DAG.

RMK. Using HTT 4.3.1.9 the above pullback diagram in $\mathcal{M}^T(\mathcal{C})$ is equiv. to the pullback diagram in $\mathcal{C}$

$$\begin{tikzcd}
A^{\eta} \arrow[r] \arrow[d] & A \arrow[d, "d_\eta"] \\
A \arrow[r, "d_0"'] & A \oplus M
\end{tikzcd}$$

where $A \oplus M$ denotes the image of M under the functor $\Omega^\infty : Sp(\mathcal{C}_{/A}) \to \mathcal{C}$ and d_0 is the section attached to the zero derivation.

The main source of square-zero extensions in the setting of E_∞-rings is n-small extensions.

Def. For $n \geq 0$, a morphism $f : A \to B$ in $CAlg$ is called an n-small extension if $A \in CAlg^{cn}$ and

$\mathrm{fib}(f) \in \mathrm{CAlg}_{[n,2n]}$ and the multiplication

$$\mathrm{fib}(f) \otimes_A \mathrm{fib}(f) \longrightarrow \mathrm{fib}(f) \text{ is nullhomotopic.}$$

Denote $\mathrm{Fun}_{n\text{-sm}}(\Delta^1, \mathrm{CAlg}) \subset \mathrm{Fun}(\Delta^1, \mathrm{CAlg})$ the ∞-cat generated by n-small extensions.

An obj. $(A, \eta : L_A \to M[1]) \in \mathrm{Der}(\mathrm{CAlg})$ is n-small if A is connective, and $M \in \mathrm{Sp}_{[n,2n]}$. Denote $\mathrm{Der}_{n\text{-sm}} \subset \mathrm{Der}(\mathrm{CAlg})$.

Thm. The functor $\Phi : \mathrm{Der}(\mathrm{CAlg}) \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg})$ induces
$$(A, \eta) \longmapsto (A^\eta \to A)$$
for each $n \geq 0$ an equiv. of ∞-cat.

$$\mathrm{Der}_{n\text{-sm}} \xrightarrow{\sim} \mathrm{Fun}_{n\text{-sm}}(\Delta^1, \mathrm{CAlg}).$$

Cor.

1) every n-small extension in CAlg is a square-zero ext.

2) for $A \in \mathrm{CAlg}^{cn}$, every map in the Postnikov tower

$$\cdots \to \tau_{\leq 3} A \to \tau_{\leq 2} A \to \tau_{\leq 1} A \to \tau_{\leq 0} A$$

is a square-zero extension.

Application: given $A, B \in \mathrm{CAlg}^{cn}$, we can understand $\mathrm{Map}_{\mathrm{CAlg}}(A, B)$ as $\lim_n \mathrm{Map}_{\mathrm{CAlg}}(A, \tau_{\leq n} B)$. For $n=0$, $\mathrm{Map}_{\mathrm{CAlg}}(A, \tau_{\leq 0} B) \simeq \mathrm{Hom}(\pi_0 A, \pi_0 B)$. For $n>0$, we have a pullback square

$$\begin{tikzcd}
\tau_{\leq n} B \arrow[r] \arrow[d] & \tau_{\leq n-1} B \arrow[d] \\
\tau_{\leq n-1} B \arrow[r] & \tau_{\leq n-1} B \oplus (\pi_n B)[n+1]
\end{tikzcd}$$

This reduces to the study of $\mathrm{Map}_{\mathrm{CAlg}}(A, \tau_{\leq n-1} B)$ and the linear problem of derivations from A to $(\pi_n B)[n+1]$ which is controlled by the cotangent complex.

2.2 Deformation theory of $\mathbb{E}_\infty$-rings.

Def. Let A be $\mathbb{E}_\infty$-ring, $\tilde{A}$ a square-zero extension of A by an A-mod M, for $B \in \mathrm{CAlg}_A$, a deformation of B to $\tilde{A}$ is a pair $(\tilde{B}, \alpha)$ where $\tilde{B} \in \mathrm{CAlg}_{\tilde{A}}$ and α an equiv. $\tilde{B} \otimes_{\tilde{A}} A \simeq A$ in CAlg_A.

RMK. If A, M are connective, then $\tilde{B}$ flat over $\tilde{A}$ $\Leftrightarrow$ B flat over A.

Pf. "$\Leftarrow$" ETS for every discrete $\tilde{A}$-mod N, $\tilde{B} \otimes_{\tilde{A}} N$ is discrete. Let $I = \ker(\pi_0 \tilde{A} \to \pi_0 A)$, we have a s.e.s.

$$0 \to IN \to N \to N/IN \to 0 \quad \text{as } \pi_0 \tilde{A}\text{-mod}$$

reduce to show $\tilde{B} \otimes_{\tilde{A}} IN$ and $\tilde{B} \otimes_{\tilde{A}} N/IN$ discrete. Note that IN, N/IN are annihilated by I hence the tensor factors through A. ///

Let $\mathrm{Der} = \mathrm{Der}(\mathrm{CAlg})$ be the ∞-cat. of derivations in CAlg. Define a subcat. $\mathrm{Der}^{+} \subset \mathrm{Der}$:

- objs are derivations $\eta : A \to M[1]$ where both A, M are connective
- morphisms $f : (\eta : A \to M[1]) \to (\eta' : B \to N[1])$ s.t. $B \otimes_A M \xrightarrow{\sim} N$.

Prop. A connective E_∞-ring, M connective A-mod, $\eta : A \to M[1]$ a derivation. Then we have an equiv. of ∞-cat.

$$\mathrm{Der}^{+}_{\eta/} \xrightarrow{\sim} \mathrm{CAlg}^{cn}_{A^\eta}$$

$$\eta' : B \to N[1] \longmapsto B^{\eta'} = \mathrm{fib}(\eta')$$

Idea: giving a deformation $\tilde{B}$ of B over $\tilde{A}$ B equiv. to providing a factorization of $\eta_B : B \otimes_A L_A \to B \otimes_A M[1]$ as a composition $B \otimes_A L_A \to L_B \xrightarrow{\eta'} B \otimes_A M[1]$ and $\tilde{B} = B^{\eta'}$.

In particular B admits deformations over $\tilde{A}$

$\iff$ $L_{B/A}[-1] \to B \otimes_A L_A \xrightarrow{\eta_B} B \otimes_A M[1]$ vanishes

2.3 Connetivity of the cotangent complex.

Thm. For $f: A \to B$ of E_∞-rings, if $\mathrm{cofib}(f)$ is n-connective for some $n \geq 0$ then there is a natural $2n$-connective map of B-mods $\varepsilon_f : B \otimes_A \mathrm{cofib}(f) \to L_{B/A}$.

Construction of ε_f: we have $\eta: L_B \to L_{B/A}$ map of B-mods $\Rightarrow$ B^η square-zero extension of B by $L_{B/A}[-1]$.

Since η restricts to L_A being nullhomotopic, f factors as $A \xrightarrow{f'} B^\eta \xrightarrow{f''} B$ so we get a map of A-mods $\mathrm{cofib}(f) \to \mathrm{cofib}(f'')$, hence a map of B-mods

$$\varepsilon_f : B \otimes_A \mathrm{cofib}(f) \to \mathrm{cofib}(f'') \simeq L_{B/A}$$

Cor. For $f: A \to B$ of connective $\mathbb{E}_\infty$-rings, if $\operatorname{cofib}(f)$ is n-connective for $n \geq 0$, then the relative cotangent cpx $L_{B/A}$ is n-connective, the converse holds provided that f induces an isom. $\pi_0 A \xrightarrow{\sim} \pi_0 B$.

Pf. fibre sequence of B-mods

$$\operatorname{fib}(\varepsilon_f) \longrightarrow B \otimes_A \operatorname{cofib}(f) \longrightarrow L_{B/A}.$$ ///

Cor. A connective $\mathbb{E}_\infty$-ring, then L_A is connective.

Pf. Consider the unit map $S \to A$ in the case $n=0$ ///

Cor. $f: A \to B$ of connective $\mathbb{E}_\infty$-rings, $L_{B/A}$ is connective.

Cor. $f: A \to B$ of connective $\mathbb{E}_\infty$-rings. Then

$$f \text{ is an equiv.} \iff \begin{cases} f \text{ induces isom. } \pi_0 A \xrightarrow{\sim} \pi_0 B \\ L_{B/A} \simeq 0 \end{cases}$$

Cor. $f: A \to B$ of connective $\mathbb{E}_\infty$-rings s.t. $\mathrm{cofib}(f)$ is n-connective for some $n \geq 0$ then the induced map $L_f : L_A \to L_B$ has n-connective cofibre. In particular the canonical map $\pi_0 L_A \to \pi_0 L_{\pi_0 A}$ is an isom.

Cor. $f: A \to B$ of connective $\mathbb{E}_\infty$-rings s.t. $\mathrm{cofib}(f)$ is n-connective for some $n \geq 0$ then there exists a canonical $(2n-1)$-connective map of A-mods $\mathrm{cofib}(f) \to L_{B/A}$.

Prop. For $f: A \to B$ of connective $\mathbb{E}_\infty$-rings, we have $\pi_0 L_{B/A} \xrightarrow{\sim} \Omega_{\pi_0 B / \pi_0 A}$ as $\pi_0 B$-mods.

Pf. By fibre sequence for $L_{B/A}$ and s.e.s. for Ω, reduce to absolute case : A discrete $\mathbb{E}_\infty$-ring, then $\pi_0 L_A \xrightarrow{\sim} \Omega_A$ as discrete A-mods.

We show $\pi_0 L_A$, Ω_A copresent the same functor on the cat. of discrete A-mods.

For M discrete A-mod, we have

$$\mathrm{Map}_{\mathrm{Mod}_A}(\pi_0 L_A, M) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(L_A, M)$$

$$\simeq \mathrm{Map}_{\mathrm{CAlg}/A}(A, A \oplus M)$$

$$\simeq \mathrm{Map}_{\mathrm{Rings}/A}(A, A \oplus M). \qquad ///$$

2.4 Finiteness of the cotangent complex.

Thm. A connective E_∞-ring, B connective E_∞-alg. $/A$.

1) if B is almost locally of f.p. over A then $L_{B/A}$ is almost perfect B-mod. The converse holds provided $\pi_0 B$ is f.p. over $\pi_0 A$.

3. Etale morphisms.

Def. A map $\phi: A \to B$ of $\mathbb{E}_\infty$-rings is called etale if $\pi_0 A \to \pi_0 B$ is etale and B flat as A-mod.

Thm. Let A be an $\mathbb{E}_\infty$-ring. every etale map of discrete comm. rings $\pi_0 A \to \pi_0 B$ can be lifted (essen. unique) to an etale $\phi: A \to B$ of $\mathbb{E}_\infty$-rings.

Cor. The relative cotangent complex of an etale morphism of $\mathbb{E}_\infty$-rings vanishes.

DAG VII 8.9. $A \to B$ of $\mathbb{E}_\infty$-rings s.t. $L_{B/A}$ vanishes.

TFAE.

1) $\pi_0 B$ f.p. over $\pi_0 A$

2) B f.p. over A

3) B almost f.p. over A

4) $A \to B$ etale.

HKR Thm and derived de Rham cohomology.

Assume char $k = 0$.

1. Review.

X/k sm. var.

Thm. (Hochschild - Kostant - Rosenberg)

$\exists$ quasi-isom, $HH(X) = R\Gamma(X, \mathcal{O}_X \otimes^{\mathbb{L}}_{\mathcal{O}_{X\times X}} \mathcal{O}_X) \simeq R\Gamma(X, \bigoplus_{i\geq 0} \Omega^i_{X/k})$

When $X = \operatorname{Spec} A$ affine, the thm is saying

$$\operatorname{Tor}_i^{A\otimes A}(A, A) \simeq \Omega^i_A$$

Questions:

1. What happens if we drop smoothness?

2. Can we get a multiplicative statement at the level of chain cpxes?

Look at $\bigoplus_{i \geq 0} \Omega^i_{X/k}[i]$ as chain cpx:

$$\begin{array}{ccccccc} & 2 & & 1 & & 0 \\ \cdots \xrightarrow{0} & \Omega^2 & \xrightarrow{0} & \Omega^1 & \xrightarrow{0} & \mathcal{O}_X \end{array}$$

d_{dR} d_{dR}

3. What about the de Rham differentials? Can we incorporate d_{dR} in the statement?

Recall: if X (derived) scheme, we have a notion of cotangent complex $L_X \in \underline{QCoh}(X)$ (the ∞-cat. of quasi-coherent sheaves on X). If X is smooth, then $L_X \simeq \Omega^1_X$ (in general $\pi_0 L_X \simeq \Omega^1_X$).

How to compute cotangent complex in practice?

$X = \operatorname{Spec} A$, take $A \in SCR_k$ (has a model struc.)

take cofib. resolution $\tilde{A}$ of A:

- $\tilde{A} \to A$ qis.
- $\tilde{A}$ is degree-wise polynomial ring

then $L_{\tilde{A}} \overset{\text{qis}}{\simeq} L_A$.

RMK. In the ∞-cat. Mod_A, for any $M \in \mathrm{Mod}_A$, we can define $\Lambda^i M = \mathrm{Sym}^i(M[1])[-i]$ where

$$\begin{array}{ccc} M^{\otimes i} & \in & \mathrm{Fun}(B(\Sigma_i), \mathrm{Mod}_A) \\ \downarrow & & \downarrow \mathrm{colim} \\ \mathrm{Sym}^i M & \in & \mathrm{Mod}_A \end{array}$$

, Σ_i sym. grp. $B(\Sigma_i)$ the groupoid of Σ_i.

Guess: $\bigoplus_{i \geq 0} \Lambda^i L_A[i] \simeq A \underset{A \otimes A}{\otimes} A$ as E_∞-rings/cdgas/SCR_k

Strategy: pick resolutions and work, unsatisfactory as hard to generalize to non-affine setting, and to study functoriality.

Idea: find some universal property.

RMK. $\bigoplus_{i\geq 0} \wedge^i L_A[i] \simeq \mathrm{Sym}_A(L_A[1])$.

Can we get a universal property of this object that only depends on A?

L_A comes with a universal derivation $A \to L_A$.

Def. A mixed derived comm. ring is

$$(A,\ d: A \to A[-1],\ \underbrace{d^2 \simeq 0,\ d^3 \simeq 0, \dots}_{\text{htpy coherence data}}\ )$$

Denote by $\varepsilon\text{-}\mathrm{CAlg}_k$ the ∞-cat. of mixed algs. Then the forgetful functor $U_\varepsilon: \varepsilon\text{-}\mathrm{CAlg}_k \to \mathrm{CAlg}_k$ has a left adjoint L_ε s.t. $U_\varepsilon(L_\varepsilon(R)) \simeq \mathrm{Sym}_R(L_R[1])$.

On the other side

recall $S^1 = * \coprod_{* \amalg *} * = \Sigma(S^0)$

for any presentable ∞-cat. $\mathcal{C}$, $\exists\ \otimes : S \times \mathcal{C} \to \mathcal{C}$

characterized by: $* \otimes X \simeq X$, $K \otimes X$ commutes with colims in K

$$S^1 \otimes A \simeq A \otimes_{A \otimes A} A$$

Denote $S^1\text{-CAlg} = \mathrm{Fun}(BS^1, \mathrm{CAlg})$.

Thm. (Toen-Vezzosi) $\exists$ equiv. ϕ

$$\begin{tikzcd}
S^1\text{-CAlg}_k \arrow[rr, "\phi", "\sim"'] \arrow[dr, shift left, "U_{S^1}"] & & \varepsilon\text{-CAlg}_k \arrow[dl, shift left, "U_\varepsilon"] \\
& \mathrm{CAlg}_k \arrow[ul, shift left, "L_{S^1}"] \arrow[ur, shift left, "L_\varepsilon"] &
\end{tikzcd}$$

$L_{S^1}(R) \simeq R \otimes_{R \otimes R} R$, $L_\varepsilon(R) \simeq \mathrm{Sym}_R(L_R[1])$

s.t. ϕ commutes with both forgetful functors and their left adjoints.

Recall from HA (consequence of Barr - Beck)

$$\begin{tikzcd} \mathcal{D}' \arrow[rr, "F"] \arrow[dr, shift left, "u'"] & & \mathcal{D} \arrow[dl, shift left, "u"] \\ & \mathcal{C} \arrow[ul, shift left, "L'"] \arrow[ur, shift left, "L"] & \end{tikzcd}$$

s.t. $uF = u'$

u, u' monadic

$FL' \to LuFL' \simeq Lu'L' \to L$ equiv.

$\Rightarrow F$ equiv.

In our setting, construction of ϕ will be difficult.

2. Mixed cdgas.

Def. Let $k[\varepsilon] = \mathrm{Sym}_k(k[1])$ (self-intersection of 0 in $\mathbb{A}'$)

$$\begin{tikzcd} \mathrm{Spec}\, k[\varepsilon] \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \mathbb{A}' \end{tikzcd}$$

$\Rightarrow$ Spec $k[\varepsilon]$ has a group struc. hence $k[\varepsilon]$ has a Hopf derived comm. ring struc.

Let mixed modules be the sym. monoid cat. $\mathrm{Mod}_{k[\varepsilon]}$

with $M \otimes_\varepsilon N = \Delta_*(M \otimes_k N) \in \Delta_*(\mathrm{Mod}_{k[\varepsilon]\otimes_k k[\varepsilon]})$

$= \mathrm{Mod}_{k[\varepsilon]}$

Let $\mathcal{E}\text{-CAlg}_k = \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Mod}_{k[\varepsilon]}, \otimes_\varepsilon)$.

We have an alternative description. Let $k[\eta] = \mathrm{Sym}_k(k[-1])$ be a non-connective alg.

In general for non-connective A.

$$\underbrace{\cdots A^2 \to A^1 \to A^0}_{\text{infinitesimal information}} \to \underbrace{A^{-1} \to A^{-2} \cdots}_{\text{stacky information}}$$

$k[\eta]$ should be thought as $R\Gamma(B\mathbb{A}^1_k, \mathcal{O}_{B\mathbb{A}^1_k}) \simeq R\Gamma(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$

$\Rightarrow k[\eta]$ is Hopf.

Thm.
$$\begin{tikzcd} \mathcal{E}\text{-CAlg}_k \arrow[rr, "\sim"] \arrow[dr, "U_\varepsilon"'] & \arrow[d, phantom, "\circlearrowright"] & k[\eta]\text{-coMod}(\mathrm{CAlg}_k) \arrow[dl, "U_\eta"] \\ & \mathrm{CAlg}_k & \end{tikzcd}$$

Idea: U_ε is strong-monoidal $\Rightarrow$ U_ε is comonadic and compute the comonad. Thm follows from Barr-Beck.

Look at an obj. in $k[\eta]$- coMod(CAlg$_k$):

- $A \in$ CAlg$_k$
- $A \xrightarrow{c} A \otimes_k k[\eta]$ coaction
- lots of htpy coherences

counit:

$$\begin{tikzcd} A \arrow[r, "c"] \arrow[dr, bend right, "\mathrm{id}_A"'] & A \otimes_k k[\eta] \simeq A \oplus A[-1] \arrow[d] \\ & A \end{tikzcd}$$

split square-zero extension

so c is by definition a derivation d of A into $A[-1]$

$\Leftrightarrow L_A[1] \to A$. $(\varepsilon^2 = 0 \Rightarrow d^2 \simeq 0)$

$k[\varepsilon] \otimes A \to A \overset{\text{dual.}}{\Leftrightarrow} A \to A \otimes k[\eta]$

$B\mathbb{G}_a$ in char p: $B\mathbb{G}_a = \mathrm{colim}(\cdots \mathbb{G}_a \times \mathbb{G}_a \rightrightarrows \mathbb{G}_a \rightrightarrows *)$

$\Gamma(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$ using Čech coh:

$$k \rightrightarrows k[x] \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; k[x,y] \cdots$$

$$\Big\Downarrow \, DK$$

$$k \xrightarrow{0} k[x] \longrightarrow k[x,y] \cdots$$

$$f(x) \longmapsto f(x) + f(y) - f(x+y)$$

has lots of terms in char p.

Cor. The forgetful functor $U_\varepsilon : \varepsilon\text{-}\mathrm{CAlg}_k \to \mathrm{CAlg}_k$ is both monadic and comonadic. In particular it has both left and right adjoints.

Problem: characterize $U_\varepsilon \circ L_\varepsilon$ as $DR(-) = \mathrm{Sym}_-(L_-[1])$.

RMK.
$$\begin{tikzcd} C \arrow[r, "U"] & D \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"] \end{tikzcd} \qquad L \dashv U \dashv R \Rightarrow UL \dashv UR.$$

Observation: Since $\varepsilon\text{-}\mathrm{CAlg}_k \simeq k[\eta]\text{-}\mathrm{coMod}(\mathrm{CAlg}_k)$, then

$R_\varepsilon(A) = A \otimes k[\eta] \simeq A \oplus A[-1]$ split square-zero extension

$U_\varepsilon R_\varepsilon(A) = A \oplus A[-1]$ with no extra struc.

Now need to understand left adjoint of $U_\varepsilon R_\varepsilon$.

For $A, B \in \mathrm{CAlg}_k$, $\mathrm{Map}_{\mathrm{CAlg}_k}(A, U_\varepsilon R_\varepsilon B)$

$$\begin{tikzcd}
 & \mathrm{Map}_{\mathrm{CAlg}_k}(A, U_\varepsilon R_\varepsilon B) \arrow[d, phantom, "\parallel"] \\
\mathrm{Map}_{\mathrm{CAlg}_{k/B}}(A, B \oplus B[-1]) \arrow[d, mapsto] \arrow[r, phantom, "\subset"] & \mathrm{Map}_{\mathrm{CAlg}_k}(A, B \oplus B[-1]) \arrow[d] \\
\{f : A \to B\} \arrow[r, phantom, "\in"] & \mathrm{Map}_{\mathrm{CAlg}_k}(A, B)
\end{tikzcd}$$

$\simeq$

$$\mathrm{Der}(A, f_* B[-1]) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(L_A, f_* B[-1])$$

On the other side

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{CAlg}_{k\,A/}}(\mathrm{Sym}_A(L_A[1]), B) \arrow[d, mapsto] \arrow[r, phantom, "\subset"] & \mathrm{Map}_{\mathrm{CAlg}_k}(\mathrm{Sym}_A(L_A[1]), B) \arrow[d] \\
\{f : A \to B\} \arrow[r, phantom, "\in"] & \mathrm{Map}_{\mathrm{CAlg}_k}(A, B)
\end{tikzcd}$$

$A \to \mathrm{Sym}_A(L_A[1])$

$\simeq$

$$\mathrm{Map}_{\mathrm{Mod}_A}(L_A[1], f_* B) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(L_A, f_* B[-1])$$

Note that there is a canonical map

$$A \to \mathrm{Sym}_A(L_A[1]) \oplus \mathrm{Sym}_A(L_A[1])[-1]$$
$$\| \qquad\qquad$$
$$U_\Sigma R_\Sigma(DR(A))$$

candidate unit map

$\Rightarrow$ get map of fibre sequences

$$\begin{tikzcd}
\mathrm{Map}(L_A[1], f_*B) \arrow[r] \arrow[d, "\wr"] & \mathrm{Map}(DR(A), B) \arrow[r] \arrow[d, "\wr"] & \mathrm{Map}(A, B) \arrow[d, equal] \\
\mathrm{Map}(L_A[1], f_*B) \arrow[r] & \mathrm{Map}(A, B \oplus B[-1]) \arrow[r] & \mathrm{Map}(A, B)
\end{tikzcd}$$

Conclusion: Σ-CAlg_k has all the promised properties.

Finally compare Σ-CAlg_k and S^1-CAlg_k.

$$\begin{tikzcd}
\Sigma\text{-}\mathrm{CAlg}_k \arrow[dr, "U_\Sigma"'] & & S^1\text{-}\mathrm{CAlg}_k \arrow[dl, "U_{S^1}"] \\
& \mathrm{CAlg}_k &
\end{tikzcd}$$

U_Σ monadic, U_{S^1} monadic. Can we compute the monads?

ε-monad: $A \longmapsto DR(A)$, S^1-monad: $A \longmapsto \underset{A\otimes A}{A \otimes A}$

not easy to compare $DR(-)$ and $S^1 \otimes -$, let alone with their monad struc.

Key observation: ε-CAlg_k is comonadic.

$k[\eta] \in \mathrm{CoMon}_{\mathbb{E}_\infty}(\mathrm{CAlg}_k)$.

RMK. For $\mathcal{C}^{\otimes}$ sym. monoidal ∞-cat. we have strong monoidal functor

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathrm{End}\,\mathcal{C} \\ x &\longmapsto x \otimes - \end{aligned}$$

$\Rightarrow \mathrm{CoMon}_{\mathbb{E}_1}(\mathcal{C}) \longrightarrow \mathrm{CoMon}_{\mathbb{E}_1}(\mathcal{C}) = \mathrm{CoMonads}(\mathcal{C})$

Key point: the comonad of ε-$\mathrm{CAlg}_k \longrightarrow \mathrm{CAlg}_k$ is "representable" by $k[\eta]$ with its comultiplication.

Thm. S^1-$\mathrm{CAlg}_k \simeq C^*(S^1)$-$\mathrm{CoMod}(\mathrm{CAlg}_k)$

consider the functor $\underline{S}^1 : dAff^{op} \to \mathcal{S}$ étale sheafification of the constant functor attached to S^1.

Then $C^*(S^1) = R\Gamma(\underline{S}^1, \mathcal{O}_{\underline{S}^1})$.

Concretely as a k-mod this is $k \oplus_{k \oplus k} k \simeq k \oplus k[-1]$

$$\begin{tikzcd}
{} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & k \arrow[r] \arrow[d, "\Delta"] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
k \arrow[r, "\Delta"] & k \oplus k \arrow[r, "{(1,-1)}"] & k
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
{} \arrow[r] \arrow[d] & k[-1] \arrow[r] \arrow[d] & 0 \arrow[d] \\
k \arrow[r] & 0 \arrow[r] & k
\end{tikzcd}$$

with ring struc.
$$\begin{tikzcd}
{} \arrow[r] \arrow[d] & k \arrow[d] \\
k \arrow[r] & k \times k
\end{tikzcd}$$
, $CAlg \to \mathcal{S}$ forgetful functor creates lims.

Summary:
$$\begin{tikzcd}
\varepsilon\text{-}CAlg_k \arrow[d, "\wr"] & S^1\text{-}CAlg_k \arrow[d, "\wr"] \\
k[\eta]\text{-}coMod(CAlg_k) & C^*(S^1)\text{-}coMod(CAlg_k)
\end{tikzcd}$$

enough to prove $k[\eta] \simeq C^*(S^1)$ as comonoids.

To do this, work geometrically

$$\begin{array}{ccc} & \overset{\text{char } 0}{\simeq} \; B\mathbb{G}_a & \\ \mathrm{Spec}(k[\eta]) & & \nwarrow \\ & & \underline{S}^1 \simeq B(\underline{\mathbb{Z}}) \end{array}$$

$\underline{\mathbb{Z}} \to \mathbb{G}_a$ canonical map of groups

$\Rightarrow B(\underline{\mathbb{Z}}) \to B\mathbb{G}_a$ map of groups

$\Rightarrow$ pass to global sections. $R\Gamma(B\mathbb{G}_a, \mathcal{O}) \to R\Gamma(\underline{S}^1, \mathcal{O})$

which has a canonical struc. of map of Hopf algs.

$\Rightarrow$ get isom. $k[\eta] \xrightarrow{\sim} C^*(S^1)$ as Hopf algs. by

combining computations above on chain cpxes.