_p(Y \star K, S)$$

$$= \{ f \mid f|_K = p \}$$

Pf. $(S_{/p})_n = \mathrm{Hom}_p(\Delta^n \star K, S)$. //

Prop. $p: K \to S$ map of sset, S weak Kan

$\Rightarrow S_{/p}$ weak Kan

called overcat.

$p: K = \Delta^0 \longmapsto X \in \mathcal{C}$, $\mathcal{C}_{/X} = \mathcal{C}_{/p}$

Dually replace $Y \star K$ by $K \star Y$ get undercat $\mathcal{C}_{p/}$.

Def. $\mathcal{C}$ ∞-cat. $p : K \to \mathcal{C}$ map of sSet

a colim for p is an initial obj of $\mathcal{C}_{p/}$

lim final $\mathcal{C}_{/p}$

A colim diagram is the associated $\bar{p} : K^{\triangleright} \to \mathcal{C}$

extending p, refer to $\bar{p}(\infty) \in \mathcal{C}$ as the colim

of p. Same for lim.