MODERN ALGEBRA I GU4041

Homework 1, due September 14: Sets and functions

1. Prove the second of De Morgan's Laws: if Y and Z are subsets of the set X, then
$$X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z).$$

2. Let A, B, C be subsets of the set X. Prove $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ by proving that each side is contained in the other.

3 (a) Let A be the set of numbers $\{22, 100, 5\}$. List all subsets of A. How many of the subsets contain an even number?

(b) Let B be the set of integers between 1 and 100. How many subsets of B have at most 2 elements?

4. Let $\mathbb{R}$ be the set of real numbers. Define a relation $\sim$ on $\mathbb{R}$ by saying $t \sim t'$ if and only if t radians and t' radians are the same angle.

(a) Show that $\sim$ is an equivalence relation.

(b) Let $\mathbb{R}/\sim$ be the set of equivalence classes. Let S denote the unit circle in the plane around the origin. Construct a function $f : \mathbb{R}/\sim \to S$ and show that it is bijective.

(c) Write an expression for the function f in terms of trigonometric functions.

5. Let S be a set. Using the definitions carefully, show that there is exactly one function from the empty set $\emptyset$ to S. For which sets S is this function injective? Surjective?

6. Let $a, b, c, d \in \mathbb{R}$. Consider the map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$:
$$\phi(x, y) = (ax + by, cx + dy).$$

(a) Use the definitions to prove that ϕ is surjective if and only if, for all $e \in \mathbb{R}$, $f \in \mathbb{R}$, the linear equations
$$L_1 : ax + by = e$$
$$L_2 : cx + dy = f$$
have a solution.

(b) Show that ϕ is surjective if and only if ϕ is injective.

(c) Find numbers a, b, c, d, all different from 0, such that the map ϕ is not surjective. Show in that case that the image of ϕ is a line.

7. Let $A = \{u, v, w, x\}$, and $B = \{1, 2\}$.

(a) How many functions are there from B to A? From A to B?

(b) Let F be the set of functions from A to B. Let c be the function from B to B that switches 1 and 2. For $f \in F$, let $C(f) = c \circ f$. Show that $C : F \to F$ is a bijection. Is there any function $f \in F$ such that $C(f) = f$?

Recommended reading

Sections 4 and 5 of Gallagher's notes, at `https://www.math.columbia.edu/~khovanov/modAlgSpring2017/Gallagher/`.