MODERN ALGEBRA I GU4041

Homework 11, due November 30: Solvable and nilpotent groups, composition series

1. Judson, section 13.4, exercises 4 and 12.

2. Prove that any subgroup of a solvable group is solvable.

3. Give an example of a finite solvable group whose center is just the identity element.

4. Let H be the subset of $GL(3, \mathbb{R})$ consisting of matrices of the form

$$u(x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where x, y, z are real numbers.

(a) Show that H is a group. (This is one version of what is called the *Heisenberg group.*)

(b) Determine the center $Z(H)$ of H.

(c) Show that H is a nilpotent group, and determine the descending central series of H.

(d) Find an abelian subgroup of H that is different from $Z(H)$.

5. Let p be an odd prime number. Show that any group of order $2p$ is either cyclic or isomorphic to D_{2p}. (You can use the Sylow theorems.)

Note: The proof of the Jordan-Hölder Theorem will not be covered in class. See the online notes for an explanation.

OPTIONAL PROBLEM: Let p be a prime number and let G be a group of order p^r for some $r \geq 1$. Show that every composition factor of G is isomorphic to Z_p. How many factors are there in a composition series for G?

Recommended reading

Online notes on Jordan-Hölder theorem.