MODERN ALGEBRA I GU4041

Homework 12, due December 7: Sylow theorems and groups of small order

1. Let A be a finite abelian group of order N. Let $p_1 < p_2 < \cdots < p_n$ denote the distinct prime numbers dividing N.

(a) Prove that A has a unique Sylow p-subgroup A_i of order **a power of** p_i for $i = 1, \ldots, n$.

(b) Show that

$$A \xrightarrow{\sim} A_1 \times A_2 \times \cdots \times A_n.$$

2. Construct p-Sylow subgroups of the symmetric groups S_3, S_4, S_5 for $p = 2, 3, 5$.

3. Let $p > 3$ be a prime number. Show that any group of order $3p$ is solvable.

4. Show that no group of order 64 or 96 is simple. Construct two distinct non-abelian groups of each order.

5. Show that no group of order 112 is simple. (Hint: if the group G is simple then it admits an injective homomorphism to the symmetric group S_r, where r is the number of 2-Sylow subgroups.)

6. Judson, section 14.5, exercises 11, 12; section 15.4, exercises 1, 3, 6, 7, 9, 20, 22, 23. (**This problem will not be graded.**)

Recommended reading

Gallagher notes 18, 19, 22, 23, 24; Judson, Chapter 15.