# LIE GROUPS, DELIGNE'S AXIOMS, AND SHIMURA VARIETIES

## Cartan involution and real reductive groups

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Thus $\mathfrak{g}$ is a Lie algebra such that the symmetric bilinear form

$$B(X,Y) = Tr(Ad(X) \circ Ad(Y))$$

(the *Killing form*) is non-degenerate. This condition suffices for the development of the structure theory of semisimple Lie algebras over algebraically closed fields of characteristic zero. The Killing form is invariant under any automorphism of $\mathfrak{g}$.

A *real form* of $\mathfrak{g}$ is a real Lie algebra $\mathfrak{g}_0$ and an isomorphism $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ of Lie algebras. To a real form $\mathfrak{g}_0$ we can associate a $\mathbb{C}$-conjugate linear automorphism $\sigma$ of $\mathfrak{g}$ with $\sigma^2 = 1$, such that

$$\mathfrak{g}_0 = \mathfrak{g}^\sigma = \{X \in \mathfrak{g} \mid \sigma(X) = X\}.$$

Conversely, if $\sigma$ is a $\mathbb{C}$-conjugate linear automorphism of $\mathfrak{g}$ with $\sigma^2 = 1$ then $\mathfrak{g}^\sigma$ is a real form of $\mathfrak{g}$.

Under the correspondence between Lie groups and Lie algebras, we define a semisimple real Lie algebra to be *compact* if the associated adjoint Lie group is compact.

**Lemma.** *The real form $\mathfrak{g}_0$ of $\mathfrak{g}$ is compact if and only if the Killing form $B_\mathfrak{g}$ restricts to a negative-definite form on $\mathfrak{g}_0$.*

*Proof.* Suppose $B_\mathfrak{g}$ is negative-definite on $\mathfrak{g}_0$. The adjoint group $G_0$ of $\mathfrak{g}_0$ maps faithfully to $Aut(\mathfrak{g}_0)$ and preserves $B_\mathfrak{g}$, hence is compact. Conversely, if $G_0$ is compact, then $\mathfrak{g}_0$ admits a $G_0$-invariant (positive-definite) inner product $<,>$. (Integrate any inner product over $G_0$.) Now $G_0$-invariance of $<,>$ translates to skew-symmetry of $ad(X)$, for any $X \in \mathfrak{g}_0$, with respect to $<,>$. We recall that the non-zero eigenvalues of a real skew symmetric matrix are pure imaginary. Thus $ad(X)^2$ is symmetric, relative to $<,>$, with non-positive eigenvalues, hence $B_\mathfrak{g}(X) = Tr(ad(X)^2) \leq 0$ for all $X \in \mathfrak{g}_0$. If $Tr(ad(X)^2) = 0$ then all the eigenvalues of $ad(X)^2$ are zero, and since a skew-symmetric matrix is diagonalizable this implies $ad(X) = 0$.

We admit the following theorem:

**Theorem.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, with Cartan subalgebra $\mathfrak{h}$. Then there is a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}$ such that $i\mathfrak{h}_\mathbb{R}$ is a maximal abelian subalgebra of $\mathfrak{g}_u$. Here $\mathfrak{h}_\mathbb{R}$ is the subalgebra spanned by the coroot vectors $H_\alpha$ such that $B_\mathfrak{g}(H_\alpha, H) = \alpha(H)$.*

This can in any case be verified easily for classical groups. For $\mathfrak{g} = \mathfrak{sl}(n)$, we take $\mathfrak{g}_u = \mathfrak{su}(n)$. For $\mathfrak{g}$ of type $B$ or $D$ we take the Lie algebra of the compact special orthogonal group. For $\mathfrak{g}$ of type $C$ we take the quaternionic unitary group.

**Theorem.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{g}_u$ a compact real form, and $\tau$ the conjugation of $\mathfrak{g}$ relative to $\mathfrak{g}_u$. Let $\sigma$ be a second conjugation of $\mathfrak{g}$. Then there is a one-parameter group of automorphisms $A(t)$ of $\mathfrak{g}$ such that $A(1)\tau A(1)^{-1}$ and $\sigma$ commute.*

*Proof (from Wallach).* Write $B = B_{\mathfrak{g}}$. For $X, Y \in \mathfrak{g}$, let $(X, Y) = -B(X, \tau(Y))$. This is a hermitian form on $\mathfrak{g}$, and since $B$ is negative-definite on $\mathfrak{g}^\tau = \mathfrak{g}_u$ one sees that $(,)$ is positive-definite on $\mathfrak{g}_u$.

Let $N = \sigma\tau$, a $\mathbb{C}$-linear automorphism of $\mathfrak{g}$. Now

$$(NX, Y) = -B(\sigma\tau(X), \tau(Y)) = -B(X, \tau\sigma\tau(Y)) = -B(X, \tau(NY)) = (X, NY),$$

where the second equality follows because any automorphism of $\mathfrak{g}$ preserves $B$. Thus $N$ is hermitian relative to $(,)$. Since $N$ is an automorphism, $N^2$ is positive-definite hermitian. Then it is a general fact about complex matrices that we can find a hermitian endomorphism $W$ of $\mathfrak{g}$ such that $N^2 = e^W$.

Claim: $C(t) = e^{tW}$ is an automorphism of $\mathfrak{g}$ for all $t \in \mathbb{R}$. If $\{X_i\}$ is a basis of $\mathfrak{g}$, this is equivalent to the vanishing of $F_{i,j}(t) = C(t)[X_i, X_j] - [C(t)X_i, C(t)X_j]$ for all $i, j$. Indeed, this is true for all $t \in \mathbb{Z}$. But the coefficients (in the basis $\{X_i\}$) of $F_{i,j}(t)$ are polynomial functions of $e^{tW}$, and it is a general fact, left as an exercise, that if $W$ is a hermitian matrix and $P$ a polynomial function such that $P(e^{mW}) = 0$ for all $m \in \mathbb{Z}$ then $P(e^{tW})$ is identically zero.

The same principle shows that

(a) $$C(t)\tau C(t) = \tau$$

for all $t$. As above, it suffices to check this for $t \in \mathbb{Z}$, hence that $N\tau N = \tau$; but $\tau N\tau = \tau\sigma\tau^2 = \tau\sigma = N^{-1}$, so this is clear. Similarly,

(b) $$NC(t) = C(t)N$$

for all $t \in \mathbb{R}$.

Let $\tau_t = C(t)\tau C(-t)$. Then applying (a)

$$\sigma\tau_t = \sigma C(t)\tau C(-t) = \sigma\tau C(-2t) = NC(-2t)$$

and by (a) and (b)

$$\tau_t\sigma = C(t)\tau C(-t)\sigma = C(2t)\tau\sigma = C(2t)N^{-1} = N^{-1}C(2t)$$

Thus $\sigma\tau_t = \tau_t\sigma$ if and only if $NC(-2t) = N^{-1}C(2t)$; i.e. if and only if $N^2 = C(4t)$. But $N^2 = C(1)$, hence this is true if $t = \frac{1}{4}$. It follows that we can take $A(t) = C(\frac{1}{4}t)$.

**Corollary.** *Any two compact real forms of $\mathfrak{g}$ are conjugate by an element of $Aut(\mathfrak{g})^0$ ($= Ad(G_0)$).*

*Proof.* Exercise.

**Definition.** *Let $\mathfrak{g}$ be a real semisimple Lie algebra. A Cartan involution of $\mathfrak{g}$ is an involutive automorphism $\sigma$ of $\mathfrak{g}$ so that, if $\mathfrak{k} = \mathfrak{g}^\sigma$, $\mathfrak{p} = \mathfrak{g}^{\sigma=-1}$, then $B_{\mathfrak{g}}$ is negative-definite on $\mathfrak{k}$, positive-definite on $\mathfrak{p}$. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called a Cartan decomposition.*

*More generally, if $\mathfrak{g}$ is reductive, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}$, then a Cartan involution of $\mathfrak{g}$ is an involution that acts trivially on $\mathfrak{z}$ and restricts to a Cartan involution on $[\mathfrak{g}, \mathfrak{g}]$.*

**Theorem.** *Let $\mathfrak{g}$ be a real semisimple Lie algebra. Then $\mathfrak{g}$ has a Cartan involution, and any two are conjugate by an element of $Aut(\mathfrak{g})^0$.*

*Proof.* Let $\mathfrak{g}_u$ be a compact real form of $\mathfrak{g}_{\mathbb{C}}$, and let $\sigma$, resp. $\tau$, be conjugation relative to $\mathfrak{g}$, resp. $\mathfrak{g}_u$. After conjugation, we can assume $\sigma$ and $\tau$ commute, hence $\sigma$ defines an involution of $\mathfrak{g}$. It is clear that this is a Cartan involution.

The final assertion is proved just as above.

**Deligne's axioms.**

We now consider a connected reductive group $G$ over $\mathbb{Q}$. Thus $G$ is a linear algebraic group with Lie algebra $Lie(G)$, which we denote $\mathfrak{g}$ To the Lie subalgebra $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}'$ corresponds a connected closed subgroup $G' \subset G$, which is semisimple in the sense that $\mathfrak{g}'$ is a semisimple Lie algebra. The quotient $G/G'$ is a commutative linear algebraic group, which is a torus because $G$ is assumed reductive.

We let $\mathbb{S}$ denote the torus $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$. This is the torus over $\mathbb{R}$ with character group $X^*(\mathbb{S})_{\mathbb{C}} = \mathbb{Z}^2$, on which complex conjugation acts by sending $(p,q) \in \mathbb{Z}^2$ to $(q,p)$. Let $h : \mathbb{S} \to G_{\mathbb{R}}$ be a homomorphism of real groups, and let $\rho : G \to GL(V)$ be a representation of $G$ on a $\mathbb{Q}$-vector space $V$. Then $\rho \circ h : \mathbb{S} \to GL(V_{\mathbb{R}})$ induces a Hodge structure as we have seen: the action of $\rho \circ h$ decomposes $V_{\mathbb{C}}$ as a sum $\oplus V^{p,q}$, where $\rho \circ h(z)$ acts as $z^{-p}\bar{z}^{-q}$ on $V^{p,q}$. Since $h$ is defined over $\mathbb{R}$, the fact that complex conjugation exchanges $(p,q)$ and $(q,p)$ implies that $\bar{V}^{p,q} = V^{q,p}$, as required.

There is a map $w : \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ dual to the map

$$\mathbb{Z}^2 = X^*(\mathbb{S})_{\mathbb{C}} \to X^*(\mathbb{G}_{m,\mathbb{R}}) = \mathbb{Z};\ (p,q) \to p+q.$$

Thus the image of $w$ corresponds to the elements fixed by complex conjugation, and on points $w$ takes $\mathbb{R}^\times$ to itself in $\mathbb{C}^\times$. For any integer $a$, and $h$, $\rho$ as above, the subspace of $V$ of weight $a$ is $V(a) = \sum_{p+q=a} V^{p,q}$. This is a rational subspace if and only if $\rho \circ h \circ w$ comes from a map $\mathbb{G}_m \to GL(V)$ defined over $\mathbb{Q}$. Moreover, it is a $G$-invariant subspace if and only if $w_h := h \circ w$ takes $\mathbb{G}_{m,\mathbb{R}}$ to the center $Z_G$ of $G$. We will only consider $h$ such that

**Hypothesis.** *$w_h$ is defined over $\mathbb{Q}$ and takes $\mathbb{G}_m$ to $Z_G$.*

This implies in particular that $h(-1) \in Z_G$, hence $adh(-1) = 1$.

**Definition.** *A Shimura datum is a pair $(G,X)$ where $G$ is a reductive group over $\mathbb{Q}$ and $X$ is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying the above hypothesis, as well as*

(i) *For any $h \in X$, $ad\ h(i)$ induces a Cartan involution of $G^{ad}$;*

(ii) *For any $h \in X$, the representation $ad \circ h : \mathbb{S} \to GL(\mathfrak{g}_{\mathbb{R}})$ induces a Hodge structure with the property that*

$$\mathfrak{g}^{p,q} = \{0\} \text{ unless } (p,q) \in \{(-1,1),(0,0),(1,-1)\}.$$

(iii) *We sometimes add the hypothesis that $G^{ad}$ has no non-trivial $\mathbb{Q}$-rational factor $G_0$ such that the projection of $h$ on $G_0$ is trivial; this is not essential but simplifies the adelic arguments.*

We draw some conclusions from these axioms. First the hypothesis that $w_h$ takes $\mathbb{G}_m$ to $Z_G$ follows from (ii): the Hodge structure given by $ad \circ h$ is of weight zero, hence the image of $w_h$ under $h$ is trivial, i.e. $w_h(\mathbb{G}_m) \subset Z_G$. Moreover $w_h$ is independent of $h \in X$, since $X$ is a conjugacy class and anything in the center is invariant under conjugacy, so it can be called $w_X$.

For any $h \in X$, let $K_h$ denote the centralizer in $G_{\mathbb{R}}$ of $h(\mathbb{S})$. By definition $\mathfrak{k}_h = Lie(K_h) = \mathfrak{g}^{0,0}$. Note that $ad\ h(i)$ acts as $-1$ on $\mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}$ and as $+1$ on $\mathfrak{k}_h$. We let $\mathfrak{p}_h^- = \mathfrak{g}^{1,-1}$, $\mathfrak{p}_h^+ = \mathfrak{g}^{-1,1}$, $\mathfrak{p}_{h,\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, the complexification of the $-1$ eigenspace $\mathfrak{p}$ of $ad(h(i))$ on $\mathfrak{g}_{\mathbb{R}}$. By (i) the image of $K_h$ in $G^{ad}(\mathbb{R})^0$ is a maximal compact subgroup and modulo $Lie(Z_G)$, $\mathfrak{k}_h \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. As in the elliptic modular case, (ii) is closely related to the Kodaira-Spencer map for moduli (in certain cases).

We have already seen examples with $G = GL(2)$ or $G = H_{\mathcal{K}}$. Given a Shimura datum $(G, X)$, and an open compact subgroup $K \subset G(\mathbf{A}_f)$, we define the (complex) Shimura variety

$$_KSh(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbf{A}_f)/K; Sh(G,X) = \varprojlim_K {_KSh(G,X)}.$$

The set $X$ is a $C^\infty$ manifold with finitely many connected components, all conjugate under $G(\mathbb{R})$ (by definition). As in the case already discussed of $GL(2)$, the Shimura varieties at finite level are finite unions of quotients $\Gamma\backslash X^0$, where $X^0$ is (any) connected component of $X$ and $\Gamma$ runs through certain discrete subgroups of $G(\mathbb{R}) \cap G(\mathbb{Q})$. The finiteness is a consequence of basic facts about adèle groups, and it is an important consequence of reduction theory that, in any $G(\mathbb{R})$-invariant measure on $X$, the quotients $\Gamma\backslash X^0$ all have finite volume. We will (or we may) return to these matters later. Meanwhile, here are some more examples of Shimura data:

**Tori.** Suppose $G = T$ is a torus. The set $X$ is then a single homomorphism $h_X$. There is a perfect pairing

$$<,>: X^*(T)_{\overline{\mathbb{Q}}} \otimes X_*(T)_{\overline{\mathbb{Q}}} \to \mathbb{Z}$$

where $X_*(T) = Hom(\mathbb{G}_m, T)$, $X^*(T) = Hom(T, \mathbb{G}_m)$. The duality is defined as follows: if $\mu \in X_*(T)$ and $\chi \in X^*(T)$ then $\chi \circ \mu \in Hom(\mathbb{G}_m, \mathbb{G}_m)$ is a character of the form $t \mapsto t^r$, and we let $r =< \chi, \mu >$. This duality is compatible with the $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-action on both sides.

We introduce the cocharacter $\mu : \mathbb{G}_m \to \mathbb{S}_{\mathbb{C}}$ with the property that, for any $h : \mathbb{S} \to GL(V)$, $h \circ \mu(z)$ acts as $z^{-p}$ on $V^{p,q}$; equivalently, if $e_1 = (1,0)$ and $e_2 = (0,1)$ are a basis of $X^*(\mathbb{S})_{\mathbb{C}}$, then $< e_1, \mu >= 1$, $< e_2, \mu >= 0$. Let $\mu_h = h \circ \mu$; then we can reconstruct $h$ from $\mu_h$ via $h(z) = \mu_h(z) \cdot \mu_h(\bar{z})$. In this way a Shimura datum $(T, h_X)$ is determined by the cocharacter $\mu_X = \mu_{h_X} \in X_*(T)_{\mathbb{C}}$. Moreover, any $\mu_X$ will do, since the two hypotheses are vacuous.

Some $\mu_X$ are especially interesting, however. Suppose $\mathcal{K}$ is a CM field, i.e. a totally imaginary quadratic extension of a totally real field $F$. Say $d = [F : \mathbb{Q}]$. Let $T = R_{\mathcal{K}/\mathbb{Q}}\mathbb{G}_m$. A basis of $X^*(T)_{\overline{\mathbb{Q}}} = X^*(T)_{\mathbb{C}}$ is given by $\Sigma_{\mathcal{K}} = Hom(\mathcal{K}, \overline{\mathbb{Q}}) = Hom(\mathcal{K}, \mathbb{C})$. Indeed,

$$T(\overline{\mathbb{Q}}) = (\mathcal{K} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^\times = (\oplus_{\sigma \in \Sigma_{\mathcal{K}}} \overline{\mathbb{Q}}_\sigma)^\times,$$

where $\overline{\mathbb{Q}}_\sigma$ is just $\overline{\mathbb{Q}}$ indexed by $\sigma$. Projection on each factor comes from an algebraic character which we denote $\sigma$. Let $i_\sigma, \sigma \in \Sigma_{\mathcal{K}}$ denote the dual basis of $X_*(T)_{\overline{\mathbb{Q}}}$.

**Exercise.** *Show that these are genuine algebraic characters, and that $X^*(T)_{\overline{\mathbb{Q}}}$ is canonically isomorphic to $\mathbb{Z}^{Hom(\mathcal{K},\overline{\mathbb{Q}})}$ with the canonical action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on the latter.*

A *CM type* for $\mathcal{K}$ is a set $\Phi \subset \Sigma_{\mathcal{K}}$ of $d$ elements such that $\Sigma_{\mathcal{K}} = \Phi \coprod c\Phi$. Equivalently, restriction to $F$ defines a bijection $\Phi \leftrightarrow \Sigma_F = Hom(F, \overline{\mathbb{Q}}) = Hom(F, \mathbb{R})$. To a CM type $\Sigma$ we can associate a cocharacter $\mu_\Phi = \sum_{\sigma \in \Phi} i_\sigma$. This defines a unique $\mathbb{R}$-homomorphism $h_\Phi : \mathbb{S} \to T_{\mathbb{R}}$, such that $\mathfrak{h}_\Phi \cdot \mu = \mu_\Phi$. Concretely, the CM type $\Phi$ defines an identification

$$T(\mathbb{R}) = (\mathcal{K} \otimes_{\mathbb{Q}} \mathbb{R})^\times \xrightarrow{\sim} (\mathbb{C}^\times)^\Phi$$

and the map $h_\Phi : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \to (\mathbb{C}^\times)^\Phi$ is the diagonal map $z \mapsto (z, \ldots, z)$. When $\mathcal{K}$ is imaginary quadratic and $\Phi$ is a single embedding then this notation is consistent with our previous notation in that case.

Note that $h_\Phi$ takes values in the subtorus $H_{\mathcal{K}} \subset T$ defined by the Cartesian diagram

$$\begin{array}{ccc} H_{CK} & \longrightarrow & R_{\mathcal{K}/\mathbb{Q}}\mathbb{G}_{m,\mathcal{K}} \\ \downarrow & & \downarrow{\scriptstyle N_{\mathcal{K}/F}} \\ \mathbb{G}_{m,\mathbb{Q}} & \xrightarrow{\iota} & R_{\mathcal{K}/\mathbb{Q}}\mathbb{G}_{m,\mathcal{K}} \end{array}$$

Here $N_{\mathcal{K}/F}$ is the map on tori defined by the obvious map on characters

$$\sigma \mapsto \sum_{\sigma' \mid_F = \sigma} \sigma'$$

and $\iota$ is the inclusion defined by the map $\sigma \mapsto Id$ on characters, where $Id$ is the identity character of $\mathbb{G}_m$. Concretely, $H_{\mathcal{K}}(\mathbb{Q})$ is the subgroup of elements of $y \in \mathcal{K}^\times$ such that $yy^c \in \mathbb{Q}^\times$; it is the Serre torus attached to $\mathcal{K}$. Thus there is a Shimura datum $(H_{\mathcal{K}}, h_\Phi)$.

The simplest non-trivial Shimura variety is $(\mathbb{G}_m, N)$, where $N(z) = z\bar{z} \in \mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times$.

**The Siegel upper half-space.** *All calculations in this section are left as exercises.*

Let $J$ be the $2n \times 2n$ matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The symplectic similitude group $G = GSp(2n)$ (group scheme over $\mathbb{Z}$) is the group

$$\{g \in GL(2n) \mid {}^t gJg = \nu(g)J\}$$

where $\nu(g)$ is a scalar. Thus there is a homomorphism $\nu : GSp(2n) \to \mathbb{G}_m$, with kernel $Sp(2n)$. Note that the group $Z_G$ of scalar diagonal matrices in $GL(n)$ is contained in $GSp(2n)$, and $\nu(tI_{2n}) = t^2$, whence it follows that $\det = \nu^n$ as characters of $GSp(n)$. Given any $2n$-dimensional space $V$ with alternating form $<,>$, one can define $Sp(V, <,>)$ and $GSp(V, <,>)$, isomorphic respectively to $Sp(2n)$ and $GSp(2n)$. Since $J$ is written in four $n \times n$ blocks, the elements of $GSp(2n)$ can be written $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The Lie group

$$U(n) = \{g \in GL(n, \mathbb{C}) \mid {}^t g^{-1} = \bar{g}\}$$

maps to $Sp(n,\mathbb{R})$ via
$$A+Bi \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$
Let $K_0 = Z_G(\mathbb{R})\cdot U(n) \subset GSp(n,\mathbb{R})$. There is a homomorphism $h_0 : \mathbb{S} \to K_0$ of real algebraic groups such that
$$h_0(x+iy) = \begin{pmatrix} xI_n & yI_n \\ -yI_n & xI_n \end{pmatrix}.$$
Of course this $h_0$ induces a complex structure on the space $\mathbb{R}^{2n}$ via the natural representation of $G$ on $\mathbb{R}^{2n}$. Note that when $n=1$ we have $GSp(2) = GL(2)$ and the map $h_0$ is one we encountered at the beginning of the course. One checks that the centralizer of $h_0$ is precisely $K_0$, hence the conjugacy class $X$ of $h_0$ is a $C^\infty$ manifold isomorphic to $G(\mathbb{R})/K_\infty$.

We verify that the pair $(G,X)$ satisfies the axioms for a Shimura datum. Obviously the weight is central and defined over $\mathbb{Q}$. The matrix $h_0(i)$ is just the matrix $J$, whose centralizer in $G'$ is the group
$$\{g \in GL(2n) \mid {}^t gJg = J = g^{-1}Jg\} = Sp(2n,\mathbb{R}) \cap O(2n)$$
which is easily seen to equal the image of $U(n)$, and to be a maximal compact subgroup. Indeed, $g \in Sp(2n)$ if and only if ${}^t g = Jg^{-1}J^{-1}$, i.e. if and only if ${}^t g^{-1} = JgJ^{-1} = ad(h_0(i))(g)$. Thus $ad(h_0(i))$ is the restriction to $Sp(2n,\mathbb{R})$ of $g \mapsto {}^t g^{-1}$ which can also be used to define the Cartan involution. Finally, there is a decomposition
$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^- = \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}$$
as in the case $n=1$, with $\mathfrak{k} = Lie(K_0)$ and
$$\mathfrak{p}^+ = \{\begin{pmatrix} \alpha & i\alpha \\ i\alpha & -\alpha \end{pmatrix}\}; \quad \mathfrak{p}^- = \{\begin{pmatrix} \alpha & -i\alpha \\ -i\alpha & -\alpha \end{pmatrix}\}$$
where $\alpha$ runs over the vector space $S_n(\mathbb{C})$ of *symmetric* complex matrices.

The space $X$ has two connected components, corresponding to $\nu(g) > 0$ and $\nu(g) < 0$. There is a map $G(\mathbb{R}) \to S_n(\mathbb{C})$ given by $g \mapsto g(iI_n) = (Ai+B)(Ci+D)^{-1}$. The image is the union of $\mathfrak{S}_n^+ = \{Z \in S_n(\mathbb{C}) \mid Im(Z) > 0\}$ with $\mathfrak{S}^- = -\mathfrak{S}_n^+$, and the map factors through an injective map $G(\mathbb{R})/K_0$ (because $K_0$ is the stabilizer of the point $iI_n$). Thus $X$ is identified with the *Siegel double space* $\mathfrak{S}_n^\pm$, and the Shimura variety ${}_KSh(G,X)$ is called the *Siegel modular variety* of genus $n$ and level $K$. It parametrizes principally polarized abelian varieties with level structure $K$. This is the most important case for the present course, and we will return to this case after the break.

If $F$ is a totally real field of degree $d$, and $(V,<,>)$ is a symplectic space over $F$ of dimension $2n$, we can consider $Sp(V)$ and
$$GSp_F(V,<,>) = \{g \in GL(V) \mid\ < gv, gw > = \nu(g) < v,w >, \nu(g) \in GL(1)\}$$
as well as the subgroup
$$GSp(V,<,>) = R_{F/\mathbb{Q}} GSp_F(V,<,>) \times_{R_{F/\mathbb{Q}}GL(1)} \mathbb{G}_{m,\mathbb{Q}}$$

whose similitude factors lie in $\mathbb{Q}$. Then $GSp(V,<,>)$ is a $\mathbb{Q}$-algebraic group, with

$$GSp(V,<,>)(\mathbb{R}) \xrightarrow{\sim} \{(g_i) \in GSp(2n,\mathbb{R})^d \mid \nu(g_i) = \nu(g_j), \forall i,j\}.$$

Then the conjugacy class $X_F = \mathfrak{S}^{\pm}_{n,F}$ of the homomorphism $h_0^d : \mathbb{S} \to GSp(2n,\mathbb{R})^d$, with values in $GSp(V,<,>)(\mathbb{R})$, defines a Shimura datum $(GSp(V,<,>), X_F)$ whose associated Shimura variety is called the *Hilbert-Siegel modular variety.*

**Unitary groups.** Let $p+q=n$, let $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, and define

$$GU(p,q) = \{g \in GL(n,\mathbb{C}) \mid {}^t\bar{g} I_{p,q} g = \nu(g) I_{p,q}\}$$

(unitary similitudes of signature $(p,q)$. As above, $U(p,q) = \ker \nu \subset GU(p,q)$. The same definitions can be given with $\mathbb{C}$ replaced by any imaginary quadratic field $\mathcal{K}$; then we obtain an algebraic group $G$ over $\mathbb{Q}$, since $g \mapsto \bar{g}$ can be viewed as an algebraic automorphism of $R_{\mathcal{K}/\mathbb{Q}} GL(n)_{\mathcal{K}}$. We define a map $h_0 : \mathbb{S} \to G(\mathbb{R}) = GU(p,q)$ by $h_0(z) = \begin{pmatrix} zI_p & 0 \\ 0 & \bar{z} I_q \end{pmatrix}$. Then $h_0(i) = i \cdot I_{p,q}$, and the centralizer of $h_0(i)$ in $U(p,q)$ is $U(p,q) \cap U(n) = U(p) \times U(q)$, easily seen to be a maximal compact subgroup of $U(p,q)$. Thus again $adh_0(i)$ is a Cartan involution, for which $\mathfrak{k} = Lie(K_0)$ with $K_0 = [GU(p) \times GU(q)] \cap G(\mathbb{R})$, and with $\mathfrak{p} = \{\begin{pmatrix} 0_p & Z \\ {}^t\bar{Z} & 0_q \end{pmatrix}\}$ as $Z$ varies over $p \times q$ complex matrices. As for the axiom (ii), we see that $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ with $\mathfrak{p}^+ = \{\begin{pmatrix} 0_p & W \\ 0 & 0_q \end{pmatrix}\}$, $\mathfrak{p}^+ = \{\begin{pmatrix} 0_p & 0 \\ W' & 0_q \end{pmatrix}\}$, as $W, W'$ run over matrices of the appropriate size (of total dimension m Obviously $adh_0(z)$ has eigenvalue $z\bar{z}^{-1}$ (resp. $z^{-1}\bar{z}$) on $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$), hence $\mathfrak{p}^+ = \mathfrak{g}^{-1,1}$, $\mathfrak{p}^- = \mathfrak{g}^{1,-1}$, as in the previous cases.

More generally, let $(\mathcal{K},\Phi)$ be a CM type, as above, with $[\mathcal{K}:\mathbb{Q}] = 2d$. Let $(\underline{p},\underline{q}) = (p_\sigma, q_\sigma)_{\sigma\in\Phi})$ with $p_\sigma, q_\sigma \in 0,1,\ldots, p_\sigma + q_\sigma = n$. We define $I_{\underline{p},\underline{q}} = (I_{p_\sigma,q_\sigma}) \in GL(n,\mathbb{C})^\Phi$, and let

$$GU(\underline{p},\underline{q}) = \{g \in GL(n,\mathbb{C})^\Phi \mid {}^t\bar{g} I_{\underline{p},\underline{q}} g = \nu(g) I_{\underline{p},\underline{q}}\}.$$

Here $\nu(g) \in \mathbb{R}^\times$ (necessarily real) embedded diagonally in $GL(n,\mathbb{C})$. Let $V$ be an $n$-dimensional vector space over $\mathcal{K}$ with hermitian form $<,>$ such that, for $\sigma \in \Phi$, the induced form on $V_\sigma$ has signature $(p_\sigma, q_\sigma)$. Then we can define the unitary similitude group $GU(V)$ as an algebraic group over $\mathbb{Q}$, imposing the condition that the similitude lie in $\mathbb{G}_m$ over $\mathbb{Q}$, and $GU(V)(\mathbb{R}) \xrightarrow{\sim} GU(\underline{p},\underline{q})$. There is a map $h_0 : \mathbb{S} \mapsto GU(\underline{p},\underline{q})$ by $z \mapsto (\begin{pmatrix} zI_{p_\sigma} & 0 \\ 0 & \bar{z} I_{q_\sigma} \end{pmatrix}$, and the $GU(\underline{p},\underline{q})$-conjugacy class $X = X_{\underline{p},\underline{q}}$ of $h_0$ defines a Shimura datum $(GU(V), X)$. The case where $n = 1 = p_\sigma$ for all $\sigma \in \Phi$ gives exactly the pair $(H_{\mathcal{K}}, h_\Phi)$.

A *morphism* of Shimura data $j : (G,X) \to (H,Y)$ is a homomorphism $j : G \to H$ of $\mathbb{Q}$-algebraic groups such that, for all $h \in X$, $j \circ h \in Y$. A Shimura datum $(G,X)$ is of *Hodge type* if it admits an injective homomorphism $(G,X) \to (GSp(2n), \mathfrak{S}^{\pm})$.

If $(G,X)$ is a Shimura datum, $H$ is a torus, and $j : G \to H$ is a homomorphism, then $j_X = j \circ h : \mathbb{S} \to H_{\mathbb{R}}$ is independent of $h \in X$ and we obtain a morphism $(G,X) \to (H, j_X)$ of Shimura data. For example, there are natural maps $\nu : (GSp(2n), \mathfrak{S}^{\pm}_n) \to (\mathbb{G}_m, N)$, and $GSp(2n)$ can be replaced by any of the symplectic or unitary similitude groups discussed above.

**Exercise.** *Let $(\mathcal{K}, \Phi)$ be a CM type and let $GU(V)$ be as above for some hermitian space $V$ over $\mathcal{K}$. The map $\det : GU(V) \rightarrow R_{\mathcal{K}/\mathbb{Q}}\mathbb{G}_{m,\mathcal{K}}$ takes values in the Serre torus $H_{\mathcal{K}}$. Determine the Shimura datum $(H_{\mathcal{K}}, \det_X)$ for $X = X_{\underline{p},\underline{q}}$ as a function of the signatures.*

In general, a homomorphism $G \rightarrow H$ does not define a morphism of Shimura data. Here are some examples where it does. First, let $F$ be a totally real field of degree $d$, and $(V, <,>)$ a symplectic space over $F$ of dimension $2n$. Then $V_{\mathbb{Q}} R_{F/\mathbb{Q}} V$ is a rational vector space of dimension $2nd$ and $R_{R/\mathbb{Q}} <,>$ is a (non-degenerate) alternating form on this space, invariant up to scalar multiples by $GSp(V, <,>)$ (one good reason to assume the similitude factor is in $\mathbb{Q}$!). Thus there is a homomorphism $GSp(V, <,>) \rightarrow GSp(2nd)$ defined by choice of symplectic basis of $V$ and a basis of $F$ over $\mathbb{Q}$, hence unique up to conjugation.

**Exercise.** *Show that this map defines a morphism of Shimura data*

$$(GSp(V, <,>), X_F) \rightarrow (GSp(2nd), \mathfrak{S}^{\pm}_{nd}).$$

Now suppose $\mathcal{K}$ is a CM quadratic extension of $F$, and let $V$ be an $n$-dimensional vector space over $\mathcal{K}$ with hermitian form $<,>$. Choose an element $\alpha \in \mathcal{K}^{\times}$ with $\alpha + c(\alpha) = 0$ ($\alpha$ is totally imaginary), and define

$$< v, v' >_{\alpha} = Tr_{\mathcal{K}/F} \alpha < v, v' > .$$

Note that the form $v \otimes v' \mapsto \alpha < v, v' >$ is skew-hermitian, and indeed the choice of a non-zero totally imaginary element $\alpha \in \mathcal{K}$ defines a bijection between hermitian and skew-hermitian forms. Then one verifies that $<,>_{\alpha}$, viewed as a bilinear form on $R_{\mathcal{K}/F} V$, is skew-symmetric. Thus there is a homomorphism $j : GU(V) \rightarrow GSp(R_{\mathcal{K}/F} V, <,>_{\alpha})$.

**Exercise.** *Show that this map defines a morphism of Shimura data*

$$(GU(V), X_{\underline{p},\underline{q}}) \rightarrow (GSp(R_{\mathcal{K}/F} V, <,>_{\alpha}), X_F).$$

Obviously a morphism $j : (G, X) \rightarrow (H, Y)$ of Shimura data defines a map of sets

$$G(\mathbb{Q}) \backslash X \times G(\mathbf{A}_f) \rightarrow H(\mathbb{Q}) \backslash Y \times H(\mathbf{A}_f)$$

and if $K_H \subset H(\mathbf{A}_f)$ contains $j(K)$, $K \subset G(\mathbf{A}_f)$, we get a smooth map

$${}_K Sh(G, X) \rightarrow {}_{K_H} Sh(H, Y)$$

of Shimura varieties. If $j$ is injective then in the limit this defines an injective map (this follows from more facts about adèle groups).

## The complex structure

Let $(G, X)$ be a Shimura datum. For every $h \in X$ we define a Hodge structure $\mathfrak{g}_h^{p,q}$ on $\mathfrak{g}$ via $Ad \circ h : \mathbb{S} \rightarrow GL(\mathfrak{g})$, concentrated in $(p,q) = (-1,1), (0,0), (1,-1)$.

The subscript $h$ will be omitted when convenient. As in the first lecture, there is a Hodge filtration $F^p_h\mathfrak{g} = \oplus_{p'\geq p}\mathfrak{g}^{p',w-p'}$, with

$$0 \subset F^1\mathfrak{g} = \mathfrak{g}^{1,-1} \subset F^0\mathfrak{g} \subset F^{-1}\mathfrak{g} = \mathfrak{g}.$$

It is easy to see that each $F^p_h\mathfrak{g}$ is a Lie subalgebra (the map $Ad\circ h$ respects the Lie algebra structure). Moreover, the Lie bracket of $\mathfrak{g}^{1,-1}$ with itself is contained in $\mathfrak{g}^{2,-2} = 0$, hence $F^1\mathfrak{g}$ is commutative; and $F^1\mathfrak{g}$ is an ideal in $F^0$ ($[F^0\mathfrak{g}, F^1\mathfrak{g}] \subset F^1\mathfrak{g}$). Let $P_h \subset G$ be the subgroup with Lie algebra $F^0_h\mathfrak{g}$, $U_h \subset P_h$ with $LieU_h = F^1_h\mathfrak{g}$. Then $(P_h/U_h)(\mathbb{C}) \xrightarrow{\sim} K_h(\mathbb{C})$ where $K_h \subset G(\mathbb{R})$, the centralizer of $h$, equals a maximal connected compact subgroup modulo the center $Z_G$.

**Proposition.** *$K_h$ contains a maximal torus of $G$; i.e. $\mathfrak{g}^{0,0}_h$ contains a Cartan subalgebra of $\mathfrak{g}$.*

*Proof.* The image of $h$ is a torus in $G(\mathbb{R})$. It is a theorem that any torus in a connected Lie group is contained in a maximal torus. Let $T$ be any maximal torus containing $h(\mathbb{S})$. Then in particular, $T$ commutes with $h(\mathbb{S})$, hence is necessarily contained in $K_h$.

**Fact.** *The identity component of the centralizer of a torus $S$ in a connected reductive algebraic group is reductive.*

(This can be deduced from the existence of a maximal torus containing $S$ and the properties of root systems.) In particular $K_h$ is reductive.

**Exercise.** *Let $B_0 \subset K_h$ be a Borel subgroup. Then $B = U_h \cdot B_0$ is a Borel subgroup of $G$.*

*Idea.* Since $U_h$ is abelian and normal in $B$ and $B/U_h = B_0$ is solvable, it follows that $B$ is solvable. Let $T \subset B_0$ be a maximal torus, and decompose $Lie(B)$ under the action of $T$. It suffices to show that $\dim \mathfrak{g}/Lie(B) = \dim Lie(B)/Lie(T)$ to show that the set of roots of $T$ in $Lie(B)$ is a set of positive roots.

**Fact.** *Let $G$ be a connected reductive algebraic group, $P \subset G$ a closed subgroup. The following are equivalent*

1. *$P$ contains a Borel subgroup.*
2. *$G/P$ is a projective algebraic variety.*

*If these conditions are satisfied, $P$ is called a* parabolic subgroup *of $G$, and the variety $G/P$ is a* generalized flag variety.

The points of $G/P$ parametrize the parabolic subgroups of $G$ conjugate to $P$; to any $x \in G/P$ we let $P_x \subset G$ denote its stabilizer. Letting the point $x_0 = 1\cdot P$, any $x \in G/P$ can be written $x = g\cdot x_0$ for some $g \in G$, and then $P_x = gPg^{-1}$. Let $\hat{X} = G/P_{h_0}(\mathbb{C})$ for some $h_0 \in X$. We see that there is a map $\beta : X \to \hat{X}$ (the Borel embedding) sending $h$ to $P_h$.

**Lemma.** *The Borel embedding is an open immersion of $C^\infty$-manifolds.*

*Proof.* First, the map is a local open immersion: for this it suffices to show that the map on tangent spaces $\mathfrak{p}_h \to T_{\beta(h)} = \mathfrak{g}/F^0\mathfrak{g} = \mathfrak{g}^{-1,1}_h = \mathfrak{p}^+_h$ is an isomorphism; but this is just the natural map

$$\mathfrak{p}_h \hookrightarrow \mathfrak{p}_{h,\mathbb{C}} \to \mathfrak{p}_{h,\mathbb{C}}/\mathfrak{g}^{1,-1}_h.$$

To show that $\beta$ is injective, it suffices to observe that $G(\mathbb{R}) \cap P_h = K_h$, hence $P_h$ at least determines the centralizer of $h$. Now we have to verify that there is a unique map $h : \mathbb{S} \to K_h$ so that $Ad \circ h(z)$ acts as required on $U_h$.

Since $\hat{X}$ is a $G(\mathbb{C})$-homogeneous complex manifold, the open submanifold $X$ has a natural $G(\mathbb{R})$-equivariant complex structure. It follows that the Shimura varieties ${}_K Sh(G, X)$ are complex manifolds (provided $K$ is sufficiently small, etc.) It is much more difficult to prove that

**Theorem.** ${}_K Sh(G, X)$ *is a quasi-projective complex algebraic variety.*

The proof requires (i) a lengthy analysis of relative root systems; (ii) reduction theory to construct a topological compactification (the Satake compactification); (iii) difficult theorems from complex analysis to prove that the resulting compact space is a (generally highly singular) complex analytic variety; (iv) the construction of a family of complex analytic line bundles on this analytic variety (the theory of the canonical automorphy factor; and (v) the construction of sufficiently many sections of these line bundles (automorphic forms, via the theory of Poincaré series) to show that most of them are very ample, hence define projective embeddings. The complete proof is due to Baily and Borel, though partial results were obtained by Piatetski-Shapiro.

In the cases of interest (Siegel and unitary modular varieties) we will at least define moduli spaces whose complex points are in bijection with ${}_K Sh(G, X)$.