DAG.

Adjoint functors.

Def. Consider Cart. fib $M \xrightarrow{p} \Delta^1$

$$\begin{tikzcd} \mathcal{C} & \mathcal{D} \\ 0 \arrow[r] & 1 \end{tikzcd}$$

$$\Big\updownarrow \text{St.}$$

functor $g : \mathcal{D} \to \mathcal{C}$

Let $\mathcal{C}, \mathcal{D}$ be ∞-cats. An adjunction between them is a functor $q : M \to \Delta^1$ which is both Cart. fib and coCart. fib. together with equiv. $\mathcal{C} \simeq M_0$ and $\mathcal{D} \simeq M_1$.

Let $f : \mathcal{C} \rightleftarrows \mathcal{D} : g$ be functors associated with M by St. We say f is left adjoint to g and ...

HTT 5.2.2.8 $\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$. TFAE.

- f left adjoint to g

02EJ

- $\exists$ unit transformation $u : id_{\mathcal{C}} \to g \circ f$ s.t.

$\forall c \in \mathcal{C}$, $d \in \mathcal{D}$,

$$\begin{array}{c} \mathrm{Map}_{\mathcal{D}}(f(c), d) \to \mathrm{Map}_{\mathcal{C}}(gf(c), g(d)) \\ \downarrow u(c) \\ \mathrm{Map}_{\mathcal{C}}(c, g(d)) \end{array}$$

is htpy equiv. (of Kan cpxes).

HTT 5.2.3.5 $\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$ adjunction. Then

f preserves all colims in $\mathcal{C}$

g lims $\mathcal{D}$.

Prop. $f : \mathcal{C} \to \mathcal{D}$ ∞-cat. $\mathcal{E}$ ∞-cat.

Composition with f gives functor

$$f^* : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

The left / right adjoint to f^* is the left / right Kan extension functor. These adjoints exist iff every functor $\mathcal{C} \to \mathcal{E}$ has a Kan ext.

Def. A full subcat $\mathcal{C}^\circ \subset \mathcal{C}$ is called a localization of $\mathcal{C}$ if the inclusion has a left adjoint.

RMK. $\mathcal{C}_0 \underset{i_0}{\overset{L}{\rightleftarrows}} \mathcal{C}$, $\mathcal{C}_0 \xrightarrow{i_0} \mathcal{C} \to \mathcal{C}[W^{-1}]$ is equiv.

where W collection of α sent by L to equiv.

set-theory notion

Def. An ∞-cat is called presentable if it is accessible and admits small colims.

HTT 5.5.2.4 A presentable ∞-cat. admits all small lims.

HTT 5.5.1.1 (Simpson) An ∞-cat. is presentable $\Leftrightarrow$ it arises as an accessible localization of an ∞-cat. of presheaves.

Adjoint functor theorem (HTT 5.5.2.9)

Let $F: \mathcal{C} \to \mathcal{D}$ of presentable ∞-cat.

(1) F has right adjoint $\Leftrightarrow$ it preserves small colims

(2) F has left adjoint $\Leftrightarrow$ it preserves small lims and κ-filtered colims for some regular cardinal κ

HTT 5.5.4.15 $\mathcal{C}$ presentable ∞-cat. Then

accessible localizations of $\mathcal{C}$

$\Updownarrow$

inverting collection of morphisms that are strongly saturated and of small generation.

8. Stable ∞-cat. "linearized ∞-cats for doing alg."

Def. An ∞-cat. $\mathcal{C}$ is stable if

(1) $\exists$ zero object $0 \in \mathcal{C}$

(2) every morphism g in $\mathcal{C}$ admits a fibre and a cofibre, i.e.

$\exists$ pullback
$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ 0 \arrow[r] & Y \end{tikzcd}$$
$\exists$ pushout
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

W fibre of g $\qquad$ Z cofibre of g

(3) a triangle in $\mathcal{C}$ is a fibre sequence $\Leftrightarrow$

it is a cofibre sequence, i.e.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

is pullback $\Leftrightarrow$ pushout

fibre seq. cofibre seq.

Def. $\mathcal{C}$ stable ∞-cat., $X \in \mathcal{C}$, form (co)fibre seq

$$\begin{tikzcd} \Omega X \quad X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \\ & X \end{tikzcd}$$

called suspension of X

loop of X

HTT 4.3.2.15 $\exists$ suspension functor Σ : $\mathcal{C} \to \mathcal{C}$

loop functor Ω

and they are mutually inverse equiv.

$\Rightarrow$ adjoint to each other

For $n \geq 0$, let $X[n] = \Sigma^n X$, $X[-n] = \Omega^n X$.

RMK. $\mathcal{C}$ stable ∞-cat. , $f: X \to Y$ in $\mathcal{C}$, form pullback diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & \Sigma X = X[1] \end{tikzcd}$$

Then the image of $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ in htpy cat. $h\mathcal{C}$ is called a distinguished triangle and the collection of dist. triangle endows $h\mathcal{C}$ the struc. of a triangulated cat.

HA 1.1.3.4. A stable ∞-cat. admits all finite lims and colims. and all pullbacks coincide with pushouts.

Equivalent def. of stable ∞-cat.

HA 1.4.2.27 Let $\mathcal{C}$ be a pointed ∞-cat. (i.e. has a zero obj.). TFAE.

(1) $\mathcal{C}$ stable

(2) $\mathcal{C}$ has finite colims, $\Sigma: \mathcal{C} \to \mathcal{C}$ is equiv.

(3) $\mathcal{C}$ has finite lims, $\Omega: \mathcal{C} \to \mathcal{C}$ is equiv.

HA 1.1.4.1 $F: \mathcal{C} \to \mathcal{D}$ of stable ∞-cat. is called exact iff the following equiv. condition hold:

(1) F preserves fibre seq.

(2) F left exact, i.e. preserving finite lims

(3) F right exact, i.e. preserving finite colims

HA 1.1.4.4, 1.1.4.6 Let $\mathrm{Cat}_\infty^{\mathrm{Ex}} \subset \mathrm{Cat}_\infty$ be the subcat. spanned by stable ∞-cats. and exact functors.

Then $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ admits small lims and small filtered colims and they are preserved by the inclusion.

Def. A t-struc. on a stable ∞-cat $\mathcal{C}$ is a pair of full subcats $\mathcal{C}_{\geq 0}$, $\mathcal{C}_{\leq 0}$ s.t.

(1) $\forall\, X \in \mathcal{C}_{\geq 0}$, $Y \in \mathcal{C}_{\leq 0}$, $\pi_0 \mathrm{Map}(X, Y[-1]) = 0$

(2) $\mathcal{C}_{\geq 1} \subset \mathcal{C}_{\geq 0}$, $\mathcal{C}_{\leq -1} \subset \mathcal{C}_{\leq 0}$ where $\mathcal{C}_{\geq n} = \mathcal{C}_{\geq 0}[n]$

$\mathcal{C}_{\leq n} = \mathcal{C}_{\leq 0}[n]$

(3) $\forall\, X \in \mathcal{C}$, $\exists$ fibre seq. $X' \to X \to X''$ where

$X' \in \mathcal{C}_{\geq 0}$, $X'' \in \mathcal{C}_{\leq -1}$

HA 1.2.1.5 $\forall n \in \mathbb{Z}$, $\mathcal{C}_{\le n} \subset \mathcal{C}$ is a localization

$\rightsquigarrow$ left adjoint $\tau_{\le n}$ truncation

dually $\mathcal{C}_{\ge n} \subset \mathcal{C}$ has right adjoint $\tau_{\ge n}$

Def. The heart $\mathcal{C}^{\heartsuit}$ = full subcat $\mathcal{C}_{\ge 0} \cap \mathcal{C}_{\le 0} \subset \mathcal{C}$

$\pi_0 = \tau_{\le 0} \circ \tau_{\ge 0} \simeq \tau_{\ge 0} \circ \tau_{\le 0} : \mathcal{C} \to \mathcal{C}^{\heartsuit}$

$\pi_n = \pi_0 \circ [-n]$

$\pi_0 Map(X, \Omega^n Y) \simeq \pi_0 Map(\Sigma^n X, Y) \simeq \pi_0 \Omega^n Map(X, Y)$

RMK. For $X, Y \in \mathcal{C}^{\heartsuit}$, $\pi_n Map(X, Y) \simeq \pi_0 Map(X, Y[-n])$

$= 0$ for $n > 0$

$\Rightarrow \mathcal{C}^{\heartsuit}$ is a 1-cat., in fact an abelian cat.

Ex. $\mathcal{A}$ Grothendieck abelian cat.

$Ch(\mathcal{A})$ cat. of chain cpxes

$D(\mathcal{A}) = Ch(\mathcal{A})[qis^{-1}]$ inverting all quasi-isoms

called the derived ∞-cat.

HA 1.3.5 $D(\mathcal{A})$ is presentable stable ∞-cat.

It has t-struc. $D(\mathcal{A})_{\geq 0}$, $D(\mathcal{A})_{\leq 0}$

spanned by chain cpxes with vanishing homology for $i<0$

The proof uses dg structure and model struc. to study the localization.

Notation.

Right bounded derived ∞-cat. $D^{-}\mathcal{A} \subset D\mathcal{A}$ spanned by chain cpxes M with $H_i M = 0$, $i \gg 0$

Left bounded derived ∞-cat. $D^{+}\mathcal{A} \subset D\mathcal{A}$

Classically $D^{-}\mathcal{A}$ is constructed from proj. resolutions

$D^{+}\mathcal{A}$ inj.

9. Spectra and stabilization.

Want to construct stable ∞-cat. from ∞-cats. with finite lims or "linearize" ∞-cat.

Idea: formally invert the loop functor.

Def. $\mathcal{C}$ ∞-cat. admitting finite lims. Let $\mathcal{C}_* = \mathcal{C}_{*/}$ the ∞-cat. of pointed objs of $\mathcal{C}$. The ∞-cat. of spectrum objs of $\mathcal{C}$

$$\mathrm{Sp}\,\mathcal{C} = \lim\left(\cdots \longrightarrow \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \right)$$

Def. ∞-cat. of finite spaces $\mathcal{S}^{fin} \subset \mathcal{S}$ full subcat. generated by $*$ under finite colims.

$\mathcal{C}$ ∞-cat. admitting finite lims, equivalently define a spectrum obj. of $\mathcal{C}$ a reduced, excisive functor $X : \mathcal{S}^{fin} \to \mathcal{C}$

take final to final | take pushouts to pullbacks

$\mathrm{Sp}\,\mathcal{C} = \mathrm{Exc}_*(\mathcal{S}_*^{fin}, \mathcal{C}) \subset \mathrm{Fun}(\mathcal{S}_*^{fin}, \mathcal{C})$ full

subcat. spanned by spectrum objs.

HA 1.4.2.17 $\mathrm{Sp}\,\mathcal{C}$ is stable ∞-cat.

Notation. $\Omega^\infty : \mathrm{Sp}\,\mathcal{C} \to \mathcal{C}$ evaluation at 0-sphere S^0

$\forall n \in \mathbb{Z}$, $\Omega^{\infty-n} = \Omega^\infty \circ [n]$

Adjoint functor theorem $\Rightarrow$ Ω^∞ has left adjoint

$\Sigma_+^\infty : \mathcal{C} \to \mathrm{Sp}\,\mathcal{C}$.

Next specialize to $\mathcal{C} = \mathcal{S}$.

Def. A spectrum is a spectrum obj. of $\mathcal{S}$

The ∞-cat. of spectra $\mathrm{Sp} = \mathrm{Sp}\,\mathcal{S}_*$

sphere spectrum $\mathbb{S} = \Sigma_+^\infty * \in \mathrm{Sp}$

More concretely, a spectrum is a sequence of pointed spaces $\{X_n\}_{n\geq 0}$ equipped with htpy equiv. $X_n \xrightarrow{\simeq} \Omega X_{n+1}$

HA 1.4.3.6. The full subcat. $Sp_{\leq -1} \subset Sp$ spanned by X s.t. $\Omega^{\infty} X \in \mathcal{S}$ is contractible. This gives a t-struc. on Sp whose heart is equiv. to the cat. of abelian groups.