DAG.

IV. Cotangent complexes

1. The cotangent complex formalism

Goal: derive Kahler differentials

Recall: A comm. ring, M A-mod, a derivation from A to M is a map $d: A \to M$ s.t.

$$d(x+y) = d(x) + d(y), \quad d(xy) = x\,dy + y\,dx$$

Let $\mathrm{Der}(A, M)$ be the abelian gp of derivations from A to M. The functor $\mathrm{Der}(A, -)$ is corep. by an A-mod Ω_A. Explicitly $\Omega_A = \mathrm{Free}(dx)_{x \in A} / \text{relations}$.

Reformulation: equip $B = A \oplus M$ the ring struc.

$$(a, m)(a', m') = (aa', am' + a'm)$$

called a trivial square-zero extension.

Then $\mathrm{Der}(A, M) =$ sections of the projection $A \oplus M \to A$

Let Ring = cat. of comm. rings

$\mathrm{Ring}^+ = \{(A, M),\ A$ comm. ring, M A-mod$\}$

$\mathrm{Mor}_{\mathrm{Ring}^+}((A,M),(B,N)) = (f, f')$ where

$f : A \to B$ ring map and $f' : M \to N$ A-mod map

$G : \mathrm{Ring}^+ \to \mathrm{Ring},\quad (A, M) \longmapsto A \oplus M$ trivial square-zero extension

G admits a left adjoint $F : A \longmapsto (A, \Omega_A)$

Steps for generalizing the above constructions to derived geom:

1) generalize trivial square-zero extension

2) generalize Ring^+ to $\mathcal{C}^+$ for any presentable ∞-cat $\mathcal{C}$

$\mathcal{C}^+$ called the tangent bundle $T_{\mathcal{C}}$ to $\mathcal{C}$.

3) define cotangent complex functor $L : \mathcal{C} \to T_{\mathcal{C}}$ via adjunction

4) define derivation by tangent correspondence to $\mathcal{C}$

1.1 Trivial square-zero extension.

Goal: A $\mathbb{E}_\infty$-ring, $M \in \mathrm{Mod}_A$, construct "$A \oplus M$".

Want a functorial functor $G: \mathrm{Mod}_A \to \mathrm{CAlg}_{/A}$.

Construction: Let X be an obj. of $\mathrm{Sp}(\mathrm{CAlg}_{/A})$.

Then the 0th-space $\Omega^\infty X$ is a pointed object of $\mathrm{CAlg}_{/A}$

i.e. an $\mathbb{E}_\infty$-ring B fitting into a comm. diagram

$$\begin{tikzcd} & B \arrow[dr, "f"] & \\ A \arrow[ur] \arrow[rr, "id"] & & A \end{tikzcd}$$

Note that the fibre of f inherits the struc. of an

A-mod $\rightsquigarrow$ functor $F': \mathrm{Sp}(\mathrm{CAlg}_{/A}) \to \mathrm{Mod}_A$

HA 7.3.4.14: F' is an equiv. of ∞-cats.

Define the trivial square-zero extension functor G to be

$$\mathrm{Mod}_A \xrightarrow{\sim} \mathrm{Sp}(\mathrm{CAlg}_{/A}) \xrightarrow{\Omega^\infty} \mathrm{CAlg}_{/A}$$

denote by $G(M) = A \oplus M$.

HA 7.3.4.15. Forgetting the alg. struc. $A \oplus M$ is canonically identified with the coproduct of A and M.

HA 7.3.4.17. The multiplication on $\pi_*(A \oplus M)$ is given on homogeneous elements by

$$(a, m)(a', m') = (aa', am' + (-1)^{|a'||m|} a'm)$$

In particular if A, M discrete, recover the classical one.

1.2 Stable envelopes and tangent bundles.

Idea: make the above construction in families, i.e. fibrewise stabilization.

Characterization of stabilization $Sp(\mathcal{C})$.

Def. $\mathcal{C}$ presentable ∞-cat., a stable envelope of $\mathcal{C}$ is a functor $u: \mathcal{C}' \to \mathcal{C}$ s.t.

1) $\mathcal{C}'$ presentable stable ∞-cat.

2) u admits a left adjoint

3) $\forall$ presentable stable ∞-cat. $\mathcal{E}$

$\mathrm{RFun}(\mathcal{E}, \mathcal{C}') \xrightarrow{u\circ} \mathrm{RFun}(\mathcal{E}, \mathcal{C})$ is an equiv.

functors admitting left adjoints of ∞-cats.

Ex. $\Omega^{\infty}_{\mathcal{C}} : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ exhibits $\mathrm{Sp}(\mathcal{C})$ as a stable envelope of $\mathcal{C}$.

Def. $p : X \to S$ of ∞-cats. is called a presentable fib if both Cart. and coCart. and every fibre is a presentable ∞-cat.

A stable envelope of a presentable fib. is a functor $u: \mathcal{C}' \to \mathcal{C}$ s.t.

1) $p \circ u$ is presentable fib

2) u carries $(p\circ u)$-Cart. morphisms to p-Cart. morphisms

3) $\forall\, s \in S$, $\mathcal{C}'_s \to \mathcal{C}_s$ is a stable envelope

Def. $\mathcal{C}$ presentable ∞-cat., a tangent bundle to $\mathcal{C}$ is a functor $T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ which exhibits $T_{\mathcal{C}}$ as the stable envelope of the presentable fib.

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}$$

Idea: objs of $T_{\mathcal{C}}$ are (A, M), $A \in \mathcal{C}$, $M \in \mathrm{Sp}(\mathcal{C}_{/A})$

For $\mathcal{C} = \mathrm{CAlg}$, $M \in \mathrm{Sp}(\mathrm{CAlg}_{/A}) \simeq \mathrm{Mod}_A$. The functor $T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ sends (A, M) to the projection $A \oplus M \to A$.

Explicit construction of $T_{\mathcal{C}}$:

$$\mathrm{Exc}(\mathcal{S}_*^{fin}, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$$

$$(X : \mathcal{S}_*^{fin} \to \mathcal{C}) \mapsto (X(S^0) \to X(*))$$

Def. $\mathcal{C}$ presentable ∞-cat., the absolute cotangent complex functor L is a left adjoint to

$$T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}$$

RMK. relative adjunction

$$\begin{tikzcd} T_{\mathcal{C}} \arrow[rr] \arrow[dr] & & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

RMK. For $A \in \mathcal{C}$, the object $L_A \in Sp(\mathcal{C}_{/A}) \simeq (T_{\mathcal{C}})_A$ corresponds to the image of $id_A \in \mathcal{C}_{/A}$ under the suspension spectrum functor $\Sigma^\infty_+ : \mathcal{C}_{/A} \to Sp(\mathcal{C}_{/A})$.

1.3 The relative cotangent complex.

$\mathcal{C}$ presentable ∞-cat., $A \in \mathcal{C}$ $\rightsquigarrow$ absolute cotangent complex $L_A \in Sp(\mathcal{C}_{/A})$

Goal: define a relative cotangent complex $L_{B/A}$ for a morphism $A \to B$ in $\mathcal{C}$.

Idea: for Kahler diff. we have an exact sequence

$$\Omega_A \otimes_A B \to \Omega_B \to \Omega_{B/A} \to 0$$

for $A \to B$.

Want to define $L_{B/A}$ via some cofibre sequence.

Def. $\mathcal{C}$ presentable ∞-cat., $p: T_{\mathcal{C}} \to \mathcal{C}$ tangent bundle. A relative cofibre sequence in $T_{\mathcal{C}}$ is a pushout square in $T_{\mathcal{C}}$

$$\begin{tikzcd} & X \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Sp}(\mathcal{C}/_{p(X)}) \ni & 0 \arrow[r] & Z \end{tikzcd}$$

s.t. each column lies in a fibre of p.

Let $\mathcal{E} \subset \mathrm{Fun}(\square, T_{\mathcal{C}}) \times_{\mathrm{Fun}(\square, \mathcal{C})} \mathrm{Fun}(\to, \mathcal{C})$

spanned by relative cofibre sequences.

The relative cotangent complex functor is

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{L} \mathrm{Fun}(\Delta^1, T_{\mathcal{C}}) \longrightarrow \mathcal{E} \longrightarrow T_{\mathcal{C}}$$

make relative cofibre sequence

take lower right corner

$$(f: A \to B) \longmapsto L_{B/A} \in \begin{matrix} (T_{\mathcal{C}})_B \\ \cong \\ \mathrm{Sp}(\mathcal{C}/_B) \end{matrix}$$

RMK. We have a relative cofibre sequence in $T_{\mathcal{C}}$

$$\begin{tikzcd} L_A \arrow[r] \arrow[d] & L_B \arrow[d] \\ 0 \arrow[r] & L_{B/A} \end{tikzcd}$$

HTT 4.3.1.9 $\Rightarrow$ cofibre sequence $f_! L_A \to L_B \to L_{B/A}$

in $(T_{\mathcal{C}})_B \simeq Sp(\mathcal{C}_{/B})$

where $f_! : Sp(\mathcal{C}_{/A}) \to Sp(\mathcal{C}_{/B})$ denotes

the functor induced by cocart. fib. p.

Ex. For $f: A \to B$, A initial obj. of $\mathcal{C}$, we have

$L_B \xrightarrow{\sim} L_{B/A}$.

For $f: A \xrightarrow{\sim} B$ equiv., $L_{B/A} \simeq 0 \in Sp(\mathcal{C}_{/B})$.

Prop. $\mathcal{C}$ presentable ∞-cat., $\begin{tikzcd} & B \arrow[dr] & \\ A \arrow[ur] \arrow[rr] & & C \end{tikzcd}$ comm.

diagram in $\mathcal{C}$ $\Rightarrow$ pushout diagram $\begin{tikzcd} L_{B/A} \arrow[r] \arrow[d] & L_{C/A} \arrow[d] \\ 0 \simeq L_{B/B} \arrow[r] & L_{C/B} \end{tikzcd}$ in $T_{\mathcal{C}}$

also a relative cofibre sequence

$\Rightarrow$ cofibre sequence $f_! L_{B/A} \to L_{C/A} \to L_{C/B}$ in $Sp(\mathcal{C}/_C)$

Prop. Given a pushout

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow f \\ A' & \longrightarrow & B' \end{array}$$

in $\mathcal{C}$

we have an equiv. $f_! L_{B/A} \xrightarrow{\sim} L_{B'/A'}$, i.e.

$L_{B/A} \longrightarrow L_{B'/A'}$ is a p-CoCart. morphism in $T_{\mathcal{C}}$.

2. Deformation theory.

2.1 Square-zero extensions.

Recall: R comm. ring, a square-zero extension of R is a comm. ring $\tilde{R}$ with surj. $\phi: \tilde{R} \twoheadrightarrow R$ s.t. $(\ker\phi)^2 = 0$. In this case, $M = \ker\phi$ has an R-mod struc.

Def. $\mathcal{C}$ presentable ∞-cat., $L: \mathcal{C} \to T_{\mathcal{C}}$ a cotangent complex functor. By unstraightening we get coCart. fib.

$$\begin{tikzcd} \mathcal{M}^T(\mathcal{C}) \arrow[d] & = \{ \mathcal{C} \arrow[r, "L"] & T_{\mathcal{C}} \arrow[l, bend left, dashed, "p"] \} \\ \Delta^1 & = \{ 0 \arrow[r] & 1 \} \end{tikzcd}$$

We call $\mathcal{M}^T(\mathcal{C})$ the tangent correspondence to $\mathcal{C}$.

It has a projection map $p: \mathcal{M}^T(\mathcal{C}) \to \Delta^1 \times \mathcal{C}$.

Def. A derivation in $\mathcal{C}$ is a morphism $\eta: A \to M$ in $\mathcal{M}^T(\mathcal{C})$ where $A \in \mathcal{C}$, $M \in (T_{\mathcal{C}})_A$. By coCart. property it can also be identified with a map $d: L_A \to M$ in $(T_{\mathcal{C}})_A$.

Let $\mathrm{Der}(\mathcal{C})$ be the ∞-cat. of derivations in $\mathcal{C}$.

Def. $\mathcal{C}$ presentable ∞-cat., for every derivation $\eta: A \to M$ form a pullback diagram

$$\begin{tikzcd} A^\eta \arrow[r] \arrow[d] & A \arrow[d] \\ 0 \arrow[r] & M \end{tikzcd}$$

in ∞-cat. $\mathcal{M}^T(\mathcal{C})$.

A morphism $f: \tilde{A} \to A$ in $\mathcal{C}$ is called a square-zero extension if there exists a derivation η in $\mathcal{C}$ and an equiv. $\tilde{A} \simeq A^{\eta}$ in $\mathcal{C}_{/A}$.

We also call $\tilde{A}$ a square-zero extension of A by $M[-1]$ for $\eta: A \to M$.