DAG.

4. Examples.

∞-cat. arise naturally by inverting a collection of morphisms in an ord. cat.

Given ∞-cat. $\mathcal{C}$ and collection of morphisms W in $\mathcal{C}$ we can construct an ∞-cat. $\mathcal{C}[W^{-1}]$ with $\alpha: \mathcal{C} \to \mathcal{C}[W^{-1}]$ satisfying universal property $\forall$ ∞-cat. $\mathcal{D}$, composition with α gives fully faithful embedding

$$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

whose essential image consists of those functors which carry morphism in W to an equiv. in $\mathcal{D}$.

Ex.

1. Kan : cat. of Kan cpxes

W : collection of htpy equiv.

$\mathcal{S} = \mathrm{Kan}[W^{-1}]$ called the ∞-cat. of spaces

2. WKan: cat. of weak Kan cpxes

(model for our ∞-cat.)

W : collection of equiv. of ∞-cat.

$\mathrm{Cat}_\infty = \mathrm{WKan}[W^{-1}]$ called the ∞-cat. of (small) ∞-cats

In practice, W comes from model cat.

5. Fibrations of ∞-cat.

Idea: want a family of ∞-cat. parametrized by an ∞-cat.

$$\mathcal{D}_{/\bar f} = \left\{ \begin{tikzcd}[column sep=small] & z \arrow[dl] \arrow[dr] & \\ \bar x \arrow[rr, "\bar f"'] & & \bar y \end{tikzcd} \right\}$$

$$\begin{tikzcd} \mathcal{C} \arrow[d, "p"] \\ \mathcal{D} \end{tikzcd}$$

$\forall d \in \mathcal{D}$, $\mathcal{C}_d = \mathcal{C} \times_{\mathcal{D}} \{d\}$ ∞-cat.

$\forall\, d \xrightarrow{f} d'$ in $\mathcal{D}$, functor $f^*: \mathcal{C}_{d'} \to \mathcal{C}_d$

so for each $y \in \mathcal{C}_{d'}$, we need to choose

a morphism $x \xrightarrow{\tilde f} y$ in $\mathcal{C}$ lifting f

and set $f^* y = x$.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "p"'] & y \arrow[d, "p"] \\ \bar x \arrow[r, "\bar f"] & \bar y \end{tikzcd}$$

Def. $p: \mathcal{C} \to \mathcal{D}$ ∞-cats.

$f: x \to y$ in $\mathcal{C}$ is called p-Cartesian if

it is a final obj. in $\mathcal{C}_{/y} \times_{\mathcal{D}_{/\bar y}} \mathcal{D}_{/\bar f}$ where the

bars denote image by p.

If f is p-Cart. it is called a p-Cart. lift

of $\bar y$ at y.

p is called a Cart. fib. iff $\forall\, y \in \mathcal{C}$ and

$\bar f: \bar x \to p(y)$ there is a p-Cart. lift at y.

Dually f is called p-coCart. if it is Cart. w.r.t.

$$p^{op}: \mathcal{C}^{op} \to \mathcal{D}^{op}$$

p is called coCart. if p^{op} is Cart.

HTT 2.4.1.10

f is p-Cart. $\Leftrightarrow$ $\forall z \in \mathcal{C}$, we have a pullback of mapping spaces

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(z, x) \arrow[r, "f\circ"] \arrow[d, "p"] & \mathrm{Map}_{\mathcal{C}}(z, y) \arrow[d, "p"] \\
\mathrm{Map}_{\mathcal{D}}(\bar{z}, \bar{x}) \arrow[r, "\bar{f}\circ"] & \mathrm{Map}_{\mathcal{D}}(\bar{z}, \bar{y})
\end{tikzcd}$$

Families of initial objs in a Cart. fib.

HTT 2.4.4.9 $p: \mathcal{C} \to \mathcal{D}$ Cart. fib. of ∞-cat.

assume $\forall d \in \mathcal{D}$, $\mathcal{C}_d$ has an initial obj.

Then $\exists$ functor $g : \mathcal{D} \to \mathcal{C}$ which is a section of p

s.t. $g(d)$ is an initial obj. , $\forall\, d \in \mathcal{D}$.

Ex. of Cart. fib.

HTT 2.4.7.12 $f : \mathcal{C} \to \mathcal{D}$ ∞-cats.

The projection

$$\begin{array}{c} \mathrm{Fun}(\Delta^1, \mathcal{D}) \times_{\mathrm{Fun}(\{1\}, \mathcal{D})} \mathcal{C} \\ p \downarrow \\ \mathrm{Fun}(\{0\}, \mathcal{D}) \end{array}$$

is a Cart fib.

if f $\mathrm{id}_{\mathcal{C}}$, taking source is Cart. fib.

Moreover a morphism in $\mathrm{Fun}(\Delta^1, \mathcal{D}) \times_{\mathrm{Fun}(\{1\}, \mathcal{D})} \mathcal{C}$ is

p-Cart. $\Leftrightarrow$ its image in $\mathcal{C}$ is equiv.

$\mathcal{C}$ ∞-cat.

$ev_0 : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ is Cart. fib.

$ev_1 : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ is coCart. fib.

Moreover if $\mathcal{C}$ has pushouts, ev_0 is also coCart.
pullbacks, ev_1 Cart.

Straightening and unstraightening.

HTT 3.2.0.1 $\mathcal{C}$ ∞-cat. We have equiv. of ∞-cat.

$$\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat}_\infty) \underset{St.}{\overset{Un.}{\rightleftarrows}} (\mathrm{Cat}_\infty)^{Cart.}_{/\mathcal{C}}$$

where RHS is subcat. of $(\mathrm{Cat}_\infty)_{/\mathcal{C}}$ spanned by Cart. fib. and functors preserving Cart. fibs.

Ex. Apply straightening, get functors

$$\mathcal{C} \longrightarrow \mathrm{Cat}_\infty \qquad\qquad \mathcal{C} \longrightarrow \mathrm{Cat}_\infty$$

$$x \longmapsto \mathcal{C}_{/x} \qquad\qquad x \longmapsto \mathcal{C}_{x/}$$

$$(x \xrightarrow{f} y) \longmapsto (\mathcal{C}_{/x} \xrightarrow{f\circ} \mathcal{C}_{/y}) \qquad (x \xrightarrow{f} y) \longmapsto (\mathcal{C}_{y/} \xrightarrow{\circ f} \mathcal{C}_{x/})$$

if $\mathcal{C}$ has pushouts

$$\mathcal{C} \longrightarrow \mathrm{Cat}_\infty$$

$$x \longmapsto \mathcal{C}_{x/}$$

$$(x \xrightarrow{f} y) \longmapsto (\mathcal{C}_{x/} \xrightarrow{- \amalg_x y} \mathcal{C}_{y/})$$

$\mathcal{C}$ has pullbacks

$$\mathcal{C} \longrightarrow \mathrm{Cat}_\infty$$

$$x \longmapsto \mathcal{C}_{/x}$$

$$(x \xrightarrow{f} y) \longmapsto (\mathcal{C}_{/y} \xrightarrow{- \times_y x} \mathcal{C}_{/x})$$

Unstraightening allows us explicitly construct lims and colims in Cat_∞.

HTT 3.3.3.2

$F : I \longrightarrow \mathrm{Cat}_\infty \overset{\mathrm{Un}}{\rightsquigarrow}$ Cart. fib. $p : X \longrightarrow I^{op}$

then $\lim_{I} F \simeq \mathrm{Fun}^{\mathrm{Cart.}}_{/I^{op}}(I^{op}, X)$

RHS denotes the full subcat. of $\mathrm{Fun}_{/I^{op}}(I^{op}, X)$ spanned by functors sending every morphism of I^{op} to a p-Cart. morphism in X.

Ex. Consider diagrams of ∞-cats.

$$\begin{tikzcd} & \mathcal{E} \arrow[d, "g"] \\ \mathcal{C} \arrow[r, "f"] & \mathcal{D} \end{tikzcd}$$

$\overset{\mathrm{Un.}}{\Longrightarrow}$ Cart. fib. $\mathcal{X} = \{\mathcal{C} \quad \mathcal{D} \quad \mathcal{E}\}$

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"] \\ \{*_1 & *_2 \arrow[l, "\alpha"'] \arrow[r, "\beta"] & *_3\} \end{tikzcd}$$

By HTT 2.4.1.10. $\forall$ $c \in \mathcal{C}$, $d \in \mathcal{D}$, $e \in \mathcal{E}$ we have

$$\mathrm{Map}_{\mathcal{X}}(d, c) \simeq \mathrm{Map}_{\mathcal{D}}(d, f(c))$$

$$\mathrm{Map}_{\mathcal{X}}(d, e) \simeq \mathrm{Map}_{\mathcal{D}}(d, g(e))$$

$$\mathrm{Map}_{\mathcal{X}}(c, d) \simeq \mathrm{Map}_{\mathcal{X}}(e, d) \simeq \mathrm{Map}_{\mathcal{X}}(c, e) \simeq \phi$$

a section s of p consists of :

- $c = s(*_1) \in \mathcal{C}$, $d = s(*_2) \in \mathcal{D}$, $e = s(*_3) \in \mathcal{E}$
- $s(\alpha) : d \to c$ in $\mathcal{X}$, i.e. $d \to f(c)$ in $\mathcal{D}$
- $s(\beta) : d \to e$ in $\mathcal{X}$, i.e. $d \to g(e)$ in $\mathcal{D}$

$\mathrm{Map}_{\mathcal{X}}(d', c) \simeq \mathrm{Map}_{\mathcal{D}}(d', f(c))$

s sends α, β to p-cart. $\Rightarrow$ $f(c) \simeq d \simeq g(e)$

$\Rightarrow$ Cart sections s of p $\rightsquigarrow$ obj. in $\mathcal{C} \times_{\mathcal{D}} \mathcal{E}$.

For colim we have

HTT 3.3.4.3. diagram $F: I \to Cat_\infty$

$\overset{Un.}{\rightsquigarrow}$ CoCart. fib. $p: X \to I$

Let W be the collection of all p-coCart in X

$$\operatorname{colim}_I F \simeq X[W^{-1}].$$

Fib. of spaces.

Def. A right/left fib is a Cart./coCart. fib.

$p: \mathcal{E} \to \mathcal{D}$ whose fibres are spaces i.e. $\mathcal{S}$.

St. & Un.

HTT 2.2.1.2 $\mathcal{C}$ ∞-cat. We have equiv. of

∞-cat. $$Fun(\mathcal{C}^{op}, \mathcal{S}) \underset{St.}{\overset{Un.}{\rightleftarrows}} (Cat_\infty)^{rfib.}_{/\mathcal{C}}$$

where RHS is the subcat. of $(Cat_\infty)_{/\mathcal{C}}$ spanned by right fib.

6. Kan extensions.

Def. $f : \mathcal{C} \to \mathcal{D}$ ∞-cats.

$\bar{p} : K^{\triangleright} \to \mathcal{C}$ diagram, $p = \bar{p}|_K$

$\bar{p}$ is called an f-colim of p if

$$\mathcal{C}_{\bar{p}/} \xrightarrow{\sim} \mathcal{C}_{p/} \times_{\mathcal{D}_{fp/}} \mathcal{D}_{f\bar{p}/}$$

Ex. $f : \mathcal{C} \to *$, $\bar{p} : K^{\triangleright} \to \mathcal{C}$ is an f-colim iff it is a colim.

Def. Given commutative diagram of ∞-cats.

$$\begin{tikzcd} \mathcal{C}^{\circ} \arrow[r, "F_0"] \arrow[d, "\text{full}"'] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{C} \arrow[r] \arrow[ur, dashed, "F"] & \mathcal{D}' \end{tikzcd}$$

base

want to extend F_0 to F

$c: \Delta^0 \to \mathcal{C}$, $\mathrm{id} \in \mathrm{Hom}(\mathcal{C}_{/c}, \mathcal{C}_{/c}) \rightsquigarrow c: \mathcal{C}_{/c} \star \Delta^0 = \mathcal{C}_{/c}^{\triangleright} \to \mathcal{C}$

F is called a p-left Kan ext. of F_0 at $c \in \mathcal{C}$ if the induced diagram

$$\begin{tikzcd} \mathcal{C}^0_{/c} \arrow[r, "F_c"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ (\mathcal{C}^0_{/c})^{\triangleright} \arrow[r] \arrow[ur, "F(c)"] & \mathcal{D}' \end{tikzcd}$$

exhibits $F(c)$ as a p-colim of F_c

F is called a p-left Kan ext. of F_0 if it is a p-left Kan ext. of F_0 at $\forall c \in \mathcal{C}$.

HTT 4.3.2.15 Given ∞-cat. $\mathcal{C}^0 \overset{\text{full}}{\subset} \mathcal{C}$

$$\begin{tikzcd} \mathcal{C} \arrow[dr] & & \mathcal{D} \arrow[dl, "p"] \\ & \mathcal{D}' & \end{tikzcd}$$

Let $K \subset \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}, \mathcal{D})$ be the full subcat. spanned by functors that are left Kan ext. of their restrictions to $\mathcal{C}^0$.

Let $K' \subset \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}^{\circ}, \mathcal{D})$ be the full subcat. spanned by functors $F_0 : \mathcal{C}^{\circ} \to \mathcal{D}$ s.t. $\forall\, C \in \mathcal{C}$, the induced diagram $\mathcal{C}^{\circ}_{/C} \to \mathcal{D}$ has a p-colim. Then the restriction functor $K \to K'$ is an equiv. of ∞-cats.

HTT 4.3.2.16 Assume $\forall\, F_0 \in \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}^{\circ}, \mathcal{D})$, $\exists\, F \in \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}, \mathcal{D})$ which is a p-left Kan ext. of F_0. Then the restriction map

$$i^* : \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}^{\circ}, \mathcal{D})$$

admits a section $i_!$ whose essential image consists of precisely functors F which are p-left ext. of $F|_{\mathcal{C}^{\circ}}$.

called left Kan ext. functor

left Kan ext. $\xleftrightarrow{\text{dual}}$ right Kan ext.

7. Presentable ∞-cats.

Def. $\mathcal{C}$ ∞-cat. $P(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ called ∞-cat. of presheaves on $\mathcal{C}$.

$\mathcal{D}$ ∞-cat. $P_{\mathcal{D}}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{D})$ ∞-cat. of $\mathcal{D}$-valued presheaves.

HTT 5.1.2.3. $P(\mathcal{C})$ has all small lims and colims and can be computed pointwise.

Construction of the Yoneda embedding.

Cat. of twisted arrows: I, J linearly ordered sets let $I \star J$ denote the coproduct $I \amalg J$ equipped $I \leq J$.

Δ : cat. of combinatorial simplices

Functor $Q : \Delta \longrightarrow \Delta$

$$I \longmapsto I \star I^{op}$$

$$[n] \longmapsto [2n+1]$$

For $\mathcal{C}$ ∞-cat. define $\mathrm{TwArr}(\mathcal{C})$ to be the sSet

$$[n] \longmapsto \mathcal{C}(Q[n]) = \mathcal{C}([2n+1])$$

informally objs of $\mathrm{TwArr}(\mathcal{C})$ are morphisms $f : C \to D$ in $\mathcal{C}$ and morphisms are commutative diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & D \\ C' \arrow[r, "f'"] & D' \arrow[u] \end{tikzcd}$$

We have canonical inclusions $I \hookrightarrow I \star I^{op} \hookleftarrow I^{op}$

inducing $\mathcal{C} \leftarrow \mathrm{TwArr}(\mathcal{C}) \rightarrow \mathcal{C}^{op}$

HAlg. 5.2.1.3. The canonical map

$\lambda : \mathrm{TwArr}(\mathcal{C}) \longrightarrow \mathcal{C} \times \mathcal{C}^{op}$ is a right fib

(of sSet).

St. $\Rightarrow$ functor $\mathcal{C} \times \mathcal{C}^{op} \longrightarrow \mathcal{S}$

and get Yoneda $j : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$

HTT 5.1.3.1, 2

The Yoneda j is fully faithful, preserving all small lims in $\mathcal{C}$

HTT 5.1.5.8

The Yoneda j generates $P(\mathcal{C})$ under small colims.