DAG.

Ex. The smash product sym. monoidal struc. on Sp. ∞-cat. Sp of spectra admits a sym. monoidal struc. uniquely determined by

(a) the bifunctor $\otimes : Sp \times Sp \to Sp$ preserves small colims separately in each variable

(b) the unit object of Sp is the sphere spectra S.

Lurie's construction: consider the Cart. sym. monoidal struc. on $Pr^{st} \subset Cat_\infty$ spanned by presentable stable ∞-cats. and colim-preserving functors. Realize Sp as the unit obj. of Pr^{st} $\Rightarrow$ inherits sym. monoidal struc. and uniqueness.

3. Algebra in the stable htpy cat.

3.1 Rings and modules. (HA 7)

sequences of ∞-operads : $\mathbb{E}_0^{\otimes} \hookrightarrow \mathbb{E}_1^{\otimes} \to \cdots \to \mathbb{E}_\infty^{\otimes}$

$\mathbb{E}_k^{\otimes}$ ∞-operad of little k-cubes

$\mathbb{E}_1^{\otimes} \simeq \mathrm{Assoc}^{\otimes}$, $\mathbb{E}_\infty^{\otimes} \simeq \mathrm{Comm}^{\otimes} = \mathrm{Fin}_*$

Def. An $\mathbb{E}_k$-ring is an $\mathbb{E}_k$-alg. obj. of Sp.

Let $\mathrm{Alg}^{(k)} = \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Sp})$, $\mathrm{Alg} = \mathrm{Alg}^{(1)}$, $\mathrm{CAlg} = \mathrm{Alg}^{(\infty)}$.

Def. For R $\mathbb{E}_1$-ring, $\mathrm{LMod}_R = \mathrm{LMod}_R(\mathrm{Sp})$ the ∞-cat. of left R-mods.

HA 7.1.1.5 LMod_R and RMod_R are stable.

Def. For R E_1-ring, $\forall n \in \mathbb{Z}$, let $\pi_n R$ be the n-th htpy gp of the underlying spectra.

Recall adjunction $\Sigma^\infty_+ : \mathcal{S} \rightleftarrows Sp(\mathcal{S}) : \Omega^\infty$

$\Rightarrow \pi_n R \simeq \pi_0 \mathrm{Map}_{Sp}(S[n], R)$

As $\otimes$ is exact in each variable

$\Rightarrow S[n] \otimes S[m] \simeq S[n+m]$.

$$\begin{array}{c}
\mathrm{Map}_{Sp}(S[n], R) \times \mathrm{Map}_{Sp}(S[m], R) \\
\downarrow \\
\mathrm{Map}_{Sp}(S[n] \otimes S[m], R \otimes R) \\
\downarrow \\
\mathrm{Map}_{Sp}(S[n+m], R)
\end{array}$$

$\Rightarrow$ bilinear $\pi_n R \times \pi_m R \longrightarrow \pi_{n+m} R$

$\rightsquigarrow$ graded asso. ring struc. on $\pi_* R = \bigoplus \pi_n R$.

If R is $\mathbb{E}_k$-ring, $k \geq 2$, then the multiplication on $\pi_* R$ is graded commutative i.e. $x \in \pi_n R$, $y \in \pi_m R$ $xy = (-1)^{nm} yx$. In particular $\pi_0 R$ is commutative ring and $\pi_n R$ is $\pi_0 R$-mod.

If R $\mathbb{E}_1$-ring, M left R-mod. The action $R \otimes M \to M$ gives bilinear $\pi_n R \times \pi_m M \to \pi_{n+m} M$.

$\Rightarrow$ $\pi_* M$ has the struc. of graded left $\pi_* R$-mod.

Def. A spectra X is called connective if $\pi_n X \simeq 0$ for $n < 0$. Denote $Sp^{cn} \subset Sp$ full subcat. spanned by connective spectra.

An $\mathbb{E}_k$-ring is called connective if its underlying spectra is connective.

$\mathrm{Alg}^{cn} \subset \mathrm{Alg}$. $\mathrm{CAlg}^{cn} \subset \mathrm{CAlg}$.

Notation.

R $\mathbb{E}_1$-ring, $\mathrm{LMod}_R^{\overset{\leq 0}{\geq 0}} \subset \mathrm{LMod}_R$ spanned by R-mods M with $\pi_n M = 0$, $\forall n \overset{>}{<} 0$.

$M, N \in \mathrm{LMod}_R$, $\mathrm{Ext}_R^i(M, N) = \pi_0 \mathrm{Map}_{\mathrm{LMod}_R}(M, N[i])$.

HA 7.1.1.13. R connective $\mathbb{E}_1$-ring, then $(\mathrm{LMod}_R^{\geq 0}, \mathrm{LMod}_R^{\leq 0})$ gives accessible t-struc. on LMod_R. Moreover π_0 gives equiv. of $\mathrm{LMod}_R^{\heartsuit}$ with abelian cat. of left $\pi_0 R$-mod inducing right t-exact $\theta: \mathcal{D}^- \mathrm{LMod}(\pi_0 R) \to \mathrm{LMod}_R$.

HA 7.1.1.15 If R discrete i.e. $\pi_n R = 0$, $\forall n \neq 0$ then θ induces equiv. of $\mathcal{D}^- \mathrm{LMod}(\pi_0 R)$ with the ∞-cat. of right bounded objs of LMod_R, and can be extended to equiv. of ∞-cat. $\mathcal{D}\mathrm{LMod}(\pi_0 R) \simeq \mathrm{LMod}_R$ and can be promoted to equiv. of sym. monoidal ∞-cats.

Recognition principle: when is a stable ∞-cat. of the form $LMod_R$ or $RMod_R$?

Schwede-Shipley thm: $\mathcal{C}$ stable ∞-cat. equiv. to $RMod_R$ for some E_1-ring R $\Leftrightarrow$ $\mathcal{C}$ presentable and $\exists$ compact object $C \in \mathcal{C}$ generating $\mathcal{C}$ in the sense that $\forall D \in \mathcal{C}$, $Ext^n_{\mathcal{C}}(C, D) = 0$ $\forall n \in \mathbb{Z}$ $\Rightarrow$ $D \simeq 0$.

HA 7.1.2.7: $\mathcal{C}$ sym. monoidal ∞-cat. equiv. to $Mod_R^{\otimes}$ for some E_∞-ring R $\Leftrightarrow$

(1) $\mathcal{C}$ stable, presentable, $\otimes$ preserves small colims in each variable

(2) unit obj. $1 \in \mathcal{C}$ compact generating $\mathcal{C}$

3.2 Explicit models for algs over discrete commutative rings

Def. R comm. ring, a differential graded alg. $/R$ is a graded asso. alg. A_* over R with differential $d: A_* \to A_{*-1}$ satisfying

- $d^2 = 0$
- d graded derivation, $d(xy) = d(x)\cdot y + (-1)^{|x|} x \cdot dy$

Morphism of dg algs is homomorphism of graded R-alg commuting with differentials.

$\mathrm{Alg}^{dg}(R)$ cat. of dg algs over R.

A map $\varphi: A_* \to B_*$ is quasi-isom. if it induces a quasi-isom. of chain cpxes over R.

HA 7.1.4.6. R comm. ring, we have equiv. of ∞-cats

$$\mathrm{Alg}^{dg}(R)[qis^{-1}] \simeq \mathrm{Alg}_R = \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{LMod}_R).$$

Def. A dg alg. A_* over a comm. ring R is comm. dg alg. (cdga) if $\forall$ $x \in A_m$, $y \in A_n$ we have $xy = (-1)^{mn} yx$.

$$\mathrm{CAlg}^{dg}(R) \subset \mathrm{Alg}^{dg}(R) \text{ full subcat.}$$

HA 7.1.4.11. For R comm. ring of char 0 (otherwise we have trouble with model struc. for cdgas) we have equiv. of ∞-cat.

$$\mathrm{CAlg}^{dg}(R)[\text{qis}^{-1}] \simeq \mathrm{CAlg}_R = \mathrm{CAlg}(\mathrm{LMod}_R(\mathrm{Sp}))$$

$$\simeq \mathrm{CAlg}_{R/}$$

HA 7.1.4.18 R comm. ring, Alg_R^{disc} cat. of discrete assoc. R-algs, Alg_R^{simp} cat. of simplicial objs of Alg_R^{disc} then we have equiv. of ∞-cat.

$$\mathrm{Alg}_R^{simp}[\text{htpy equiv. of underlying sset}^{-1}] \simeq \mathrm{Alg}_R^{cn}$$

HA 7.1.4.20. R comm. ring, $\mathrm{CAlg}_R^{\mathrm{disc}}$ cat. of discrete comm. R-algs, $\mathrm{CAlg}_R^{\mathrm{simp}}$ cat. of simplicial obj. in $\mathrm{CAlg}_R^{\mathrm{disc}}$.

We have a functor

$$\mathrm{CAlg}_R^{\mathrm{simp}}[\cdots^{-1}] \longrightarrow \mathrm{CAlg}_R^{\mathrm{cn}}$$

which is an equiv. if R has char 0.

3.3. Properties of rings and modules.

3.3.1 Free resolutions and spectral sequence.

Def. $\mathcal{C}$ presentable ∞-cat., S a collection of objs.

A simplicial obj. $X_\bullet$ of $\mathcal{C}$ is called S-free if $\forall n$, $\exists$ coproduct F of objs. of S and a map $F \to X_n$ in $\mathcal{C}$ inducing an equiv.

$$L_n X \amalg F \xrightarrow{\sim} X_n.$$

$\uparrow$ n-th latching obj consisting of all degenerate simplices

Let $c \in \mathcal{C}$, $X_\bullet$ simplicial obj. of $\mathcal{C}_{/c}$, $X_\bullet$ is called an S-hypercovering of c iff $\forall\, Y \in S$ corep. a functor $\chi : \mathcal{C} \to \mathcal{S}$, the simplicial obj. $\chi(X_\bullet)$ is a hypercovering in the ∞-topos $\mathcal{S}/\chi(c)$.

Ex. R asso. ring, $\mathcal{A}$ cat. of left R-mods, $S = \{R\}$.

$M_\bullet$ simplicial obj. in $\mathcal{A}$ $\xrightarrow{\text{Dold-Kan}}$ P_* chain cpx

then $M_\bullet$ is S-free $\Leftrightarrow$ P_n free left R-mod, $\forall n$

$M_\bullet$ S-hypercovering of M $\Leftrightarrow$ $P_* \to M \to 0$ exact

HA 7.2.1.4-9 $\mathcal{C}$ presentable ∞-cat., S set of objs.

Then $\forall\, c \in \mathcal{C}$, there exists an S-free S-hypercovering

$X_\bullet : \Delta^{op} \to \mathcal{C}/c$ unique up to simplicial htpy.

Def. R $\mathbb{E}_1$-ring, $N \in \mathrm{LMod}_R$ is called quasi-free if

$$N \simeq \bigoplus_{\alpha \in A} R[n_\alpha], \quad n_\alpha \in \mathbb{Z}.$$

For $M \in \mathrm{RMod}_R$, $N \in \mathrm{LMod}_R$, we can study htpy gps of $M \otimes_R N$ by resolving N by quasi-free modules.

P. S-free S-hypercovering of N where $S = \{R[n]\}$

$\rightsquigarrow$ can construct s.s. with

$$E_2^{p,q} = \mathrm{Tor}_p^{\pi_* R}(\pi_* M, \pi_* N)_q$$

converging to $\pi_{p+q}(M \otimes_R N)$.

HA 7.2.1.19