DAG

II. Derived Rings.

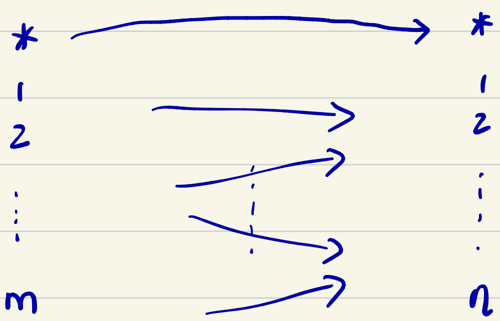
1. ∞ - Operads

Our language to make sense of commutativity and associativity up to htpy.

Def. Segal's cat. Fin_* of pointed finite sets

obj : $\langle n \rangle = \{*, 1, \dots, n\}$ $\quad n \geq 0$

Mor : $\alpha : \langle m \rangle \longrightarrow \langle n \rangle$ preserving $*$



$\forall\ 1 \le i \le n$, let $\rho^i : \langle n \rangle \to \langle 1 \rangle$ sending i to 1, others to $*$.

Def. $f : \langle m \rangle \to \langle n \rangle$ in Fin_* is inert if $\forall\ i \in \langle n \rangle^\circ = \{1, \dots, n\}$, $f^{-1}\{i\}$ has exactly one element.

An ∞-operad is a functor $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ of ∞-cats s.t.

(1) $\forall$ inert $f : \langle m \rangle \to \langle n \rangle$ in Fin_*,

$\forall\ C \in \mathcal{O}^{\otimes}_{\langle m \rangle}$, $\exists$ p-coCart. $\bar{f} : C \to C'$

in $\mathcal{O}^{\otimes}$ lifting f, any such $\bar{f}$ is called inert. In particular f induces functor

$$f_! : \mathcal{O}^{\otimes}_{\langle m \rangle} \longrightarrow \mathcal{O}^{\otimes}_{\langle n \rangle}$$

(2) let $c \in \mathcal{O}^{\otimes}_{\langle m\rangle}$, $c' \in \mathcal{O}^{\otimes}_{\langle n\rangle}$, $f: \langle m\rangle \to \langle n\rangle$.

Let $\mathrm{Map}^{f}_{\mathcal{O}^{\otimes}}(c, c') \subset \mathrm{Map}_{\mathcal{O}^{\otimes}}(c, c')$ be

union of conn. comp. lying over f.

Choose p-coCart. $c' \to c_i'$ lying over inert

$\rho^i: \langle n\rangle \to \langle 1\rangle$ for $1 \le i \le n$. Then the induced

map

$$\mathrm{Map}^{f}_{\mathcal{O}^{\otimes}}(c, c') \longrightarrow \prod_{1\le i\le n} \mathrm{Map}^{\rho^i \circ f}_{\mathcal{O}^{\otimes}}(c, c_i')$$

is htpy equiv.

(3) $\forall\, n \geq 0$, the functors $\{\rho^i_! : \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O} = \mathcal{O}^{\otimes}_{\langle 1\rangle}\}_{1\le i\le n}$

determine an equiv. of ∞-cat. $\phi: \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}^{n}$.

$\mathcal{O} = \mathcal{O}^{\otimes}_{\langle 1\rangle}$ is called the underlying ∞-cat. of $\mathcal{O}^{\otimes}$.

Given $X_1, \ldots, X_n, Y \in \mathcal{O}$, let $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}, Y)$ be the union of components of $\mathrm{Mor}_{\mathcal{O}^\otimes}(X_1 \oplus \cdots \oplus X_n, Y)$ which lie over the unique $\beta : \langle n \rangle \longrightarrow \langle 1 \rangle$ s.t. $\beta^{-1}\{*\} = \{*\}$.

Idea. We think of an ∞-operad as a cat. $\mathcal{O}$ together with "multi-morphisms" $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}, Y)$ whose compositions are associative up to htpy.

Ex. The commutative ∞-operad $\mathrm{Comm}^\otimes = \mathrm{Fin}_* \xrightarrow{id} \mathrm{Fin}_*$.

Def. Let $\mathcal{O}^\otimes$, $\mathcal{O}'^\otimes$ ∞-operads. An ∞-operad map is a functor $f : \mathcal{O}^\otimes \longrightarrow \mathcal{O}'^\otimes$ of ∞-cats over Fin_* carrying inert mor to inert mor.

Let $\mathrm{Alg}_{O}(O')$ denote the full subcat. of

$\mathrm{Fun}_{\mathrm{Fin}_*}(O^{\otimes}, O'^{\otimes})$ spanned by ∞-operad maps.

More generally

$$\begin{tikzcd} O^{\otimes} \arrow[dr] & & O'^{\otimes} \arrow[dl] \\ & O''^{\otimes} \arrow[d] & \\ & \mathrm{Fin}_* & \end{tikzcd}$$

, $\mathrm{Alg}_{O/O''}(O')$.

Def. $O^{\otimes}$ ∞-operad, a coCart. fib. $p: \mathcal{C}^{\otimes} \to O^{\otimes}$ is called a coCart. fib. of ∞-operads if the comp. $\mathcal{C}^{\otimes} \to O^{\otimes} \to \mathrm{Fin}_*$ is ∞-operad and $\mathcal{C}$ is called an O-monoidal ∞-cat.

Idea. $\forall f \in \mathrm{Mul}_{O}(\{X_i\}, Y)$, the coCart. fib. p determines a functor $\otimes_f : \prod \mathcal{C}_{X_i} \to \mathcal{C}_Y$.

Ex. A symmetric monoidal ∞-cat. $\mathcal{C}$ is a Fin_*-monoid ∞-cat., i.e. a coCart. fib. $\mathcal{C}^\otimes \to \mathrm{Fin}_*$ s.t. $\forall\, n \geq 0$ the maps $\{\rho^i : \langle n \rangle \to \langle 1 \rangle\}$ induces functors $\rho^i_! : \mathcal{C}^\otimes_{\langle n \rangle} \to \mathcal{C}^\otimes_{\langle 1 \rangle}$ which determines an equiv. $\mathcal{C}^\otimes_{\langle n \rangle} \xrightarrow{\sim} (\mathcal{C}^\otimes_{\langle 1 \rangle})^n$.

$\mathrm{CAlg}_{\mathcal{C}} = \mathrm{Alg}_{\mathrm{Fin}_*}\mathcal{C}$ called ∞-cat. of comm. alg. objs in $\mathcal{C}$.

The morphisms $\alpha : \langle 0 \rangle \to \langle 1 \rangle$, $\beta : \langle 2 \rangle \to \langle 1 \rangle$ in Fin_* determines functors $\Delta^0 \to \mathcal{C}$, $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ corresponding to unit obj. $1 \in \mathcal{C}$ and tensor product.

Def. $\mathcal{C}^\otimes$ $\mathcal{D}^\otimes$ $\mathcal{O}$-monoidal ∞-cats.

$$\begin{array}{ccc} \mathcal{C}^\otimes & & \mathcal{D}^\otimes \\ p \downarrow & & \downarrow q \\ & \mathcal{O}^\otimes & \end{array}$$

$f \in \mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$ is an $\mathcal{O}$-monoidal functor if it carries p-coCart. mor to q-coCart. mor.

If $\mathcal{O}^\otimes = \mathrm{Fin}_*$, we say sym. monoidal functor.

Def. $\mathcal{C}$ ∞-cat., a sym. monoidal struc. on $\mathcal{C}$ is called Cart. iff

- the unit obj. $1_{\mathcal{C}} \in \mathcal{C}$ is final
- $\forall C, D \in \mathcal{C}$, the canonical maps

$$C \simeq C \otimes 1_{\mathcal{C}} \leftarrow C \otimes D \rightarrow 1_{\mathcal{C}} \otimes D \simeq D$$

exhibit $C \otimes D$ as a product of C and D in $\mathcal{C}$.

Dually we have coCart. sym. monoidal struc.

HA 2.4.1. $\mathcal{C}$ ∞-cat. admitting finite products. Then $\mathcal{C}$ admits a Cart. sym. monoidal struc. unique up to equiv.

In this case, the ∞-cat. $\mathrm{CAlg}\,\mathcal{C}$ admits a more direct description in terms of monoids.

2. Algebras and modules.

$\mathcal{C}^{\otimes}$ sym. monoidal ∞-cat. $\mathcal{O}^{\otimes} = \mathrm{Fin}_*$

$\mathrm{CAlg}\,\mathcal{C} = \mathrm{Alg}_{\mathcal{O}}\,\mathcal{C}$

evaluate at $\langle 1 \rangle \in \mathrm{Fin}_*$ $\rightsquigarrow$ forgetful functor

$\theta : \mathrm{CAlg}\,\mathcal{C} \to \mathcal{C}$

HA 3.1.3.14 assume $\mathcal{C}$ admits countable colims and $\otimes$ preserves countable colims. Then θ admits a left adjoint which is informally given by

$$C \longmapsto \mathrm{Sym}^* C = \coprod_n C^{\otimes n}/\Sigma_n$$

∞-categorical quotient by the sym. group

HA 3.2

1) If $\mathcal{C}$ complete (has small lims) so is $\mathrm{CAlg}\,\mathcal{C}$ and θ detects lims (preserves and reflects)

$\mathcal{C}$ cat. of abelian grps, CAlg $\mathcal{C}$ cat. of comm. rings

lims are same.

2) If $\mathcal{C}$ is cocomplete (has small colims) and $\otimes$ is compatible with small colims then CAlg $\mathcal{C}$ is cocomplete and θ detects only sifted colims.

A simplicial set K is sifted if it is nonempty and the diagonal map $K \longrightarrow K \times K$ is cofinal.

Idea: sifted $\supset$ filtered + quotient by equiv. relations

3) If $\mathcal{C}$ is presentable and $\otimes$ is compatible with colims then CAlg $\mathcal{C}$ is presentable.

4) $\exists$ coCart. sym. monoidal struc. on CAlg $\mathcal{C}$ s.t. θ is monoidal functor. (Informally the tensor product on $\mathcal{C}$ induces tensor product on comm. alg. objs).

Def. Associative operad. Assoc.

It is a colored operad having a single object $\mathfrak{a}$.

$\forall$ finite set I, the set of operations

$\mathrm{Mul}_{\mathrm{Assoc}}(\{\mathfrak{a}\}_{i \in I}, \mathfrak{a})$ is the set of

linear orderings on I.

Composition of linear orderings.

$\rightsquigarrow$ ∞-operad $\mathrm{Assoc}^{\otimes} \to \mathrm{Fin}_*$ called associative ∞-operad.

If $\mathcal{C}^{\otimes}$ ∞-operad equipped with fib. $q: \mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$

define $\mathrm{Alg}\,\mathcal{C} = \mathrm{Alg}_{/\mathrm{Assoc}}\,\mathcal{C}$ ∞-operad sections of q.

called ∞-cat. of associative alg. objs of $\mathcal{C}$.

A monoidal ∞-cat. is a coCart. fib. of ∞-operads

$\mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$.

HA 4.1.1.14 Let $p: \mathcal{C}^{\otimes} \longrightarrow \mathrm{Fin}_*$ ∞-operad, then $\mathcal{C}^{\otimes}$ is a sym. monoidal ∞-cat. if and only if the induced map $p': \mathrm{Assoc}^{\otimes} \times_{\mathrm{Fin}_*} \mathcal{C}^{\otimes} \longrightarrow \mathrm{Assoc}^{\otimes}$ is a monoidal ∞-cat.

Def. left module operad LM.

It is a colored operad having two elements $\mathfrak{a}$, $\mathfrak{m}$

$\forall$ objs $\{X_i\}_{i \in I}$ and Y of LM

$$\mathrm{Mul}_{LM}(\{X_i\}_I, Y) = \begin{cases} X_i = Y = \mathfrak{a}, & \text{collection of all linear orderings on } I \\ Y = \mathfrak{m}, \text{ exactly one } X_i = \mathfrak{m}, & \text{collection of linear orderings on } I \text{ s.t. the last is } X_i = \mathfrak{m} \\ \text{empty set} & \text{all other cases} \end{cases}$$

$\rightsquigarrow$ ∞-operad $LM^{\otimes}$.

We have natural embeddings $\mathrm{Assoc}^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes}$

and fib. $\mathcal{LM}^{\otimes} \longrightarrow \mathrm{Assoc}^{\otimes}$

If we have $\mathcal{C}^{\otimes} \longrightarrow \mathrm{Assoc}^{\otimes}$ fib. of ∞-operads,

the ∞-cat. of left modules $\mathrm{LMod}\,\mathcal{C} = \mathrm{Alg}_{\mathcal{LM}/\mathrm{Assoc}}\,\mathcal{C}$

For $A \in \mathrm{Alg}\,\mathcal{C}$, define ∞-cat. of left A-mod

$$\mathrm{LMod}_A\,\mathcal{C} = \mathrm{LMod}\,\mathcal{C} \times_{\mathrm{Alg}\,\mathcal{C}} \{A\}$$

HA 4.2. $\mathcal{C}$ monoidal ∞-cat., $A \in \mathrm{Alg}\,\mathcal{C}$. forgetful

functor $\theta : \mathrm{LMod}_A\,\mathcal{C} \to \mathcal{C}$.

(1) θ has a left adjoint given by free module

(2) $\mathcal{C}$ complete, so is $\mathrm{LMod}_A\,\mathcal{C}$ and θ detects lims

(3) $\mathcal{C}$ cocomplete, $\otimes$ preserves colim, then $\mathrm{LMod}_A\,\mathcal{C}$ cocomplete and θ detects colims.

(4) $\mathcal{C}$ presentable, $\otimes$ preserves colims, then $\mathrm{LMod}_A \mathcal{C}$ is presentable.

(5) $\mathcal{C}$ sym. monoidal, admits simplicial colims, $\otimes$ preserves simplicial colims, $A \in \mathrm{CAlg}\,\mathcal{C}$. Then $\mathrm{Mod}_A \mathcal{C} = \mathrm{LMod}_A \mathcal{C}$ is sym. monoidal ∞-cat. and θ is lax-monoidal (i.e. only a map of ∞-operads).