MODERN ALGEBRA I GU4041

Homework 3, due September 28: Basic properties of groups

1. Let X be a set with two elements e, f.

(a) Can you define a binary operation

$$\star : X \times X \to X$$

that is not associative?

(b) Now suppose X has three elements e, f, g, and e is a two-sided identity:

$$e \star e = e; e \star f = f \star e = f; e \star g = g \star e = g.$$

Is $\star$ necessarily associative?

2. List all subgroups of the cyclic groups $\mathbb{Z}/5\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$. How many subgroups contain 3 elements in each case?

3. Let $n \geq 3$ be an integer. Let Δ_n be a regular polygon with n sides in the complex plane, with one vertex at the point 1 and the other vertices on the circle $x^2 + y^2 = 1$. Let μ_n denote the set of vertices of Δ_n.

(a) Use either the exponential function or trigonometric functions to list the coordinates of the points in μ_n.

(b) Show that the subset $\mu_n \subset \mathbb{C}$ is a group under multiplication.

(c) Define an isomorphism of groups $f : \mathbb{Z}/n\mathbb{Z} \to \mu_n$.

(d) How many solutions does part (c) have? Explain.

4. (a) Show that the set of 2×2 matrices with real coefficients and determinant different from 0 forms a group $GL(2, \mathbb{R})$, where the operation is matrix multiplication. Show by an example that it is not commutative.

(b) Define subgroups of $GL(2, \mathbb{R})$ of order 2, 3, and 4.

5. Judson book, Exercises 2 and 10, section 3.5.

Recommended reading

Howie book, Chapter 1; you should do as many exercises as you can.