DAG.

Applications.

1) if R, M, N all discrete, then $\pi_n(M \otimes_R N) \simeq \mathrm{Tor}_n^R(M, N)$

2) if R, M, N all connective, so is $M \otimes_R N$ and

$\pi_0(M \otimes_R N) \simeq \pi_0 M \otimes_{\pi_0 R} \pi_0 N$.

Flat and proj. mods over connective $\mathbb{E}_1$-rings.

Def. R $\mathbb{E}_1$-ring, a left R-mod M is called free if it is equiv. to a coproduct of copies of R.

A free left module is called f.g. if it is equiv. to a finite coproduct of R.

Def. Let $\mathcal{C}$ be an ∞-cat. admitting geom. realization of simplicial objs, $X \in \mathcal{C}$ is called proj. if $\mathrm{Map}_{\mathcal{C}}(X, \cdot): \mathcal{C} \to \mathcal{S}$ commutes with geom. real.

Def. R connective $\mathbb{E}_1$-ring, a left mod is proj. if it is a proj. obj. of the ∞-cat. LMod_R^{cn}.

Def. $\mathcal{C}$ ∞-cat., $X, Y \in \mathcal{C}$, Y is called a retract of X if $\exists$ 2-simplex $\Delta^2 \to \mathcal{C}$ corresponding to

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"'] & & Y \end{tikzcd}$$

HA 7.2.2.7-8 R connective $\mathbb{E}_1$-ring, $P \in \mathrm{LMod}_R^{cn}$, then

P proj. $\Leftrightarrow$ P retract of a free module

P proj., $\pi_0 P$ f.g. over $\pi_0 R$ $\Leftrightarrow$ P compact proj. obj. of LMod_R^{cn}

$\Leftrightarrow$ P retract of a f.g. free module

Def. R $\mathbb{E}_1$-ring, $M \in \mathrm{LMod}_R$ is called flat if

$\pi_0 M$ is flat over $\pi_0 R$ and $\forall n \in \mathbb{Z}$,

$$\pi_n R \underset{\pi_0 R}{\otimes} \pi_0 M \xrightarrow{\sim} \pi_n M.$$

Easy consequence.

1) R $\mathbb{E}_1$-ring, $f: M \to N$ flat R-mods then

f is an equiv. $\Leftrightarrow$ f induces $\pi_0 M \xrightarrow{\sim} \pi_0 N$.

2) R connective $\mathbb{E}_1$-ring, flat R-mod M is proj.

$\Leftrightarrow$ $\pi_0 M$ proj. over $\pi_0 R$

3) R $\mathbb{E}_1$-ring, $M \in \mathrm{RMod}_R$, $N \in \mathrm{LMod}_R$ flat, $\forall n \in \mathbb{Z}$

we have $\mathrm{Tor}_0^{\pi_0 R}(\pi_n M, \pi_n N) \xrightarrow{\sim} \pi_n (M \underset{R}{\otimes} N)$.

Lazard's thm: R connective $\mathbb{E}_1$-ring, $N \in \mathrm{LMod}_R^{cn}$, then TFAE:

1) N filtered colim of f.g. free mods

2) N filtered colim of proj. left R-mods

3) N flat

4) $- \otimes_R N$ is left t-exact

5) M discrete $\Rightarrow$ $M \otimes_R N$ discrete

Finiteness properties of rings and modules (HA 7.2.4)

Def. R $\mathbb{E}_1$-ring, a left module M is called perfect if $M \in \mathrm{LMod}_R^{perf.}$, smallest stable subcat. of LMod_R containing R and closed under retracts.

Similarly $\mathrm{RMod}_R^{perf.}$

Idea: build from finitely many R by shifting, extensions and retracts.

Recall: $\mathcal{C}$ ∞-cat. admitting filtered colims, $X \in \mathcal{C}$ is called compact if $\mathrm{Map}_{\mathcal{C}}(X, \bullet)$ commutes with filtered colims.

Prop. R $\mathbb{E}_1$-ring, $M \in \mathrm{LMod}_R$ compact $\Leftrightarrow$ perfect.

Cor. R connective $\mathbb{E}_1$-ring, $M \in \mathrm{LMod}_R^{\mathrm{perf.}}$, then

1) $\pi_n M = 0$, $n \ll 0$

2) if $\pi_m M = 0$ for all $m < k$, then $\pi_k M$ is f.p. $/\pi_0 R$

Prop. Duality between left and right modules

R $\mathbb{E}_1$-ring. $- \otimes_R - : \mathrm{RMod}_R \times \mathrm{LMod}_R \to \mathrm{Sp}$ induces fully faithful embeddings

$\theta: \mathrm{RMod}_R \to \mathrm{Fun}(\mathrm{LMod}_R, \mathrm{Sp})$, $\theta': \mathrm{LMod}_R \to \mathrm{Fun}(\mathrm{RMod}_R, \mathrm{Sp})$

with essential image functors preserving small colims.

Prop. R E_1-ring, $M \in LMod_R$, M perfect $\Leftrightarrow \exists M^\vee \in RMod_R$

s.t. $LMod_R \xrightarrow{M^\vee \otimes} Sp \xrightarrow{\Omega^\infty} \mathcal{S}$ is equiv. to $Mor(M, -)$

and in this case $M^\vee$ also perfect.

Def. $\mathcal{C}, \mathcal{D}$ ∞-cats. $F: \mathcal{C} \times \mathcal{D} \to \mathcal{S}$ is a perfect

pairing if it satisfies equivalently

1) the induced $f: \mathcal{C} \to Fun(\mathcal{D}, \mathcal{S}) = P(\mathcal{D}^{op})$ is

fully faithful, with essential image the same as

Yoneda embedding $\mathcal{D}^{op} \to P(\mathcal{D}^{op})$

2) ... $\mathcal{D} \to P(\mathcal{C}^{op})$.

Perfect pairing $\rightsquigarrow$ $\mathcal{C} \simeq \mathcal{D}^{op}$.

Prop. R $\mathbb{E}_1$-ring, $\mathrm{RMod}_R^{\mathrm{perf}} \times \mathrm{LMod}_R^{\mathrm{perf}} \xrightarrow{\otimes_R} \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S}$

is a perfect pairing.

Warning: R comm. Noeth. ring, M discrete f.g. R-mod, then M in general not perfect (needs finite proj. dim).

Def. $\mathcal{C}$ compactly generated ∞-cat. (presentable and ω-accessible), $C \in \mathcal{C}$ is called almost compact if $\tau_{\leq n} C$ is a compact obj. of $\tau_{\leq n}\mathcal{C}$ for all $n > 0$.

R connective $\mathbb{E}_1$-ring. a left R-mod M is called almost perfect if $\exists\, k \in \mathbb{Z}$ s.t. $M \in (\mathrm{LMod}_R)_{\geq k}$ and is almost compact as an obj. of $(\mathrm{LMod}_R)_{\geq k}$.

Denote $\mathrm{LMod}_R^{\mathrm{aperf}} \subset \mathrm{LMod}_R$.

Prop. $M \in (\mathrm{LMod}_R^{\mathrm{aperf}})_{\geq 0}$, then M is the geom. real. of a simplicial left R-mod $P_\cdot$ where each P_n is free and f.g.

Recall: an assoc. ring R is left coherent if every f.g. left ideal of R is f.p.

Def. R $\mathbb{E}_1$-ring, R is called left coherent if the following holds:

(1) R connective

(2) $\pi_0 R$ left coherent

(3) For each $n \geq 0$, $\pi_n R$ is f.p. as left $\pi_0 R$-mod.

Prop. R left coherent $\mathbb{E}_1$-ring, $M \in \mathrm{LMod}_R$, then M is almost perfect $\iff$ $\pi_m M = 0$, $\forall\, m \ll 0$

$\pi_m M$ is f.p. over $\pi_0 R$, $\forall\, m \in \mathbb{Z}$

Prop. R left con. $\mathbb{E}_1$-ring, then t-struc. on LMod_R gives t-struc. on $\mathrm{LMod}_R^{\mathrm{aperf.}}$

Prop. R connective $\mathbb{E}_1$-ring, M connective left R-mod. TFAE.

(1) M retract of a f.g. free R-mod.

(2) M flat and almost perfect.

Def. R connective $\mathbb{E}_1$-ring, a left R-mod M has Tor-amplitude $\leq n$ if for every discrete R-mod N, $\pi_i(N \otimes_R M)$ vanishes for $i > n$. M is of finite Tor-amplitude if it has Tor-amplitude $\leq n$ for some n.

RMK. A connective left R-mod M has Tor-amplitude ≤ 0 $\Leftrightarrow$ M flat.

Prop. R connective $\mathbb{E}_1$-ring, $M \in \mathrm{LMod}_R$ almost perfect.

Then M perfect $\Leftrightarrow$ M has finite Tor-amplitude.

Def. R connective $\mathbb{E}_\infty$-ring, $\mathrm{Free}: \mathrm{LMod}_R \to \mathrm{CAlg}_R$

Let A be a connective $\mathbb{E}_\infty$-alg. over R. We say A

- f.g. and free if $A \simeq \mathrm{Free}\, M$ for some f.g. free $M \in \mathrm{LMod}_R$
- of finite presentation if A is a finite colim of f.g. and free algs
- locally of finite presentation if A is a compact object of CAlg_R
- almost of finite presentation if A is an almost compact object of CAlg_R.