MODERN ALGEBRA I GU4041

Homework 10, due November 20: Group actions and conjugacy classes

1. Judson, section 14.5, exercises 2, 3.

2. List the conjugacy classes of the groups Q_8, $\mathbb{Z}_{12}$, and D_{14}. Determine the number of elements in each conjugacy class and verify the class equation for each group.

3. Let A be a set with n elements, and let $P(A)$ denote the set of subsets of A.

(i) How many elements does $P(A)$ have?

(ii) Number the elements of A from 1 to n and let the symmetric group Σ_n act on A by permuting the elements. Thus if $n = 5$, the element

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

takes the first element to the second element, the second element to the third element, and so on. How many orbits does this action have?

(iii) Show that the action defined in (ii) defines an action of Σ_n on $P(A)$. How many orbits does this action have? How many elements are in each orbit? Justify your answer.

(iv) Write $P(A)$ as the union of the orbits described in (iii):

$$P(A) = \bigcup_{i \in I} O_i.$$

Write $|P(A)| = \sum_i |O_i|$. Use the binomial theorem to provide another proof of this equality.

4. Describe all of the finite groups with exactly two conjugacy classes, and prove your claim.

Recommended reading

Judson, Chapter 14. Gallagher's notes, 16, 17.